# Stable forbidden subgraphs - what is behind?

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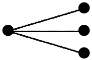

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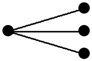

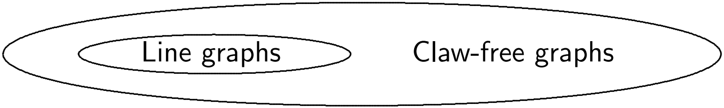

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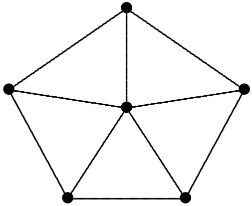
A vertex $x \in V(G)$ is *locally connected* if its neighborhood $N_G(x)$ induces in $G$ a connected graph.



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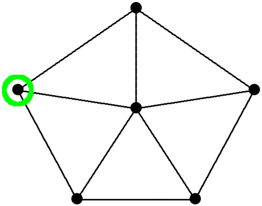

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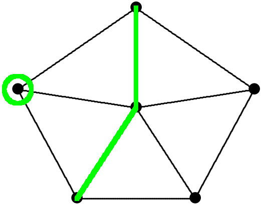

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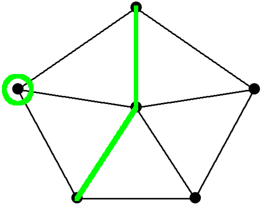

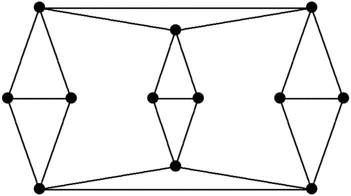

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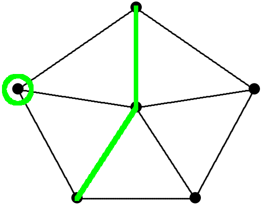

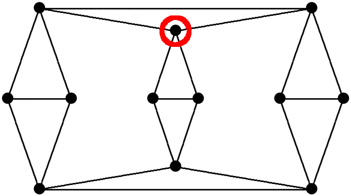

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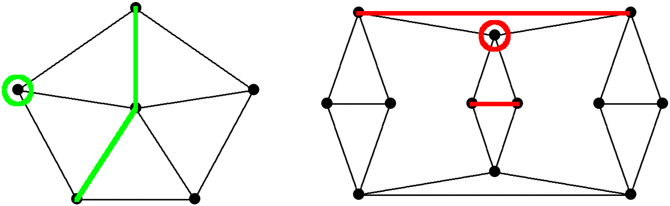

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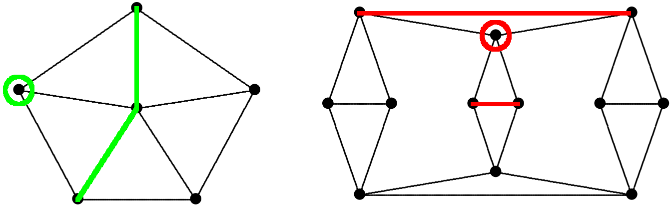

A locally connected vertex with noncomplete neighborhood is called *eligible*.

Let $x \in V(G)$ be an eligible vertex.
The *local completion* of a graph $G$ at $x$: the graph $G^*_x$ with
$V(G^*_x) = V(G)$,
$E(G^*_x) = E(G) \cup \{xy | \ x, y \in N(x)\}$
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$c(G)$: the *circumference* of $G$ (the length of a longest cycle in $G$)

**Theorem [ZR 1997].** *Let $x$ be an eligible vertex of a claw-free graph $G$ and let $G_x^*$ be the local completion of $G$ at $x$. Then*
$(i)$ $G_x^*$ *is claw-free,*
$(ii)$ $c(G_x^*) = c(G)$.

**Corollary.** $G_x^*$ *is hamiltonian if and only if $G$ is hamiltonian.*

The *closure of* $G$: the (?) graph $\mathrm{cl}(G)$ obtained from $G$ by recursively performing the local completion operation at eligible vertices, as long as this is possible.

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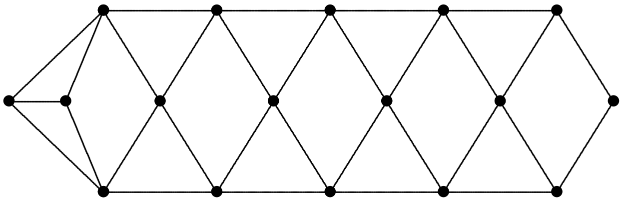

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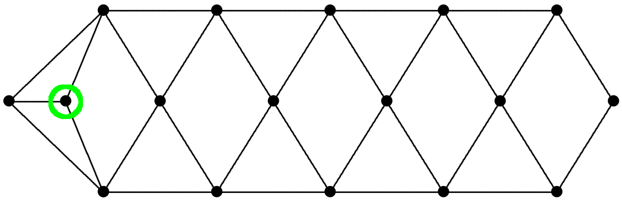

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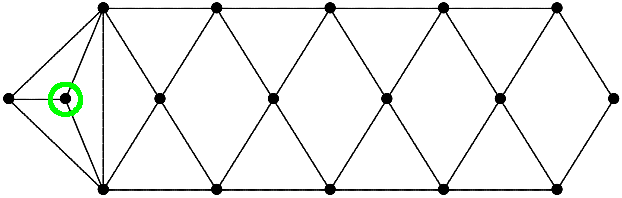

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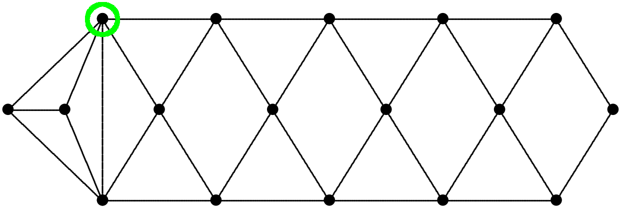

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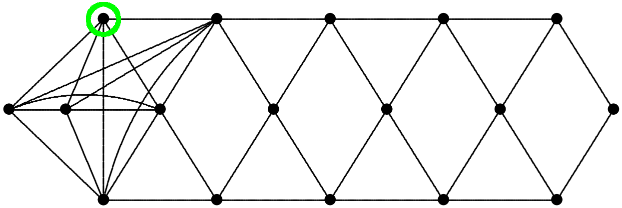

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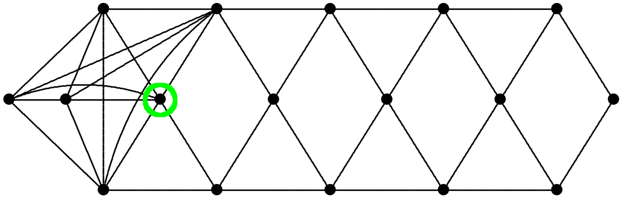

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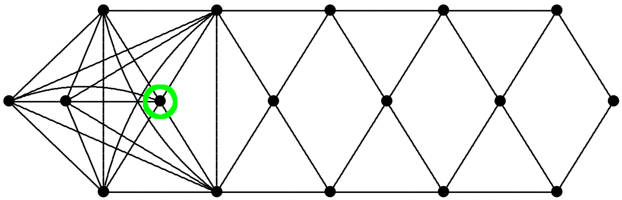

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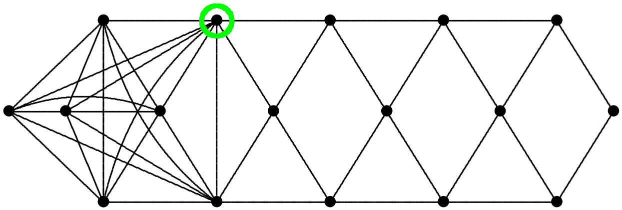

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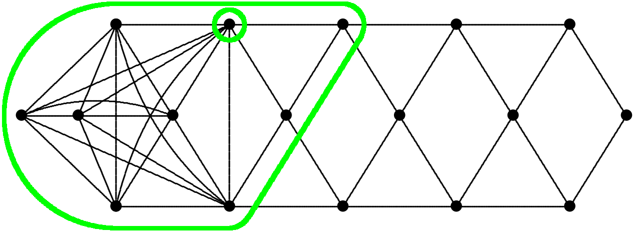

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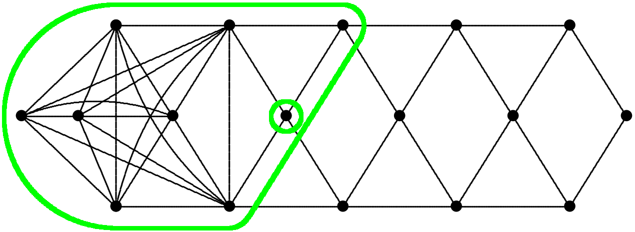

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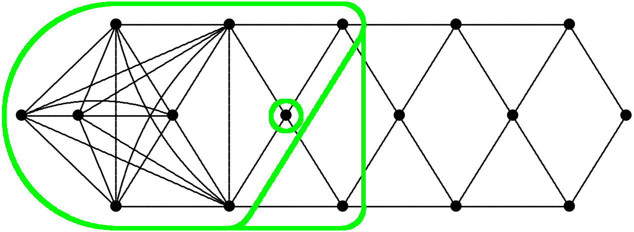

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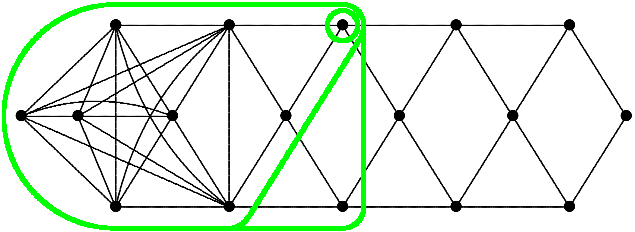

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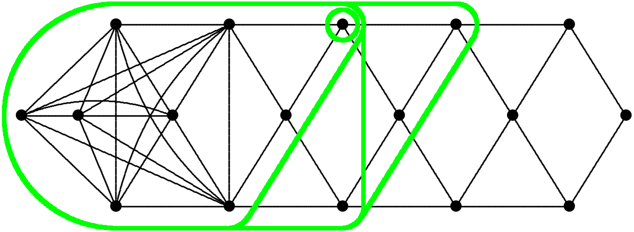

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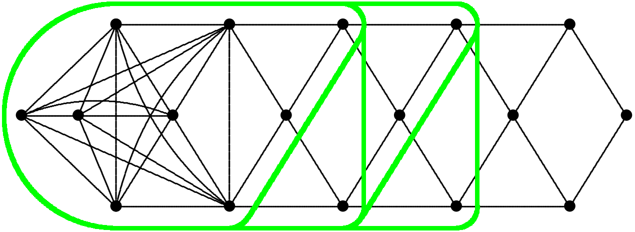

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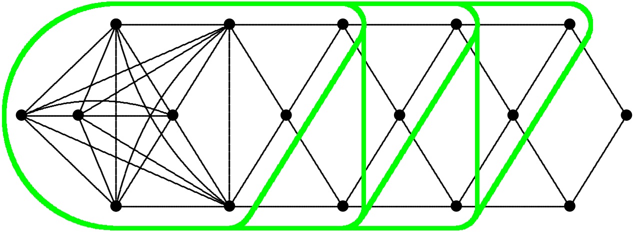

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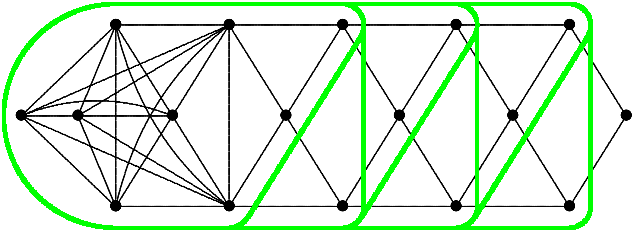

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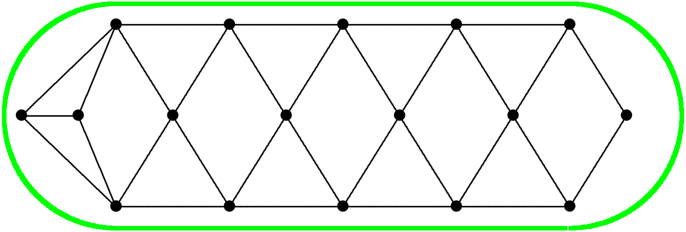

**$\mathrm{cl}(G)$ is complete**

**Theorem [ZR 1997].** *Let $G$ be a claw-free graph. Then*

(*i*) $cl(G)$ *is uniquely determined,*

(*ii*) $c(cl(G)) = c(G)$,

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We say that

- Hamiltonicity is a *stable property*
- $c(G)$ is a *stable invariant*

in the class of claw-free graphs.

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$L(H)$: the *line graph* of a graph $H$
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(An eulerian subgraph $T \subset H$ is *dominating* if every edge of $H$ has at least one vertex on $T$.)

| Property / invariant | Stable | Connectivity |
|---|---|---|
| Circumference | YES | 1 |
| Hamiltonicity | YES | 1 |
| Having a 2-factor with $\leq k$ components | YES | 1 |
| Minimum number of components in a 2-factor | YES | 1 |
| Having a cycle cover with $\leq k$ cycles | YES | 1 |
| Minimum number of cycles in a cycle cover | YES | 1 |
| (Vertex) pancyclicity | NO | any $\kappa \geq 2$ |
| (Full) cycle extendability | NO | any $\kappa \geq 2$ |
| Length of a longest path | YES | 1 |
| Traceability | YES | 1 |
| Having a path factor with $\leq k$ components | YES | 1 [Ishizuka] |
| Minimum number of components in a path factor | YES | 1 [Ishizuka] |
| Having a path cover with $\leq k$ paths | YES | 1 [Ishizuka] |
| Minimum number of paths in a path cover | YES | 1 [Ishizuka] |
| Hamilton-connectedness | NO | 3 |
| | ??? | $4 \leq \kappa \leq 5$ |
| | YES | 6 |

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| Homogeneous traceability | NO<br>???<br>YES | 3<br>$4 \leq \kappa \leq 5$<br>6 |
| Having a $P_3$-factor | NO<br>YES | 1<br>2 [Kaneko et al.] |
| Flower property | YES | 1 |
| Hamiltonian index | YES | 1 |
| 2-factor index | YES | 1 [Xiong, Saito] |
| Minimum number of components in a 2-factor in $k$-th iterated line graph | YES | 1 [Xiong, Saito] |
| Supereulerian index | YES | 1 [Xiong, Li] |
| Having a spanning trail with one prescribed endvertex | YES | 1 [Xiong, Fu] |
| Having hamiltonian prism | YES | 1 [Čada] |
| Tutte orientability | YES | 1, $\delta \geq 7$ [Kriesell] |

## Forbidden subgraphs for hamiltonicity

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**Theorem [Bedrossian 1991; Faudree, Gould 1997].**
*Let $X, Y$ be connected graphs with $X, Y \not\simeq P_3$ and let $G$ be a 2-connected graph of order $n \geq 10$ that is not a cycle. Then, $G$ being $XY$-free implies $G$ is hamiltonian if and only if (up to a symmetry) $X = C$ and $Y$ is an induced subgraph of some of the graphs $P_6$, $Z_3$, $B_{1,2}$ or $N_{1,1,1}$.*

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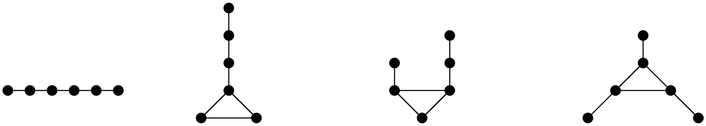

$P_6$ $\quad$ $Z_3$ $\quad$ $B_{1,2}$ *The wounded (generalized bull)* $\quad$ $N_{1,1,1}$ *The net*

**Question:** For which connected graphs $Y$, the class of $CY$-free graphs is stable?

A class of graphs $\mathcal{C}$ is *stable under cl* if, for any $x \in V(G)$,

$$G \in \mathcal{C} \Rightarrow G_x^* \in \mathcal{C}.$$

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**Theorem [Brousek, Favaron, ZR, Schiermeyer 1999].**
*Let $Y$ be a closed connected claw-free graph. Then the class of $CY$-free graphs is stable if and only if*
$Y \in \{T\} \cup \{P_i |\ i \geq 3\} \cup \{N_{i,j,k} |\ i,j,k \geq 1\}$.

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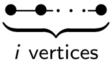
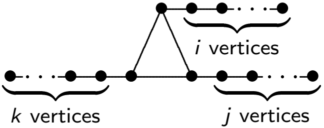
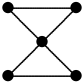
$P_i$




$N_{i,j,k}$ (generalized net)



Example



The hourglass $H$

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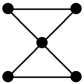

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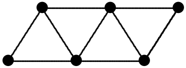
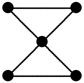
$G$



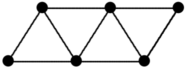
$G$ is $(C, H)$-free

Example



The hourglass $H$

$G$



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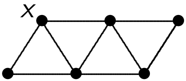

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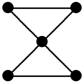

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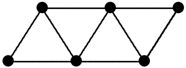

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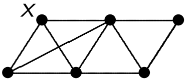

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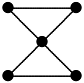

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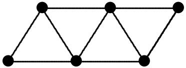

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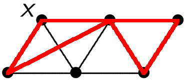


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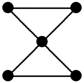

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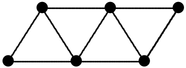

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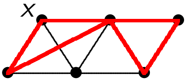


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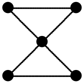
The class of $(C, H)$-free graphs is not stable.

Example



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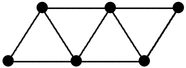

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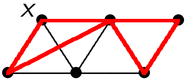


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BUT: $\mathrm{cl}(G)$ is complete $\Rightarrow$ $\mathrm{cl}(G)$ is $(C, H)$-free

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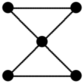

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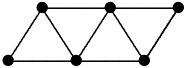

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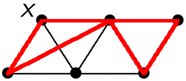


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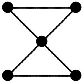

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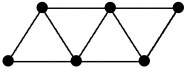

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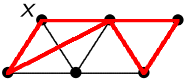


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Can be proved: the class of $(C, H)$-free graphs is weakly stable.

**Question:** For which connected graphs $Y$, the class of $CY$-free graphs is weakly stable?

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**Theorem [Brousek, Favaron, ZR, Schiermeyer 1999].**
*Let $Y$ be a closed connected claw-free graph. Then the class of $CY$-free graphs is weakly stable if and only if*
$Y \in \{H, T\} \cup \{P_i | \ i \geq 3\} \cup \{Z_i | \ i \geq 1\} \cup \{N_{i,j,k} | \ i, j, k \geq 1\}$.

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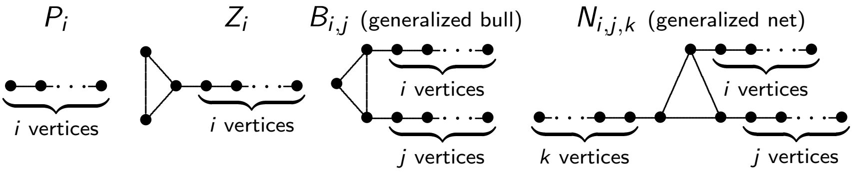

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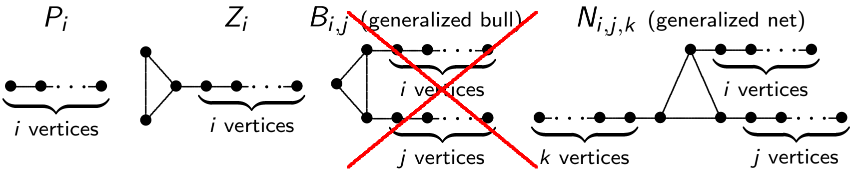

## Comparison

Forbidden subgraphs for hamiltonicity:
$X = C, \quad Y \in \{P_6, Z_3, B_{1,2}, N_{1,1,1}\}$.

Stable classes:
$X = C$,
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$G$ is $CB_{i,j}$-free while
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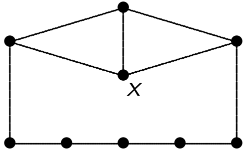

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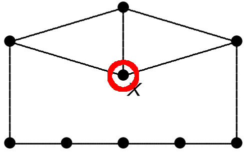

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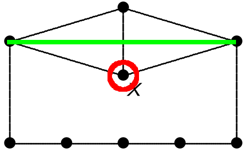

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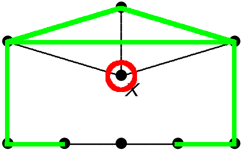

**Theorem [ZR 2002].** *Let $G$ be a 2-connected graph. Then*

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**Theorem [ZR 2002].** *Let $G$ be a 2-connected $XY$-free graph of order $n \geq 11$, where $X, Y$ is a pair of connected graphs such that $G$ being $XY$-free implies $G$ is hamiltonian. Then $G$ is claw-free and $cl_C(G)$ is $N_{1,1,1}$-free.*

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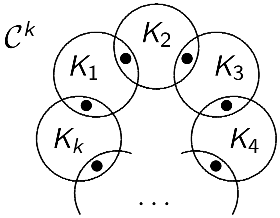
*Moreover, $cl_C(G)$ is either complete or $cl_C(G) \in \mathcal{C}^{\geq 4}$.*



## Cycle closure

$G$ closed, $H = L^{-1}(G)$, $C_k \subset G$ induced, $4 \leq k \leq 6$.
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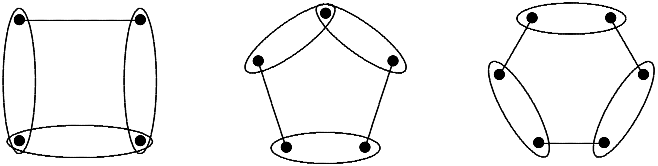
In $G$:



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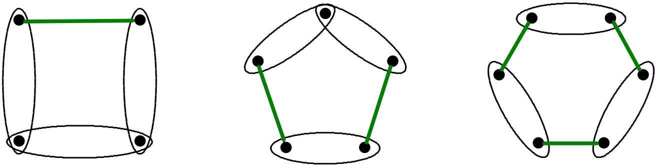

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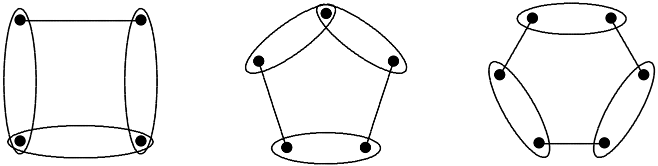

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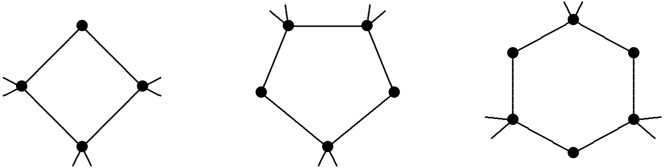

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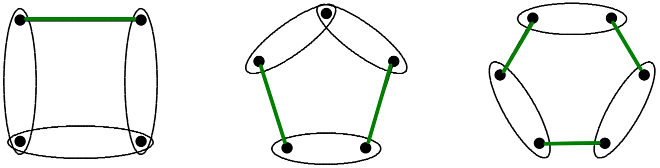
In $G$: 

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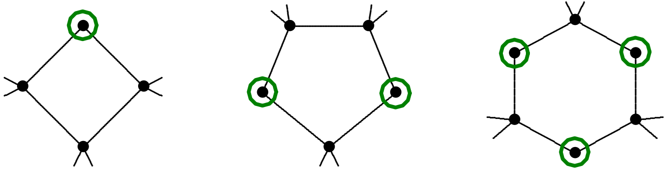
In $H$: 

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$G'_C$ – *$C$-completion* of $G$ at $C$: cover $N(C)$ with a clique

The *cycle closure* $\mathrm{cl}_C(G)$ of $G$: repeat

- local completions at eligible vertices
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**Theorem [Broersma, ZR 2001].** *Let $G$ be a claw-free graph. Then*

(i) $cl_C(G)$ *is well-defined,*
(ii) $c(G) = c(cl_C(G))$.

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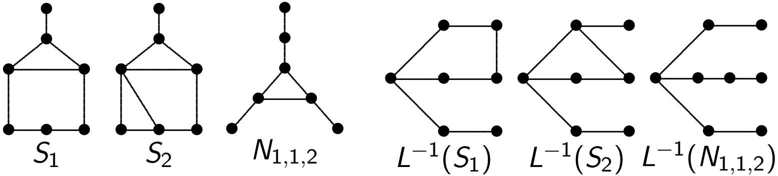

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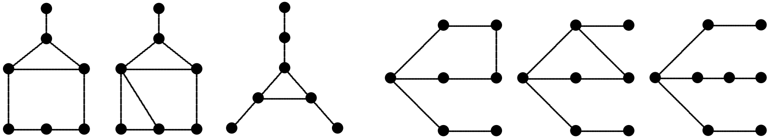

$S_1$ $S_2$ $N_{1,1,2}$ $L^{-1}(S_1)$ $L^{-1}(S_2)$ $L^{-1}(N_{1,1,2})$

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$(i)$ If $G_x^*$ contains $S_1, S_2$ or $N_{1,1,2}$, then so does $G$
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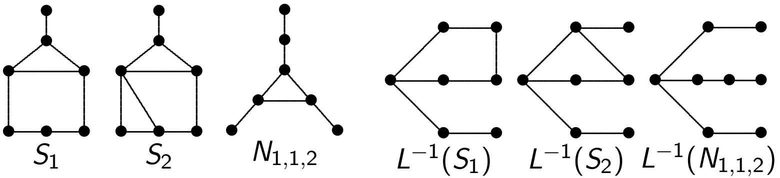


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($iii$) *$G$ is $CZ_3$-free $\Rightarrow$ $cl_C(G)$ is $CN_{1,1,1}$-free.*

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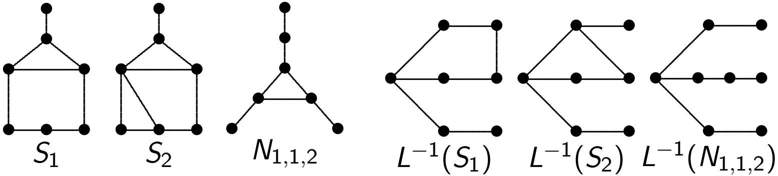
Idea of proof of ($ii$):




**Claims:**

($i$) If $G^*_x$ contains $S_1, S_2$ or $N_{1,1,2}$, then so does $G$ (i.e., the $CS_1S_2N_{1,1,2}$-free class is stable).

($ii$) Each of $S_1, S_2, N_{1,1,2}$ contains an induced $B_{1,2}$.

($iii$) If cl($G$) contains $N_{1,1,1,}$, then cl($G$) contains $S_1, S_2$ or $N_{1,1,2}$.

## 2-factors

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**Theorem [ZR, Saito, Schelp 1999].**
*Let $G$ be a claw-free graph. Then $cl(G)$ has a 2-factor with at most $k$ components if and only if $G$ has a 2-factor with at most $k$ components.*

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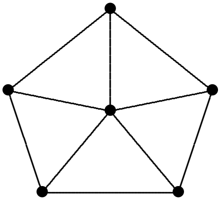

No 2-factor with 2 components
$\quad$ cl($G$) complete

## Closure for 2-factors

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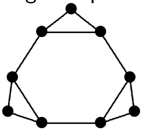
*2f-eligible cycles* in $G$: induced $C_4$, $C_5$ or edge-antipodal $C_6$

*Edge-antipodal cycle*: for any pair of antipodal edges, one of them is not in a larger clique

Edge-antipodal



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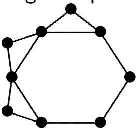
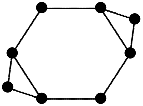


Not edge-antipodal



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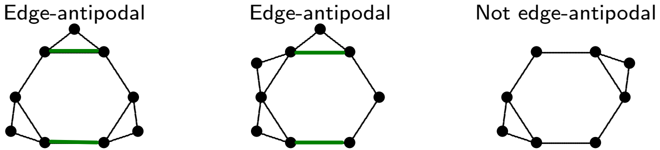

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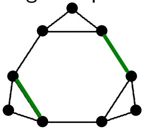

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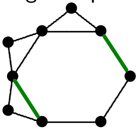

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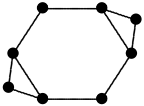

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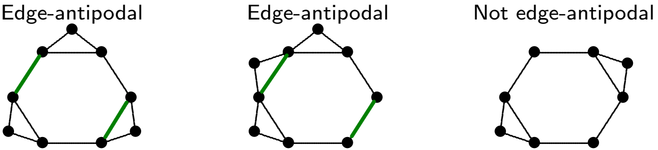

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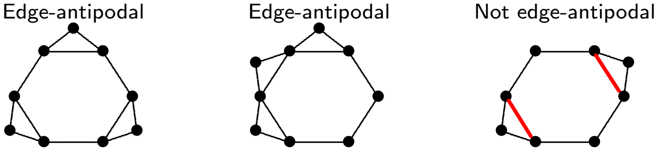

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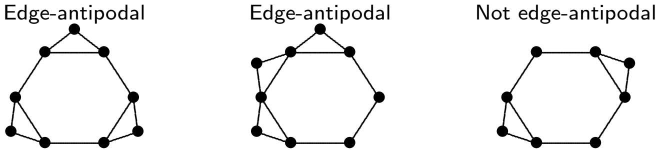


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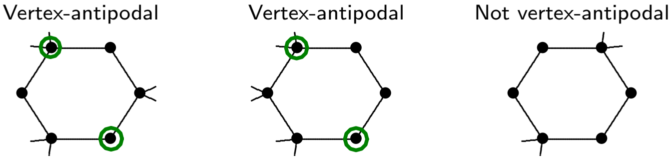
In $H = L^{-1}(G)$: $C_4$, $C_5$ or vertex-antipodal $C_6$



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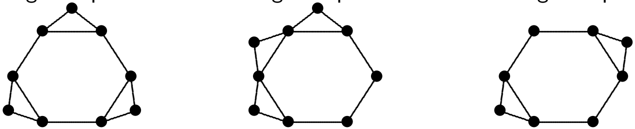

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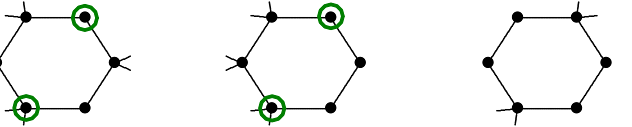

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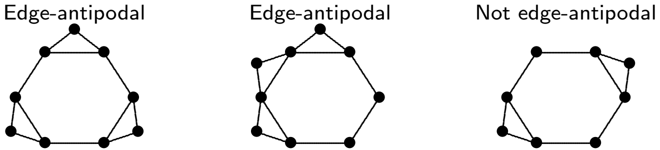


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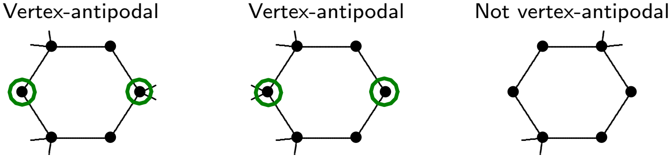

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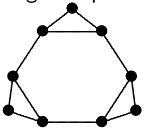

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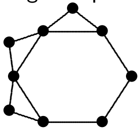

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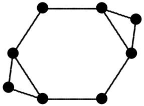

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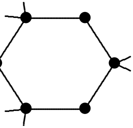

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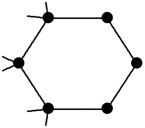

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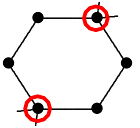

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*2-factor-closure* $\mathrm{cl}^{2f}(G)$ of $G$: repeat local completions at 2f-eligible vertices as long as this is possible.

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  ($\alpha$) $L(H) = cl^{2f}(G)$,

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Note: $\mathrm{cl}^{2f}(G)$ does not preserve

– (non)-hamiltonicity

– minimum number of components of a 2-factor

# Application: forbidden subgraphs for 2-factors

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**Theorem [Faudree, Faudree, ZR 2008].**
*Let $X$ and $Y$ be connected graphs with $X, Y \not\simeq P_3$, and let $G$ be a 2-connected graph of order $n \geq 10$. Then, $G$ being $XY$-free implies that $G$ has a 2-factor if and only if, up to the order of the pairs, either*

$(i)$ $\{X, Y\} = \{K_{1,4}, P_4\}$, *or*

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**Theorem ZR, Saburov 2010].**
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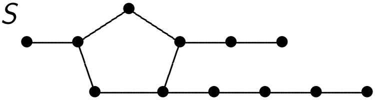
Crucial part:
*If $G$ is 2-connected and $CB_{1,4}$-free, then $cl^{2f}(G)$ is $N_{1,1,3}$-free.*

---

Idea of proof:
*Good sun*: contains an $L^{-1}(B_{1,4})$ (as a subgraph).



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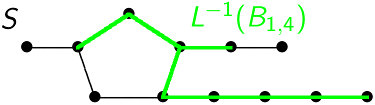

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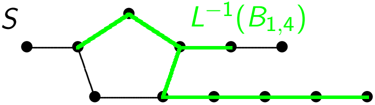


**Claims:**

($i$) If $G_x^*$ contains $L(S)$ for some good sun $S$, then so does $G$ (i.e., the $CL(S)$-free class is stable).

($ii$) The line graph of every good sun contains an induced $B_{1,4}$.

($iii$) If $\mathrm{cl}^{2f}(G)$ contains $N_{1,1,3}$, then $\mathrm{cl}^{2f}(G)$ contains $L(S)$ for some good sun $S$.

**More closures:**

- Cycle closure (for hamiltonicity, closes *some* cycles $C_4$, $C_5$, $C_6$
- Contraction closure (for hamiltonicity, generalizes the Catlin's collapsibility technique)
- Multigraph closure (for Hamilton-connectedness, turns a claw-free graph into a line graph of a multigraph)
- Strong multigraph closure (for Hamilton-connectedness, turns a claw-free graph into a line graph of a multigraph with 'few' triangles or multiedges)
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All of them use local completions, with a different definition of eligibility.

# Time for generalization

## Some definitions

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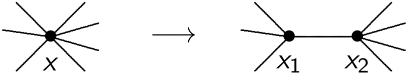
Splitting of type 1 of a vertex $x$:



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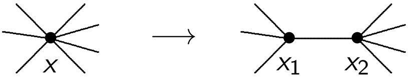


Splitting of type 2
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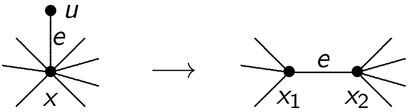

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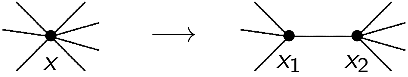


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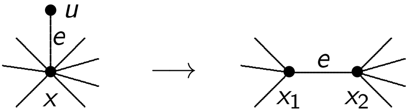
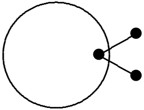



Cherry: a pair of pendant edges
with a common vertex

No cherry: *cherry-free*



**Proposition 1.** *Let $G$ be a claw-free graph and $x \in V(G)$. If $G_x^*$ contains an induced subgraph $X$ such that $X = L(Y)$, where $Y$ is a triangle-free and cherry-free graph, then $G$ contains an induced subgraph $X'$ such that $X' = L(Y')$, where either*

- *$Y$ is isomorphic to $Y'$, or*
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$K$ – a largest clique in $X$ containing $B$

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It needs some work to get the following:

**Claim.** *The graph $K_S$ satisfies one of the following:*

(*i*) *$K_S$ has two components and both are cliques,*

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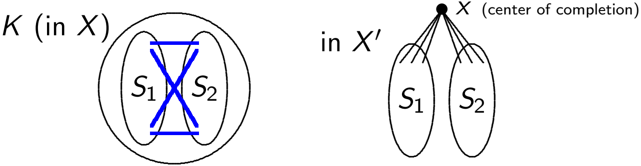

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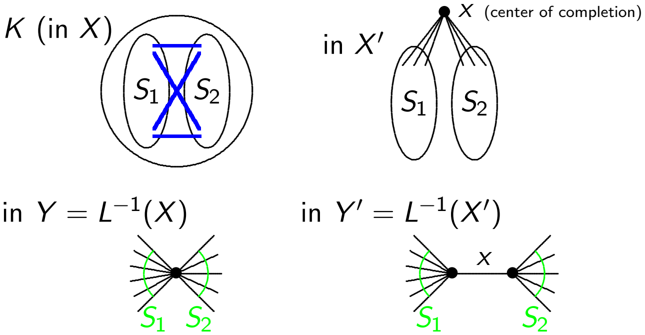


Vertex splitting of type 1

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Case (*ii*):

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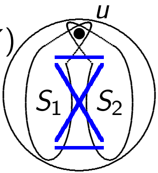


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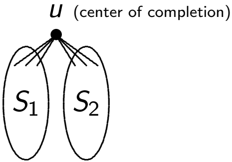

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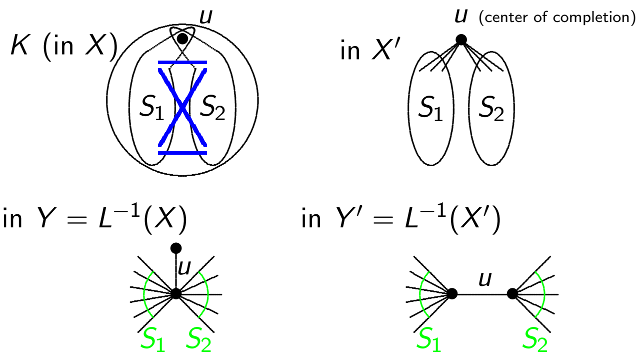


Vertex splitting of type 2

**Examples.**

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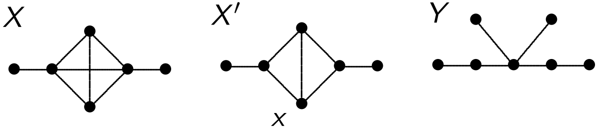


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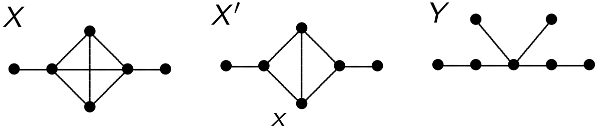


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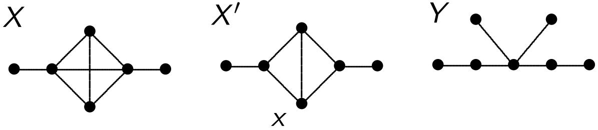


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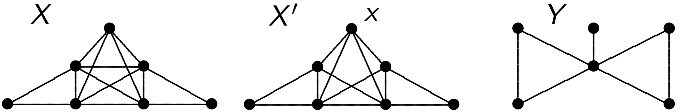


Similarly, if $X' \overset{\text{IND}}{\subset} G$, then $X \overset{\text{IND}}{\subset} G_x^*$ and $X = L(Y)$, but $X'$ (and also $X' - x$) is not a line graph.
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A family of graphs $\mathcal{Y}$ is *closed under vertex splitting* (or briefly *split-closed*), if, for any $Y \in \mathcal{Y}$ and any $Y'$ obtained from $Y$ by vertex splitting (of type 1 or 2), $Y'$ contains a subgraph $Y'' \in \mathcal{Y}$.

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The following result characterizes families of closed forbidden subgraphs that yield stable classes.

**Theorem 2.** *Let $\mathcal{H}$ be the class of all triangle-free and cherry-free graphs and let $\mathcal{Y} \subset \mathcal{H}$ and $\mathcal{X} = L(\mathcal{Y})$. Then the class $\mathrm{Forb}(C, \mathcal{X})$ is stable if and only if $\mathcal{Y}$ is split-closed.*

**Examples. 1.** $\mathrm{Forb}(C, \mathcal{X})$, where

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**2.** The graphs $S_1$, $S_2$ and $N_{1,1,2}$:
Splitting of any vertex in any of the graphs $L^{-1}(S_1)$, $L^{-1}(S_2)$ or $L^{-1}(N_{1,1,2})$ gives a graph containing $L^{-1}(N_{1,1,2})$, hence $\mathrm{Forb}(C, S_1, S_2, N_{1,1,2})$ is stable.

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**3.** $L(\mathcal{S})$, where $\mathcal{S}$ is the family of good suns:
Splitting of any vertex in a good sun gives a good sun
(this is clearly true for splitting of type 1; for splitting of type 2 needs some more work but can be done).
Hence $\mathrm{Forb}(C, L(\mathcal{S}))$ is stable.

**4.** $\mathcal{C}_k = \{C_i\}_{i=k}^{\infty}$ – all cycles of lengths $i \geq k$ ($k \geq 3$ fixed). Splitting of any vertex in a cycle of length $i$ gives a cycle of length $i+1$, hence $\mathrm{Forb}(C, \{C_i\}_{i=k}^{\infty})$ is stable for any fixed $k \geq 3$.

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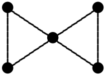
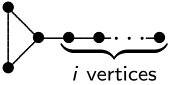
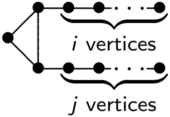


$Z_i$




$B_{i,j}$ (generalized bull)



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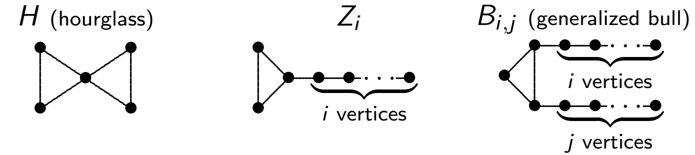
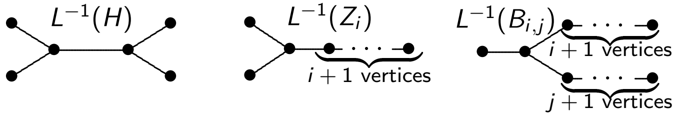



If $Y$ is any of the graphs $L^{-1}(H)$, $L^{-1}(Z_i)$ or $L^{-1}(B_{i,j})$,




then splitting of type 2 of a vertex of degree 3 yields a graph not containing $Y$ as a subgraph. Hence none of the classes $\mathrm{Forb}(C, H)$, $\mathrm{Forb}(C, Z_i)$, $\mathrm{Forb}(C, B_{i,j})$ is stable.

## Minimal families of forbidden subgraphs

We know: $\mathrm{Forb}(C, S_1, S_2, N_{1,1,2})$ is a stable class.

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$Y$ is a *subdivision of a star* : $Y$ can be obtained from a star $K_{1,r}$ by adding at least one vertex of degree 2 to each of its edges.

Special case $r = 3$: $Y$ is a *subdivision of the claw*.

Note: $Y$ is a subdivision of the claw $\Leftrightarrow$ $L(Y)$ is a generalized net.

**Proposition 3.** *Let* $\mathcal{Y} = \{Y_1, \ldots, Y_k\}$, $k \geq 1$, *be split-closed . Then some* $Y_i \in \mathcal{Y}$ *is a path or a subdivision of the claw.*

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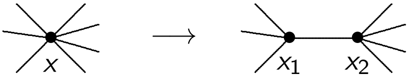

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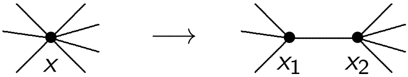


A contradiction with the choice of $T$.

From Proposition 3 immediately:

**Theorem 4.** *Let $\mathcal{X}$ be a finite family of line graphs of triangle-free and cherry-free graphs such that $\mathrm{Forb}(C, \mathcal{X})$ is stable. Then $\mathcal{X}$ contains a path or a generalized net.*

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**Proposition 5.** *Let $\mathcal{Y}$ be a (possibly infinite) split-closed family of graphs. Then $\mathcal{Y}$ contains a proper subfamily $\mathcal{Y}' \subset \mathcal{Y}$, $\mathcal{Y}' \neq \mathcal{Y}$, such that $\mathcal{Y} \setminus \mathcal{Y}'$ is split-closed, unless $|\mathcal{Y}| = 1$ and $\mathcal{Y} = \{Y\}$, where $Y$ is a path or a subdivision of the claw.*

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Remains: $\eta(T) \leq 1$ and $|V_{\geq 4}(T)| = 0$, ie., $T$ is a path or a subdivision of the claw.

Immediately from Proposition 5:

**Theorem 6.** *Let* $\mathcal{X}$ *be a family of line graphs of triangle-free and cherry-free graphs such that* $\mathrm{Forb}(C, \mathcal{X})$ *is stable. Then* $\mathcal{X}$ *contains a subfamily* $\mathcal{X}' \subset \mathcal{X}$, $\mathcal{X}' \neq \mathcal{X}$, *such that* $\mathrm{Forb}(C, \mathcal{X} \setminus \mathcal{X}')$ *is stable, unless* $|\mathcal{X}| = 1$ *and* $\mathcal{X}$ *contains a path or a generalized net.*

## Unstable classes and stabilizers

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$\mathfrak{cl}$ – a closure operation that

(*i*) turns a claw-free graph into a line graph by a series of local completions at eligible vertices

(*ii*) the definition of eligibility can vary in specific cases.

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## Unstable classes and stabilizers

$\mathfrak{cl}$ – a closure operation that

(*i*) turns a claw-free graph into a line graph by a series of local completions at eligible vertices

(*ii*) the definition of eligibility can vary in specific cases.

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- Closure for hamiltonicity
- Cycle closure (for hamiltonicity)
- Closure for 2-factor
- Contraction closure (for hamiltonicity)
- Multigraph closure (for Hamilton-connectedness)
- Strong multigraph closure (for Hamilton-connectedness)
- Closure for 1-Hamilton-connectedness

$\mathcal{B}$ – a family of connected line graphs of triangle-free and cherry-free graphs

$\mathcal{G}$ – a family of connected closed (under $\mathfrak{cl}$) claw-free graphs

$k \geq 2$ – an integer

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A family $\mathcal{S}$ of connected triangle-free and cherry-free graphs s.th.

(*i*) $\mathcal{S}$ is split-closed,

(*ii*) every $S \in \mathcal{S}$ contains a subgraph isomorphic to $L^{-1}(B)$ for some $B \in \mathcal{B}$,

(*iii*) every $\mathfrak{cl}$-closed $k$-connected $(C, L(\mathcal{S}))$-free graph is $(C, \mathcal{G})$-free,

is called a *$k$-stabilizer for $\mathcal{B}$ into $\mathcal{G}$ under* $\mathfrak{cl}$, or briefly a $(k, \mathcal{B}, \mathcal{G}, \mathfrak{cl})$-*stabilizer*.

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In the special case when $\mathcal{B} = \{B\}$ and $\mathcal{G} = \{Q\}$ we simply say that $\mathcal{S}$ is a $(k, B, Q, \mathfrak{cl})$*-stabilizer*.

**Theorem.** *Let $k \geq 1$, let $\mathcal{B}$, $\mathcal{G}$ be classes of graphs such that there is a $(k, \mathcal{B}, \mathcal{G}, \mathfrak{cl})$-stabilizer, and let $G$ be a $k$-connected $(C, \mathcal{B})$-free graph. Then $\mathfrak{cl}(G)$ is $(C, \mathcal{G})$-free.*

Equivalently,

$$\mathfrak{cl}(\mathrm{Forb}_k(C, \mathcal{B})) \subset \mathrm{Forb}_k(C, \mathcal{G}).$$

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**Examples. 1.** Let $\mathcal{S} = \{S_1, S_2, N_{1,1,2}\}$. Then $\mathcal{S}$ is a $(2, B_{1,2}, N_{1,1,1}, \mathrm{cl})$-stabilizer.

**2.** Similarly, the family $\mathcal{S}$ of all good suns is a $(2, B_{1,3}, N_{1,1,2}, \mathrm{cl}^{2f})$-stabilizer.

Consider again the family $\mathrm{Forb}(C, \mathcal{S})$, where $\mathcal{S} = \{S_1, S_2, N_{1,1,2}\}$.

- $\mathrm{Forb}(C, \mathcal{S})$ is stable,
- $\mathcal{S}$ is not minimal stable (since $\mathrm{Forb}(C, N_{1,1,2}\}$ is also stable),
- $\mathcal{S}$ is a $(2, B_{1,2}, N_{1,1,1}, \mathrm{cl})$-stabilizer,
- but no proper subfamily of $\mathcal{S}$ is a $(2, B_{1,2}, N_{1,1,1}, \mathrm{cl})$-stabilizer (i.e., as a stabilizer, $\mathcal{S}$ is minimal).

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**Open problem**: a characterization of minimal (in some sense) stabilizers for given families $\mathcal{B}$, $\mathcal{G}$.

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**Thank you**