Hourglasses, Hamilton-connectedness and 1-Hamilton-connectedness

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Domažlice 2013
Hamilton-connectedness is NOT STABLE in 3-connected claw-free graphs.
Hamilton-connectedness

*Internally dominating trail* (IDT): internally dominates all edges

$G = L(H)$ is Hamilton-connected $\iff H$ has $(e, f)$-IDT $\forall e, f \in E(H)$
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\[ X \]

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Hamilton-connectedness is NOT STABLE in 3-connected claw-free graphs
Closure for Hamilton-connectedness

Theorem. Hamilton-connectedness is stable under cl_{k(G)}.

Conjecture. Hamilton-connectedness is stable under cl_{k(G)}.

Theorem [ZR., Vrana, 2010]. Hamilton-connectedness is stable under cl_{k(G)}.

What is the structure of cl_{k(G)}?
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$cl_k(G)$: closing only neighborhoods of locally $k$-connected vertices.
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Bollobás, Riordan, ZR., Saito, Schelp, 1999:

Theorem. Hamilton-connectedness is stable under $\text{cl}_3(G)$.

Conjecture. Hamilton-connectedness is stable under $\text{cl}_2(G)$.
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What is the structure of $\text{cl}_2(G)$?
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Line graph of a multigraph?
Theorem [Hemminger 1971; Bermond, Meyer, 1973].
A graph $G$ is a line graph of a multigraph if and only if $G$ does not contain a copy of any of the following graphs as an induced subgraph.

![Graphs G1 to G7](image-url)
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A graph $G$ is a line graph of a multigraph if and only if $G$ does not contain a copy of any of the following graphs as an induced subgraph.

$G_1$ is crossed out.

$G_2$, $G_3$, $G_4$, $G_5$, $G_6$, $G_7$.
Theorem [Hemminger 1971; Bermond, Meyer, 1973].

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Thus, in $\cl_2(G)$, only $G_2$ or $G_4$ can remain.
It needs some more work to get the following:

Theorem [ZR., Vr´ ana, 2010]. Let $G$ be a claw-free graph. Then there is a graph $\text{cl}_M(G)$ such that:

(i) $\text{cl}_M(G)$ is uniquely determined,

(ii) there is a multigraph $H$ such that $\text{cl}_M(G) = L(H),

(iii) $G$ is Hamilton-connected if and only if $\text{cl}_M(G)$ is Hamilton-connected.

Hamilton-connectedness is stable under $\text{cl}_M(G)$. 
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Hamilton-connectedness is stable under $cl^M(G)$. 
Drawback: there can be multigraphs $H_1, H_2$ such that $H_1 \not\cong H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the “preimage” is not uniquely determined).
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\begin{center}
\begin{tikzpicture}
  \node (T1) at (0,0) {$T_1$};
  \node (x1) at (1,1) {$x_1$};
  \node (x2) at (1,-1) {$x_2$};
  \draw (T1) -- (x1);
  \draw (T1) -- (x2);
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**Theorem.** Let $G$ be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph $H = L_M^{-1}(G)$ such that a vertex $e \in V(G)$ is simplicial in $G$ if and only if the corresponding edge $e \in E(H)$ is a pendant edge in $H$. 
Drawback: there can be multigraphs $H_1$, $H_2$ such that $H_1 \not\cong H_2$ but $L(H_1) \simeq L(H_2)$ (i.e., the “preimage” is not uniquely determined).

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If $G$ is a line graph of a graph, then the “multigraph preimage $L_{M}^{-1}(G)$” and the obvious line graph preimage $L^{-1}(G)$ can be different!
**M-closed graphs**

**Proposition.** Let $G$ be a claw-free graph and let $T_1, T_2, T_3$ be the graphs shown in the figure. Then $G$ is M-closed if and only if $G$ is a line graph of a multigraph and $L_M^{-1}(G)$ does not contain a subgraph (not necessarily induced) isomorphic to any of the graphs $T_1, T_2$ or $T_3$. 

![Diagram of graphs T1, T2, T3]

Here are the figures of $T_1, T_2, T_3$. They represent the graphs mentioned in the proposition.
Strengthening the M-closure

Proposition [Brandt, Favaron, ZR, 2000]

$G$ claw-free, $x \in V(G)$ eligible, $a, b$ two distinct vertices of $G$. Then for every longest $(a, b)$-path $P'$ in $G'$ there is a path $P$ in $G$ such that $V(P) = V(P')$ and $P$ admits at least one of $a, b$ as an endvertex. Moreover, there is an $(a, b)$-path $P$ in $G$ such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between $a$ and $b$):

1. $c_3$ or $d_3$
2. $d_2$ or $c_1$
3. $d_1$ or $c_2$

There is an induced subgraph $H \subset G$ isomorphic to the graph $S$ such that both $a$ and $x$ are vertices of degree 4 in $H$. In this case $G$ contains a path $P_b$ such that $b$ is an endvertex of $P_b$ and $V(P_b) = V(P'_b)$. If, moreover, $b \in V(H)$, then $G$ contains also a path $P_a$ with endvertex $a$ and with $V(P_a) = V(P'_a)$.

(ii) $x = a$ and $ab \in E(G)$. In this case there is always both a path $P_a$ in $G$ with endvertex $a$ and with $V(P_a) = V(P'_a)$ and a path $P_b$ in $G$ with endvertex $b$ and with $V(P_b) = V(P'_b)$.
Strengthening the M-closure

Proposition [Brandt, Favaron, ZR, 2000] \( G \) claw-free, \( x \in V(G) \) eligible, \( a, b \) two distinct vertices of \( G \). Then for every longest \((a, b)\)-path \( P'(a, b) \) in \( G_x' \) there is a path \( P \) in \( G \) such that \( V(P) = V(P') \) and \( P \) admits at least one of \( a, b \) as an endvertex. Moreover, there is an \((a, b)\)-path \( P(a, b) \) in \( G \) such that \( V(P) = V(P') \) except perhaps in each of the following two situations (up to symmetry between \( a \) and \( b \)): 
Strengthening the M-closure

Proposition [Brandt, Favaron, ZR, 2000] \( G \) claw-free, \( x \in V(G) \) eligible, \( a, b \) two distinct vertices of \( G \). Then for every longest \((a, b)\)-path \( P'(a, b) \) in \( G'_x \) there is a path \( P \) in \( G \) such that \( V(P) = V(P') \) and \( P \) admits at least one of \( a, b \) as an endvertex. Moreover, there is an \((a, b)\)-path \( P(a, b) \) in \( G \) such that \( V(P) = V(P') \) except perhaps in each of the following two situations (up to symmetry between \( a \) and \( b \)):

(i) There is an induced subgraph \( H \subset G \) isomorphic to the graph \( S \) such that both \( a \) and \( x \) are vertices of degree 4 in \( H \). In this case \( G \) contains a path \( P_b \) such that \( b \) is an endvertex of \( P \) and \( V(P_b) = V(P') \). If, moreover, \( b \in V(H) \), then \( G \) contains also a path \( P_a \) with endvertex \( a \) and with \( V(P_a) = V(P') \).

(ii) \( x = a \) and \( ab \in E(G) \). In this case there is always both a path \( P_a \) in \( G \) with endvertex \( a \) and with \( V(P_a) = V(P') \) and a path \( P_b \) in \( G \) with endvertex \( b \) and with \( V(P_b) = V(P') \).

\[ \text{Diagram:} \]

\[ S \]

\[ \begin{array}{ccc}
S & d_3 \\
\text{c1} & \text{c2} \\
\text{d2} & \text{c3} & \text{d1} \\
\end{array} \]
We construct a graph $G^M$ by the following construction.

(i) If $G$ is Hamilton-connected, we set $G^M = \text{cl}(G)$.

(ii) If $G$ is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \cdots, G_k$ such that

1. $G_1 = G$,
2. $G_{i+1} = (G_i)_{x_i}$ for some $x_i \in V_{EL}(G_i)$, $i = 1, \cdots, k$,
3. $G_k$ has no Hamiltonian $(a, b)$-path for some $a, b \in V(G_k)$,
4. for any $x \in V_{EL}(G_k)$, $(G_k)_x$ is Hamilton-connected,

and we set $G^M = G_k$. 

$G^M$ is called a strong M-closure (or briefly an SM-closure) of $G$; a graph $G$ equal to its SM-closure is said to be SM-closed.

Note that, for a given graph $G$, its SM-closure is not necessarily uniquely determined.
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$G^M$ is called a strong M-closure (or briefly an SM-closure) of $G$; a graph $G$ equal to its SM-closure is said to be SM-closed. Note that, for a given graph $G$, its SM-closure is not necessarily uniquely determined.
Theorem. Let $G$ be a claw-free graph and let $G^M$ be its SM-closure. Then $G$ is Hamilton-connected if and only if $G^M$ is Hamilton-connected, $G^M$ is a line graph, and $H = L^{-1}(G^M)$ satisfies one of the following conditions:

1. $H$ is a triangle-free simple graph;
2. There are $e, f \in E(H)$ such that there is no $(e, f)$-IDT and either
   - (α) $H$ is triangle-free and $e, f$ is the only multiedge in $H$,
   - (β) $H$ is a simple graph containing at most 2 triangles with no common edge, and each triangle contains at least one of $e, f$.

Theorem. Let $G$ be an SM-closed graph and let $H = L^{-1}(G^M)$. Then $H$ does not contain a triangle with a vertex of degree 2.
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The *hourglass* is the unique graph $\Gamma$ with degree sequence $4, 2, 2, 2, 2$. 

The vertex $x \in V(\Gamma)$ of degree 4 is called the *center* of $\Gamma$ and we also say that $\Gamma$ is *centered at* $x$.

$\Gamma$ is a line graph and, in multigraphs, it has three nonisomorphic preimages:
**Question:** under which conditions on a claw-free graph $G$, $G$ is Hamilton-connected if and only if $\text{cl}(G)$ is Hamilton-connected?
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Proposition [M.Li. X.Chen, Broersma 2011] Let $G$ be a connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph and let $x$ be a locally connected vertex of $G$. Then $G_x^*$ is also $\{K_{1,3}, (P_6)^2, \Gamma\}$-free.
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**Theorem [M.Li. X.Chen, Broersma 2011]** Let $G$ be a $3$-connected $\{K_{1,3}, (P_6)^2, \Gamma\}$-free graph with minimum degree at least 4. Then $G$ is Hamilton-connected if and only if $\text{cl}(G)$ is Hamilton-connected.
**Theorem.** Let $G$ be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ and let $G^M$ be an SM-closure of $G$. If $G^M$ is hourglass-free, then $G^M$ is the only SM-closure of $G$, and $G^M = \text{cl}(G)$. 

**Corollary.** Let $G$ be a 2-connected claw-free graph with minimum degree $\delta(G) \geq 3$ such that $G$ has a hourglass-free SM-closure $G^M$. Then $G$ is Hamilton-connected if and only if $\text{cl}(G)$ is Hamilton-connected.

**Corollary.** Let $G$ be a 2-connected $\{K_1, 3, (P_6)^2, \Gamma\}$-free graph with minimum degree $\delta(G) \geq 3$. Then $G$ is Hamilton-connected if and only if $\text{cl}(G)$ is Hamilton-connected.
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Theorem [Folklore, 90’s]
Every 4-connected claw-free hourglass-free graph is hamiltonian.
1-Hamilton-connectedness
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$G$ is 1-Hamilton-connected if $G - x$ is Hamilton-connected for every vertex $x \in V(G)$. 
1-Hamilton-connectedness

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Note that a 1-Hamilton-connected graph must be 4-connected.
1HC-closure:

Proposition. Let $G$ be a claw-free graph. Then there is a sequence of graphs $G_0, \ldots, G_k$ such that 

(i) $G_0 = G$, 

(ii) $V(G_i) = V(G_{i+1})$ and $E(G_i) \subset E(G_{i+1}) \subset E(G_i^*)$ for some $x_i \in V(G_i)$ eligible in $G_i$, 

(iii) $G_k$ is a 1HC-closure of $G$.

Theorem. Let $G$ be a claw-free graph and let $G$ be its 1HC-closure. Then 

(i) $G$ is a line graph, 

(ii) for some $x \in V(G)$, the graph $G - x$ is SM-closed, 

(iii) $G$ is 1-Hamilton-connected if and only if $G$ is 1-Hamilton-connected.
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(iii) $G_k$ is a 1HC-closure of $G$.

**Theorem.** Let $G$ be a claw-free graph and let $\overline{G}$ be its 1HC-closure. Then

(i) $\overline{G}$ is a line graph,
(ii) for some $x \in V(\overline{G})$, the graph $\overline{G} - x$ is SM-closed,
(iii) $\overline{G}$ is 1-Hamilton-connected if and only if $G$ is 1-Hamilton-connected.
**Theorem.** Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.
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Recall: $G$ is 1-Hamilton-connected $\Rightarrow$ $G$ is 4-connected.

Thus, if $G$ is claw-free and hourglass-free, then
$G$ is 1-Hamilton-connected $\iff$ $G$ is 4-connected

**Corollary.** 1-Hamilton-connectedness is polynomial-time decidable in the class of claw-free hourglass-free graphs.

Note than an analogous result is known to be true in planar graphs (an easy consequence of a 1997 result by Sanders).
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**Corollary.** 1-Hamilton-connectedness is polynomial-time decidable in the class of claw-free hourglass-free graphs.

Note that an analogous result is known to be true in planar graphs (an easy consequence of a 1997 result by Sanders).
Lemma. Let $G$ be a claw-free graph such that every induced hourglass in $G$ is centered at an eligible vertex and let $\overline{G}$ be a $1HC$-closure of $G$ obtained by a sequence of local completions at eligible vertices. Then every induced hourglass in $\overline{G}$ is centered at an eligible vertex.
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**Lemma.** Let $G$ be a 4-connected claw-free hourglass-free graph. Then there is a 1HC-closure $\overline{G}$ of $G$ such that $L^{-1}(\overline{G})$ has at most three vertices of degree three.
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The core of a graph $H$: the graph $\text{co}(H)$ obtained by deleting all vertices of degree 1 and suppressing all vertices of degree 2.

Clearly, $\delta(\text{co}(H)) \geq 3$. 
Theorem [Lai, Liang, Shao 2008]. Let $H$ be a graph such that $\text{co}(H)$ has two edge-disjoint spanning trees and $G = L(H)$ is 3-connected. Then, for any pair of edges $e_1, e_2 \in E(H)$, $H$ has an internally dominating $(e_1, e_2)$-trail.

Theorem [Nash-Williams 1961, Tutte 1961]. A graph $G$ has $k$ edge-disjoint spanning trees if and only if
\[ |E_0| \geq k(\omega(G - E_0) - 1) \]
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Claim 3. $\text{co}(H - f) = \text{co}(\text{co}(H) - f)$.
Thank you