Hourglasses, Hamilton-connectedness and 1-Hamilton-connectedness

Zdeněk Ryjáček

Department of Mathematics, University of West Bohemia and Institute for Theoretical Computer Science, Charles University, Plzeň, Czech Republic

Joint work with Liming Xiong, Jun Yin and Petr Vrána

Domažlice 2013

Hamilton-connectedness

Hamilton-connectedness

Internally dominating trail (IDT): internally dominates all edges

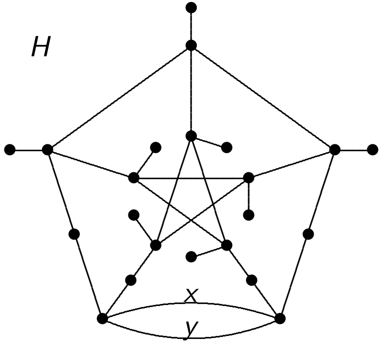
$G = L(H)$ is Hamilt.-connected $\Leftrightarrow$ H has (e, f)-IDT $\forall e, f \in E(H)$

Hamilton-connectedness

Internally dominating trail (IDT): internally dominates all edges

$G = L(H)$ is Hamilt.-connected $\Leftrightarrow$ H has (e, f)-IDT $\forall e, f \in E(H)$

$G = L(H)$:



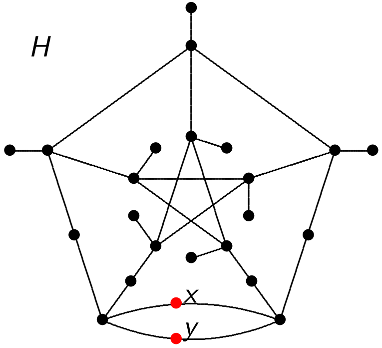
Hamilton-connectedness

Internally dominating trail (IDT): internally dominates all edges

$G = L(H)$ is Hamilt.-connected $\Leftrightarrow$ H has (e, f)-IDT $\forall e, f \in E(H)$

$G = L(H)$:

H has no (x, y)-IDT



Hamilton-connectedness

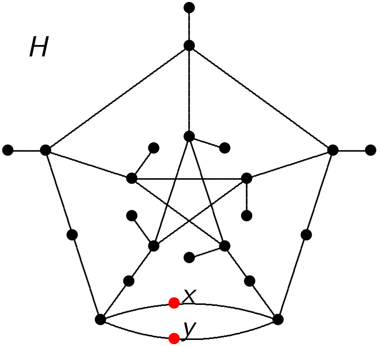
Internally dominating trail (IDT): internally dominates all edges

$G = L(H)$ is Hamilt.-connected $\Leftrightarrow$ H has (e, f)-IDT $\forall e, f \in E(H)$

$G = L(H)$:

H has no (x, y)-IDT

G is not Hamilton-connected



Hamilton-connectedness

Internally dominating trail (IDT): internally dominates all edges

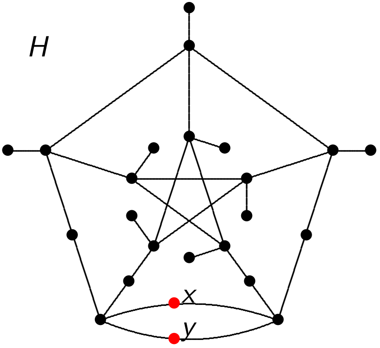
$G = L(H)$ is Hamilt.-connected $\Leftrightarrow$ H has (e, f)-IDT $\forall e, f \in E(H)$

$G = L(H)$:

H has no (x, y)-IDT

G is not Hamilton-connected

G'_x is Hamilton-connected



Hamilton-connectedness

Internally dominating trail (IDT): internally dominates all edges

$G = L(H)$ is Hamilt.-connected $\Leftrightarrow$ H has (e, f)-IDT $\forall e, f \in E(H)$

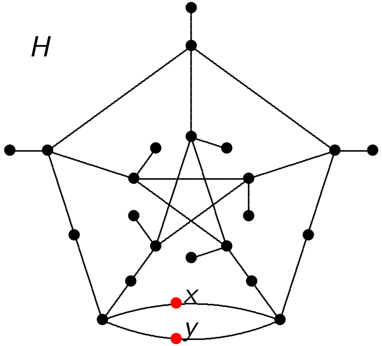
$G = L(H)$:

H has no (x, y)-IDT

G is not Hamilton-connected

G'_x is Hamilton-connected

Hamilton-connectedness
is NOT STABLE
in 3-connected claw-free graphs



Closure for Hamilton-connectedness

Closure for Hamilton-connectedness

$\mathrm{cl}_k(G)$: closing only neighborhoods of locally k-connected vertices.

Closure for Hamilton-connectedness

$\mathrm{cl}_k(G)$: closing only neighborhoods of locally k-connected vertices.

Bollobás, Riordan, ZR., Saito, Schelp, 1999:
Theorem. *Hamilton-connectedness is stable under* $cl_3(G)$.
Conjecture. *Hamilton-connectedness is stable under* $cl_2(G)$.

Closure for Hamilton-connectedness

$\mathrm{cl}_k(G)$: closing only neighborhoods of locally k-connected vertices.

Bollobás, Riordan, ZR., Saito, Schelp, 1999:
Theorem. *Hamilton-connectedness is stable under* $cl_3(G)$*.*
Conjecture. *Hamilton-connectedness is stable under* $cl_2(G)$*.*

Theorem [ZR., Vrána, 2010].
Hamilton-connectedness is stable under $cl_2(G)$*.*

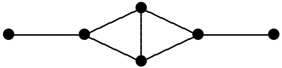
Closure for Hamilton-connectedness

$\mathrm{cl}_k(G)$: closing only neighborhoods of locally k-connected vertices.

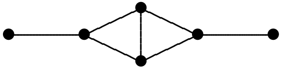
Bollobás, Riordan, ZR., Saito, Schelp, 1999:
Theorem. *Hamilton-connectedness is stable under $cl_3(G)$.*
Conjecture. *Hamilton-connectedness is stable under $cl_2(G)$.*

Theorem [ZR., Vrána, 2010].
Hamilton-connectedness is stable under $cl_2(G)$.

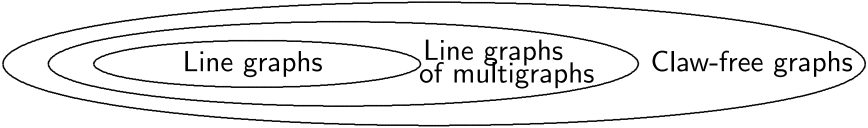
What is the structure of $\mathrm{cl}_2(G)$?

Not a line graph: 

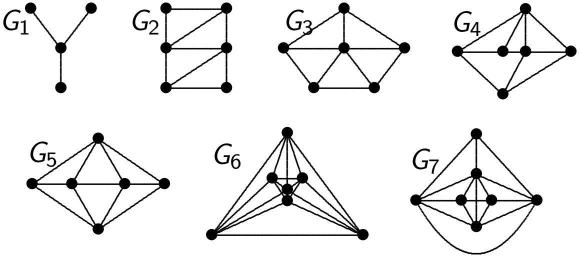
Not a line graph:



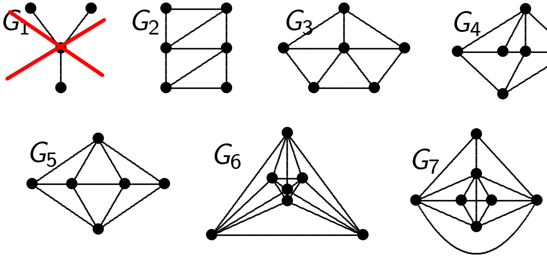
Line graph of a multigraph?



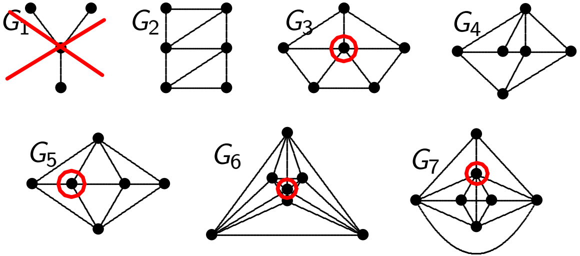
Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



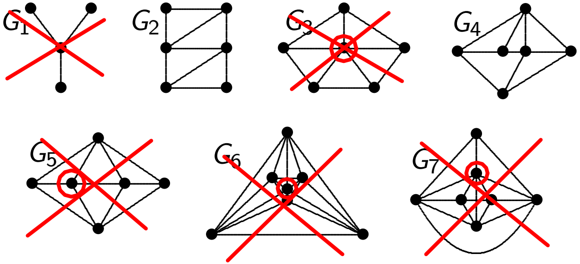
Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



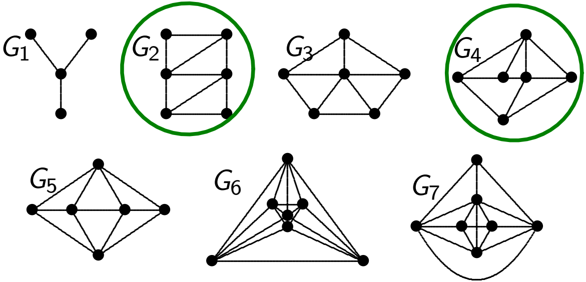
Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph G is a line graph of a multigraph *if and only if G does not contain a copy of any of the following graphs as an* induced *subgraph.*



Thus, in $\mathrm{cl}_2(G)$, only G_2 or G_4 can remain.

It needs some more work to get the following:

It needs some more work to get the following:

Theorem [ZR., Vrána, 2010].
Let G be a claw-free graph. Then there is a graph $cl^M(G)$ such that:

(*i*) $cl^M(G)$ *is uniquely determined,*
(*ii*) *there is a multigraph H such that* $cl^M(G) = L(H)$,
(*iii*) G *is Hamilton-connected if and only if* $cl^M(G)$ *is Hamilton-connected.*

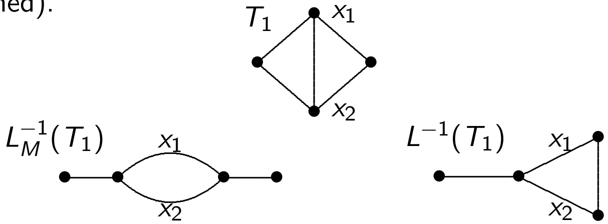
It needs some more work to get the following:

Theorem [ZR., Vrána, 2010].
Let G be a claw-free graph. Then there is a graph $cl^M(G)$ such that:

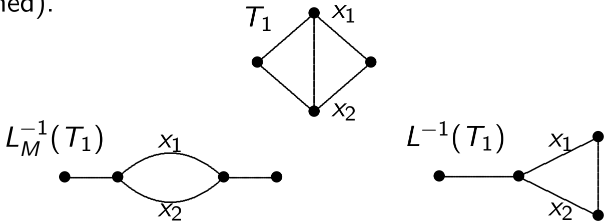
(*i*) *$cl^M(G)$ is uniquely determined,*
(*ii*) *there is a multigraph H such that $cl^M(G) = L(H)$,*
(*iii*) *G is Hamilton-connected if and only if $cl^M(G)$ is Hamilton-connected.*

Hamilton-connectedness is stable under $\mathrm{cl}^M(G)$.

Drawback: there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the "preimage" is not uniquely determined).

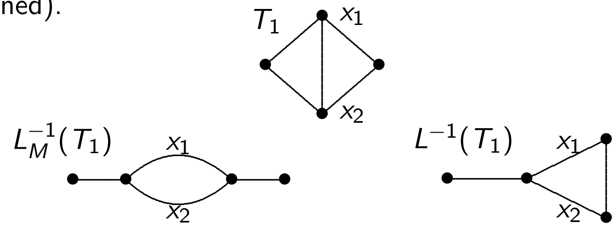


Drawback: there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the "preimage" is not uniquely determined).



Theorem. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

Drawback: there can be multigraphs H_1, H_2 such that $H_1 \not\simeq H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the "preimage" is not uniquely determined).

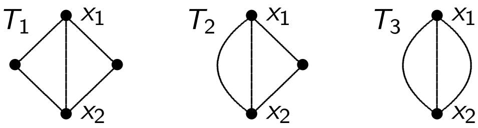


Theorem. *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

If G is a line graph of a graph, then the "multigraph preimage $L_M^{-1}(G)$" and the obvious line graph preimage $L^{-1}(G)$ can be different!

M-closed graphs

Proposition. *Let G be a claw-free graph and let T_1, T_2, T_3 be the graphs shown in the figure. Then G is M-closed if and only if G is a line graph of a multigraph and $L_M^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs T_1, T_2 or T_3.*



Strengthening the M-closure

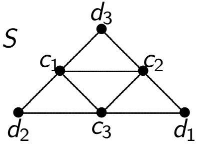
Strengthening the M-closure

Proposition [Brandt, Favaron, ZR, 2000] *G claw-free, $x \in V(G)$ eligible, a, b two distinct vertices of G. Then for every longest (a,b)-path $P'(a,b)$ in G'_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a,b)-path $P(a,b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

Strengthening the M-closure

Proposition [Brandt, Favaron, ZR, 2000] *G claw-free, $x \in V(G)$ eligible, a, b two distinct vertices of G. Then for every longest (a,b)-path $P'(a,b)$ in G'_x there is a path P in G such that $V(P) = V(P')$ and P admits at least one of a, b as an endvertex. Moreover, there is an (a,b)-path $P(a,b)$ in G such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between a and b):*

- (i) *There is an induced subgraph $H \subset G$ isomorphic to the graph S such that both a and x are vertices of degree 4 in H. In this case G contains a path P_b such that b is an endvertex of P and $V(P_b) = V(P')$. If, moreover, $b \in V(H)$, then G contains also a path P_a with endvertex a and with $V(P_a) = V(P')$.*
- (ii) *$x = a$ and $ab \in E(G)$. In this case there is always both a path P_a in G with endvertex a and with $V(P_a) = V(P')$ and a path P_b in G with endvertex b and with $V(P_b) = V(P')$.*



We construct a graph G^M by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \cdots G_k$ such that

  (1) $G_1 = G$,
  (2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i), i = 1, \cdots, k$,
  (3) G_k has no Hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
  (4) for any $x \in V_{EL}(G_k), (G_k)^*_x$ is Hamilton-connected,

  and we set $G^M = G_k$.

We construct a graph G^M by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \cdots G_k$ such that

(1) $G_1 = G$,
(2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i), i = 1, \cdots, k$,
(3) G_k has no Hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
(4) for any $x \in V_{EL}(G_k), (G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

G^M is called *a strong M-closure* (or briefly an *SM-closure*) of G; a graph G equal to its SM-closure is said to be *SM-closed*.

We construct a graph G^M by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.

(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \cdots G_k$ such that

(1) $G_1 = G$,
(2) $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{EL}(G_i), i = 1, \cdots, k$,
(3) G_k has no Hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
(4) for any $x \in V_{EL}(G_k), (G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

G^M is called *a strong M-closure* (or briefly an *SM-closure*) of G; a graph G equal to its SM-closure is said to be *SM-closed*. Note that, for a given graph G, its SM-closure is not necessarily uniquely determined.

Theorem. *Let G be a claw-free graph and let G^M be its SM-closure. Then G is Hamilton-connected if and only if G^M is Hamilton-connected, G^M is a line graph, and $H = L^{-1}(G^M)$ satisfies one of the following conditions:*

(1) *H is a triangle-free simple graph;*

(2) *There are $e, f \in E(H)$ such that there is no (e, f)-IDT and either*

  (α) *H is triangle-free and e, f is the only multiedge in H,*

  (β) *H is a simple graph containing at most 2 triangles with no common edge, and each triangle contains at least one of e, f.*

Theorem. *Let G be a claw-free graph and let G^M be its SM-closure. Then G is Hamilton-connected if and only if G^M is Hamilton-connected, G^M is a line graph, and $H = L^{-1}(G^M)$ satisfies one of the following conditions:*

(1) *H is a triangle-free simple graph;*

(2) *There are $e, f \in E(H)$ such that there is no (e, f)-IDT and either*

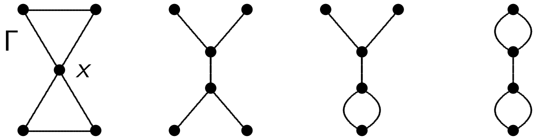
   (α) *H is triangle-free and e, f is the only multiedge in H,*

   (β) *H is a simple graph containing at most 2 triangles with no common edge, and each triangle contains at least one of e, f.*

Theorem. *Let G be an SM-closed graph and let $H = L^{-1}(G)$. Then H does not contain a triangle with a vertex of degree 2.*

The *hourglass*: is the unique graph Γ with degree sequence $4, 2, 2, 2, 2$.
The vertex $x \in V(\Gamma)$ of degree 4 is called the *center* of Γ and we also say that Γ is *centered at* x.

Γ is a line graph and, in multigraphs, it has three nonisomorphic preimages:



Question: under which conditions on a claw-free graph G, G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected?

Question: under which conditions on a claw-free graph G, G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected?

Proposition [M.Li. X.Chen, Broersma 2011] *Let G be a connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph and let x be a locally connected vertex of G. Then G_x^* is also $\{K_{1,3}, (P_6)^2, \Gamma\}$-free.*

Question: under which conditions on a claw-free graph G, G is Hamilton-connected if and only if $\mathrm{cl}(G)$ is Hamilton-connected?

Proposition [M.Li. X.Chen, Broersma 2011] *Let G be a connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph and let x be a locally connected vertex of G. Then G^*_x is also $\{K_{1,3}, (P_6)^2, \Gamma\}$-free.*

Theorem [M.Li. X.Chen, Broersma 2011] *Let G be a 3-connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph with minimum degree at least 4. Then G is Hamilton-connected if and only if $cl(G)$ is Hamilton-connected.*

Theorem. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ and let G^M be an SM-closure of G. If G^M is hourglass-free, then G^M is the only SM-closure of G, and $G^M = cl(G)$.*

Theorem. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ and let G^M be an SM-closure of G. If G^M is hourglass-free, then G^M is the only SM-closure of G, and $G^M = cl(G)$.*

Corollary. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ such that G has a hourglass-free SM-closure G^M. Then G is Hamilton-connected if and only if $cl(G)$ is Hamilton-connected.*

Theorem. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ and let G^M be an SM-closure of G. If G^M is hourglass-free, then G^M is the only SM-closure of G, and $G^M = cl(G)$.*

Corollary. *Let G be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ such that G has a hourglass-free SM-closure G^M. Then G is Hamilton-connected if and only if $cl(G)$ is Hamilton-connected.*

Corollary. *Let G be a 2-connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph with minimum degree $\delta(G) \geq 3$. Then G is Hamilton-connected if and only if $cl(G)$ is Hamilton-connected.*

Theorem [Folklore, 90's]
Every 4-connected claw-free hourglass-free graph is hamiltonian

1-Hamilton-connectedness

1-Hamilton-connectedness

G is 1-*Hamilton-connected* if $G - x$ is Hamilton-connected for every vertex $x \in V(G)$.

1-Hamilton-connectedness

G is 1-*Hamilton-connected* if $G - x$ is Hamilton-connected for every vertex $x \in V(G)$.

Note that a 1-Hamilton-connected graph must be 4-connected.

1HC-closure:

1HC-closure:

Proposition. *Let G be a claw-free graph. Then there is a sequence of graphs $G_0, \ldots, G_k$ such that*

- (i) $G_0 = G$,
- (ii) $V(G_i) = V(G_{i+1})$ *and* $E(G_i) \subset E(G_{i+1}) \subset E((G_i)^*_{x_i})$ *for some* $x_i \in V(G_i)$ *eligible in* G_i,
- (iii) G_k *is a 1HC-closure of* G.

1HC-closure:

Proposition. *Let G be a claw-free graph. Then there is a sequence of graphs $G_0, \ldots, G_k$ such that*

(i) $G_0 = G$,

(ii) $V(G_i) = V(G_{i+1})$ *and* $E(G_i) \subset E(G_{i+1}) \subset E((G_i)^*_{x_i})$ *for some* $x_i \in V(G_i)$ *eligible in* G_i,

(iii) G_k *is a 1HC-closure of* G.

Theorem. *Let G be a claw-free graph and let $\overline{G}$ be its 1HC-closure. Then*

(i) $\overline{G}$ *is a line graph,*

(ii) *for some* $x \in V(\overline{G})$, *the graph* $\overline{G} - x$ *is SM-closed,*

(iii) $\overline{G}$ *is 1-Hamilton-connected if and only if* G *is 1-Hamilton-connected.*

Theorem. *Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.*

Theorem. *Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.*

Recall: G is 1-Hamilton-connected $\Rightarrow$ G is 4-connected.

Thus, if G is claw-free and hourglass-free, then
G is 1-Hamilton-connected $\iff$ G is 4-connected

Theorem. *Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.*

Recall: G is 1-Hamilton-connected $\Rightarrow$ G is 4-connected.

Thus, if G is claw-free and hourglass-free, then
G is 1-Hamilton-connected $\Longleftrightarrow$ G is 4-connected

Corollary. *1-Hamilton-connectedness is polynomial-time decidable in the class of claw-free hourglass-free graphs.*

Theorem. *Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.*

Recall: G is 1-Hamilton-connected $\Rightarrow$ G is 4-connected.

Thus, if G is claw-free and hourglass-free, then
G is 1-Hamilton-connected $\iff$ G is 4-connected

Corollary. *1-Hamilton-connectedness is polynomial-time decidable in the class of claw-free hourglass-free graphs.*

Note than an analogous result is known to be true in planar graphs (an easy consequence of a 1997 result by Sanders).

Lemma. *Let G be a claw-free graph such that every induced hourglass in G is centered at an eligible vertex and let $\overline{G}$ be a 1HC-closure of G obtained by a sequence of local completions at eligible vertices. Then every induced hourglass in $\overline{G}$ is centered at an eligible vertex.*

Lemma. *Let G be a claw-free graph such that every induced hourglass in G is centered at an eligible vertex and let $\overline{G}$ be a 1HC-closure of G obtained by a sequence of local completions at eligible vertices. Then every induced hourglass in $\overline{G}$ is centered at an eligible vertex.*

Lemma. *Let G be a 4-connected claw-free hourglass-free graph. Then there is a 1HC-closure $\overline{G}$ of G such that $L^{-1}(\overline{G})$ has at most three vertices of degree three.*

Lemma. *Let G be a claw-free graph such that every induced hourglass in G is centered at an eligible vertex and let $\overline{G}$ be a 1HC-closure of G obtained by a sequence of local completions at eligible vertices. Then every induced hourglass in $\overline{G}$ is centered at an eligible vertex.*

Lemma. *Let G be a 4-connected claw-free hourglass-free graph. Then there is a 1HC-closure $\overline{G}$ of G such that $L^{-1}(\overline{G})$ has at most three vertices of degree three.*

The *core* of a graph H: the graph $\mathrm{co}(H)$ obtained by deleting all vertices of degree 1 and suppressing all vertices of degree 2

Clearly, $\delta(\mathrm{co}(H)) \geq 3$.

Theorem [Lai, Liang, Shao 2008]. *Let H be a graph such that $co(H)$ has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem [Nash-Williams 1961, Tutte 1961]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

Theorem [Lai, Liang, Shao 2008]. *Let H be a graph such that co(H) has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem [Nash-Williams 1961, Tutte 1961]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

G – 4-connected claw-free hourglass-free, $\overline{G}$ – a 1HC-closure of G such that $H = L^{-1}(\overline{G})$ has at most three vertices of degree 3.

If $\overline{G} - x = L(H - f)$, we need to show that the graph $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees.

Theorem [Lai, Liang, Shao 2008]. *Let H be a graph such that co(H) has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem [Nash-Williams 1961, Tutte 1961]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

G – 4-connected claw-free hourglass-free, $\overline{G}$ – a 1HC-closure of G such that $H = L^{-1}(\overline{G})$ has at most three vertices of degree 3.

If $\overline{G} - x = L(H - f)$, we need to show that the graph $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees.

Claim 1. *$\mathrm{co}(H) - f$ has two edge-disjoint spanning trees.*

Theorem [Lai, Liang, Shao 2008]. *Let H be a graph such that $co(H)$ has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem [Nash-Williams 1961, Tutte 1961]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

G – 4-connected claw-free hourglass-free, $\overline{G}$ – a 1HC-closure of G such that $H = L^{-1}(\overline{G})$ has at most three vertices of degree 3.

If $\overline{G} - x = L(H - f)$, we need to show that the graph $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees.

Claim 1. $\mathrm{co}(H) - f$ *has two edge-disjoint spanning trees.*
Claim 2. $\mathrm{co}(\mathrm{co}(H) - f)$ *has two edge-disjoint spanning trees.*

Theorem [Lai, Liang, Shao 2008]. *Let H be a graph such that co(H) has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any any pair of edges $e_1, e_2 \in E(H)$, H has an internally dominating (e_1, e_2)-trail.*

Theorem [Nash-Williams 1961, Tutte 1961]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

G – 4-connected claw-free hourglass-free, $\overline{G}$ – a 1HC-closure of G such that $H = L^{-1}(\overline{G})$ has at most three vertices of degree 3.

If $\overline{G} - x = L(H - f)$, we need to show that the graph $\mathrm{co}(H - f)$ has two edge-disjoint spanning trees.

Claim 1. $\mathrm{co}(H) - f$ *has two edge-disjoint spanning trees.*
Claim 2. $\mathrm{co}(\mathrm{co}(H) - f)$ *has two edge-disjoint spanning trees.*
Claim 3. $\mathrm{co}(H - f) = \mathrm{co}(\mathrm{co}(H) - f)$.

Thank you