Tutte cycles and cycles of length linear of the order in line graphs

Kenta Ozeki (National Institute of Informatics, Japan)
(JST, ERATO, Kawarabayashi Large Graph Project)

Joint work with

Roman Čada, Petr Vrána（University of West Bohemia）
Kiyoshi Yoshimoto (Nihon University）
Shuya Chiba (Tokyo University of Science）

Main targets of this talk

The following 2 kinds of cycles

in line graphs (and claw-free graphs)

I. Tutte cycles

- ✓ Equivalent conj.s to Thomassen's conj.
- ✓ A result and problem on the Ryjáček closure
- ✓ Why are Tutte cycles good for the plane graphs?

II. Cycles of linear length with the # of vertices

- ✓ Linear length is enough for the case $\delta \geq 5$
- ✓ Relationship to Bondy's conjecture

Tutte cycles

C : Tutte cycle of G

$\Updownarrow$

$\boxed{|C| \geq 4}$ and

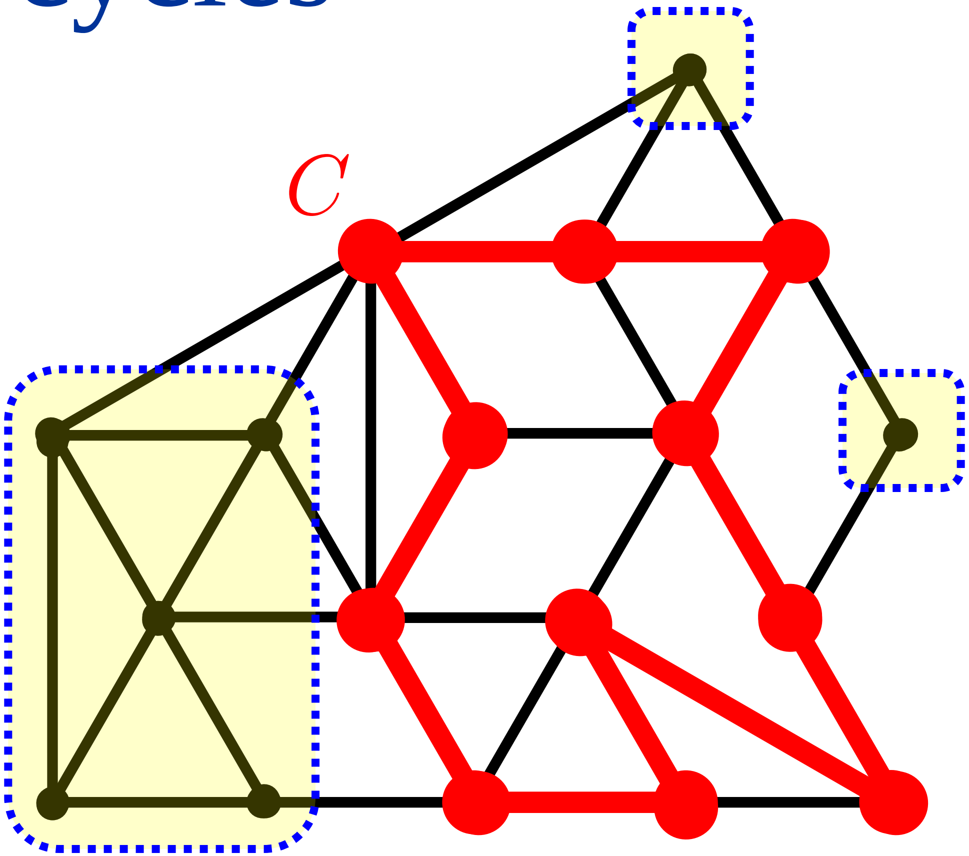
$\forall$ compo. of $G - V(C)$

has ≤ 3 neighbors on C

If G : 4-conn. graphs

$\Rightarrow$ A Tutte cycle in G

is a Hamiltonian cycle



A Tutte cycle C in G

History of Tutte cycles

Tutte cycles in graphs on surfaces

- Some ides can be found in Whitney's paper (`31)
- The main idea was posed by Tutte (`56, `77)

> Thm. G : 2-conn. plane graph $\Rightarrow \exists$ Tutte cycle

- Extended by the following researchers
  - ✓ Thomassen (`83)
  - ✓ Chiba & Nishizeki (`86)
  - ✓ Thomas &Yu (& Zang) (`94, `97, `05)
  - ✓ Sanders (`96)
  - ✓ Kawarabayashi & Ozeki (`13+)

History of Tutte cycles

Tutte cycles in graphs NOT on surfaces

- Almost nothing is known
- Jackson's conjecture (`92)

Conj.1 G : 2-conn. claw-free graph $\Rightarrow \exists$ Tutte cycle

History of Tutte cycles

<u>Tutte cycles in graphs NOT on surfaces</u>

- Almost nothing is known
- Jackson's conjecture (`92)

<u>Conj.1</u> G : 2-conn. claw-free graph$\Rightarrow \exists$ Tutte cycle

- Main Thm.

<u>Thm.</u> Jackson's conj. is equiv. to Thomassen's conj.

Easy: Jackson's conj. $\Rightarrow$ Thomassen's conj.

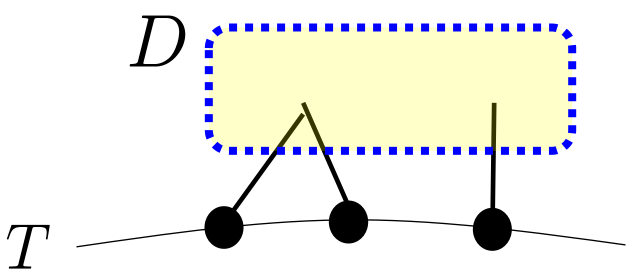
Main part: Thomassen's conj. $\Rightarrow$ Jackson's conj.

Equivalence to the line graph ver.

$\boxed{\underline{\text{Conj.2}}\ G: \text{2-conn. line graph} \Rightarrow \exists\ \text{Tutte cycle}}$

$\uparrow$ Line graph $\qquad\qquad \downarrow$ Preimage

$\boxed{\underline{\text{Conj.3}}\ H: \text{ess. 2-e-conn. graph} \Rightarrow \exists\ T: \text{Tutte closed trail}}$



T: Tutte closed trail

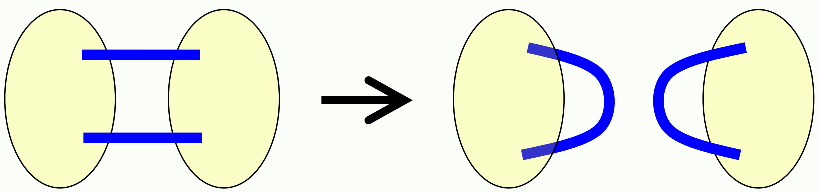
$\Leftrightarrow$

$^\forall D$: compo. of $H - V(T)$ with ≥ 1 edges,

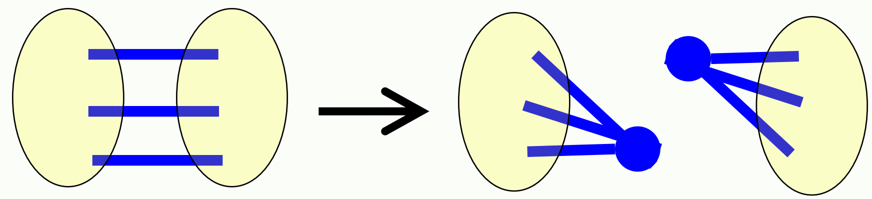
# of edges between D and T is ≤ 3

Equivalence to the line graph ver.

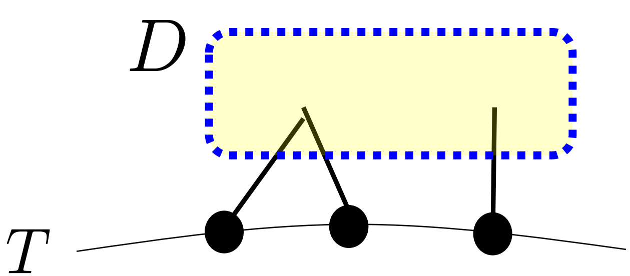
$\boxed{\text{\underline{Conj.3} } H : \text{ess. 2-e-conn. graph} \Rightarrow \exists\, T : \text{Tutte closed trail}}$



(ess.) 2-edge-cut



ess. 3-edge-cut



T : Tutte closed trail

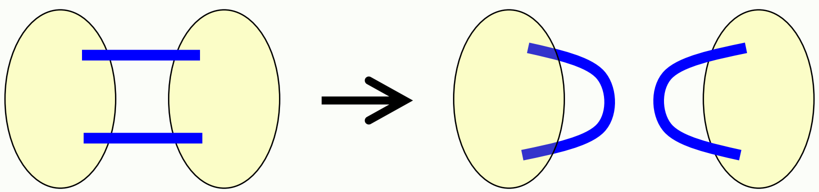
$\Leftrightarrow$

$\forall D$: compo. of $H - V(T)$ with ≥ 1 edges,

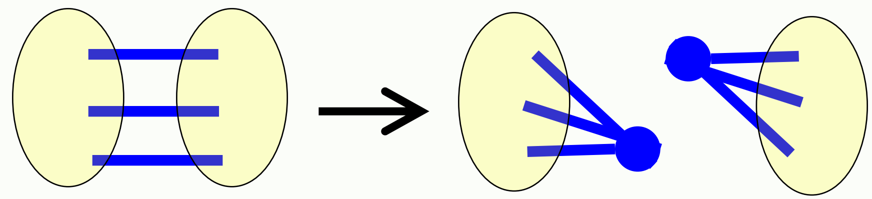
\# of edges between D and T is ≤ 3

Equivalence to the line graph ver.

<u>Conj.3</u> H : ess. 2-e-conn. graph $\Rightarrow \exists\, T$: Tutte closed trail



(ess.) 2-edge-cut

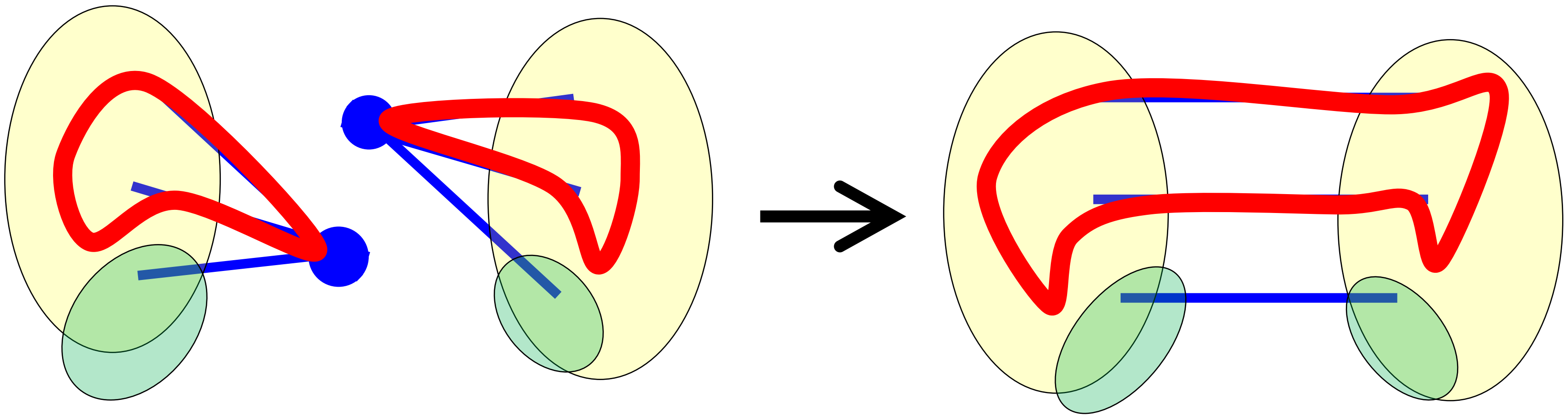


ess. 3-edge-cut

- To connect 2 Tutte CTs in each part,
  we need to specify edges that must be on T
- And more!

Equivalence to the line graph ver.

Conj.3 H : ess. 2-e-conn. graph $\Rightarrow \exists\, T$: Tutte closed trail



- To connect 2 Tutte CTs in each part,

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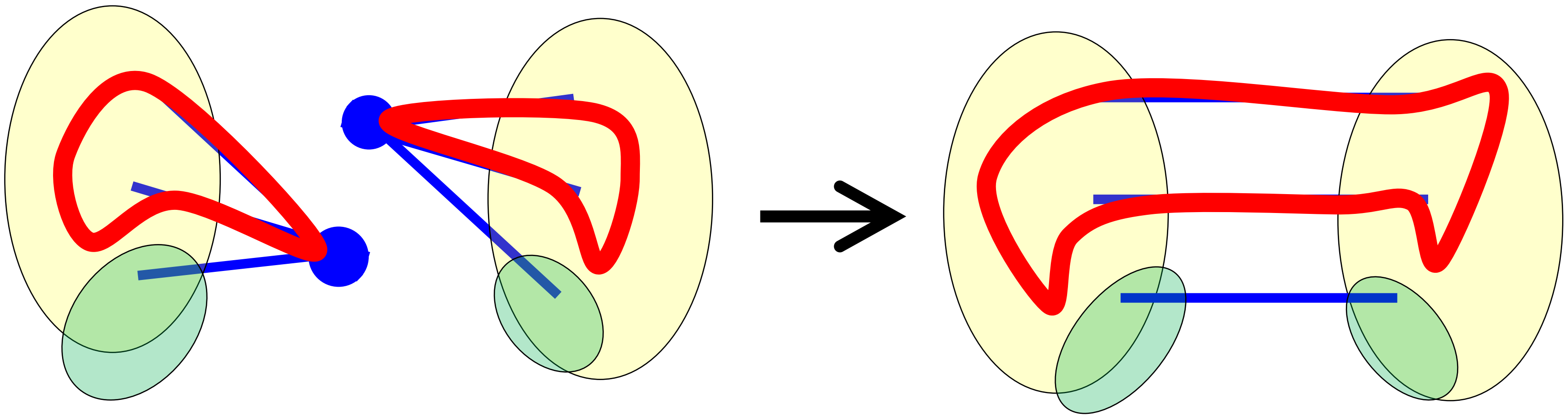
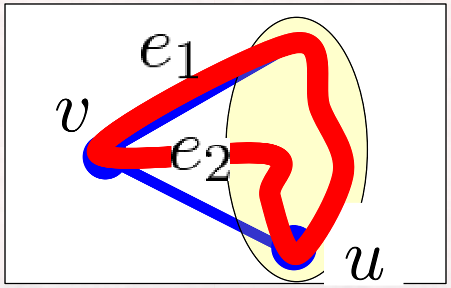


<u>Conj.4</u> H : ess. 4-e-conn. graph

v : vertex of deg. 3

$\Rightarrow \exists\, T$: Tutte CT thr. e_1, e_2, u



Equivalence to the line graph ver.

Thm. (Kuzel, `08) Thomassen's conj. is eqiv. to Conj. 5

Conj.5 H : ess. 4-e-conn. cubic graph $\Rightarrow H : V_2(H)$-dom.

Using the inflation method, we show Conj.5 $\Rightarrow$ Conj.4

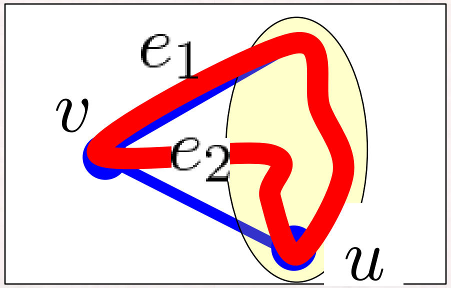
Then Thomassen's conj. is equiv. to Conj. 2

Conj.2 G : 2-conn. line graph $\Rightarrow \exists$ Tutte cycle

Conj.4 H : ess. 4-e-conn. graph

v : vertex of deg. 3

$\Rightarrow \exists\, T$: Tutte CT thr. e_1, e_2, u



Equivalence to the claw-free graph ver.

Thomassen's Conjecture

↑ Trivial ↓ We have shown

Conj.2 G : 2-conn. line graph $\Rightarrow \exists$ Tutte cycle

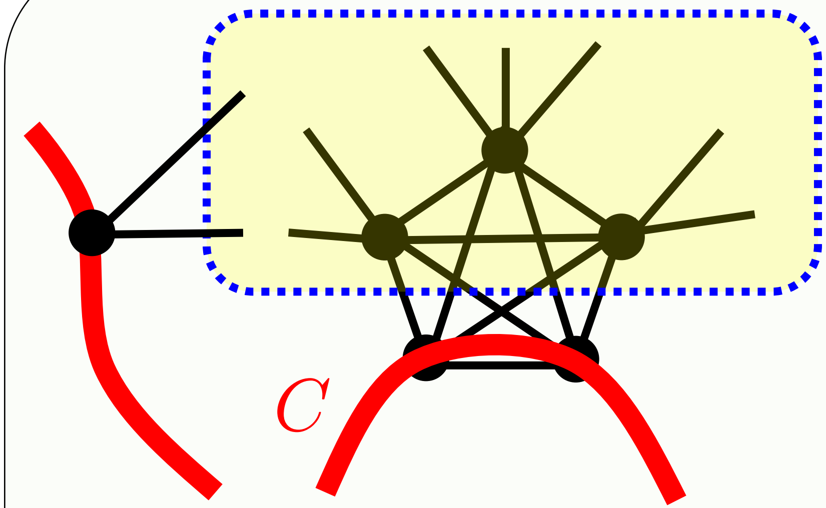
↑ Trivial ↓ Ryjacek closure ?

Conj.1 G : 2-conn. claw-free graph $\Rightarrow \exists$ Tutte cycle

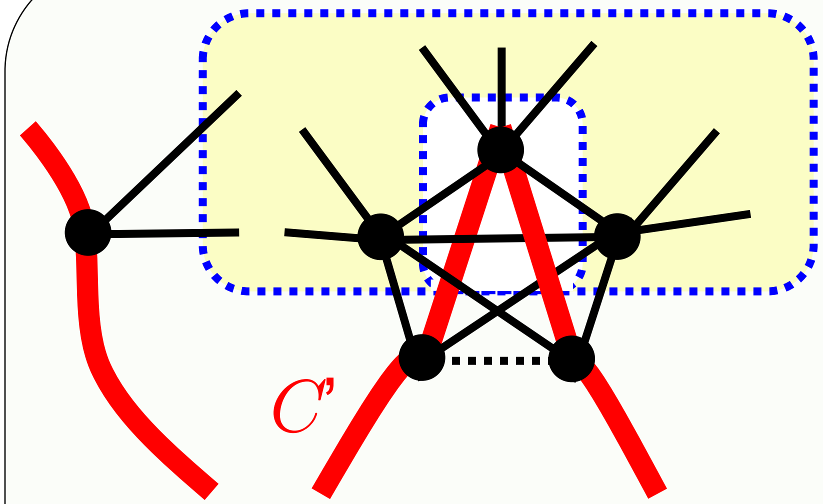
Prob. Is the property ``$\exists$Tutte cycles'' stable under closure?

Maybe YES, but we don't know!

Equivalence to the claw-free graph ver.



C : Tutte cycle in cl(G)

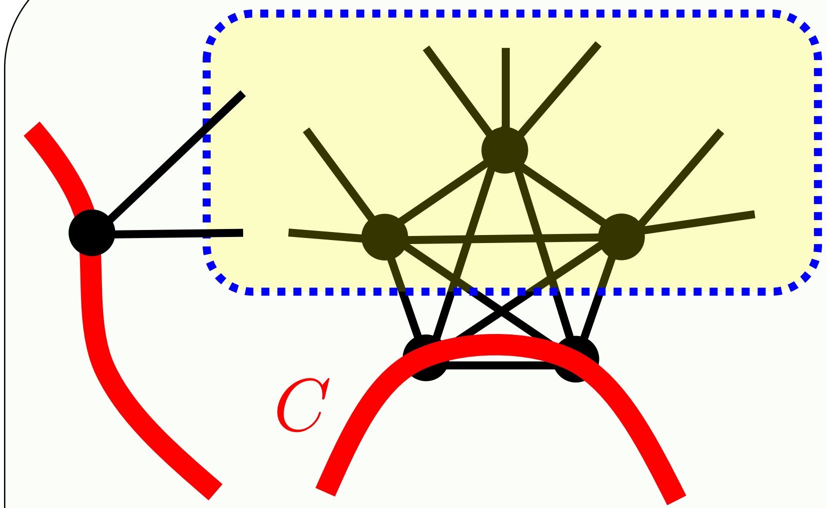


C' : Tutte cycle in G ??

<u>Prob.</u> Is the property ``$\exists$Tutte cycles'' stable under closure?

Maybe YES, but we don't know!

Equivalence to the claw-free graph ver.



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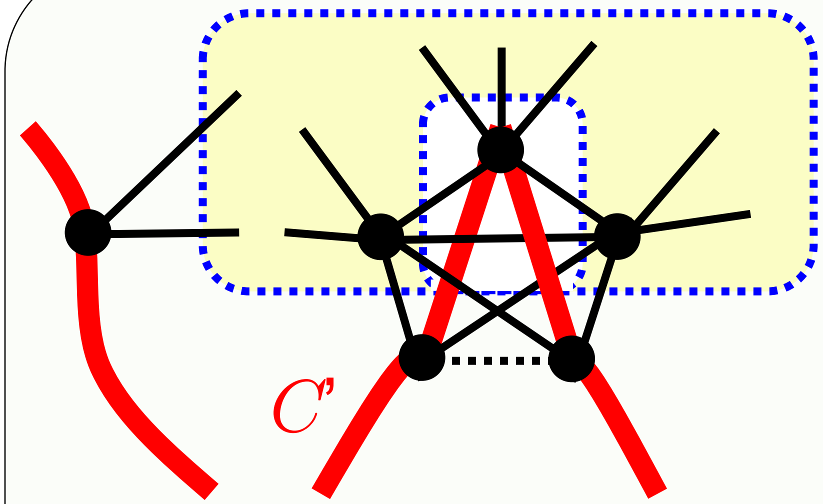


C' : Tutte cycle in G ??

This trouble does NOT happen

if C visits all vertices in the clique

Equivalence to the claw-free graph ver.

D_v : the clique of cl(G) containing v

Def.

A cycle C has (*)-property

$\Leftrightarrow$ For a vertex v : simplicial in cl(G),

if C uses an edge of D_v $\Rightarrow$ C visits all vertices in D_v

Thm. For $\forall C$ with (*) in cl(G), $\exists$ C' with $V(C') = V(C)$ in G

Cor. For any longest cycle C in cl(G),

$\exists$ C' with $V(C') = V(C)$ in G

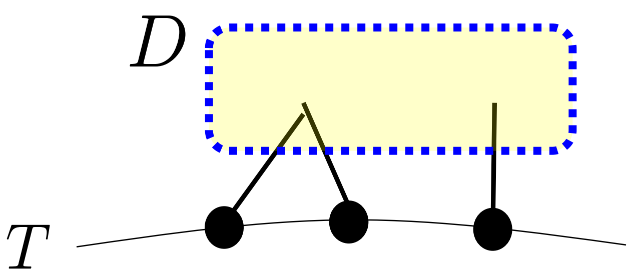
Cor. The property `` $\exists$ Tutte cycles with (*)'' is stable.

Equivalence to the claw-free graph ver.

Conj.2’ G : 2-conn. line graph $\Rightarrow \exists$ Tutte cycle with (*)

↑ Line graph!! ↓ Preimage

Conj.3 H : ess. 2-e-conn. graph $\Rightarrow \exists\, T$: Tutte closed trail



T : Tutte closed trail

$\Leftrightarrow$

$^{\forall}D$: compo. of $H - V(T)$ with ≥ 1 edges,

# of edges between D and T is ≤ 3

Equivalence to the claw-free graph ver.

Thomassen's Conjecture

$\uparrow$ Trivial $\qquad \downarrow$ Already shown

Conj.2' G : 2-conn. line graph $\Rightarrow \exists$ Tutte cycle with (*)

$\uparrow$ Trivial $\qquad \downarrow$ Ryjacek closure!!

Conj.1 G : 2-conn. claw-free graph $\Rightarrow \exists$ Tutte cycle

Prob. Is the property ``$\exists$Tutte cycles'' stable under closure?

To show the existence of Tutte cycles

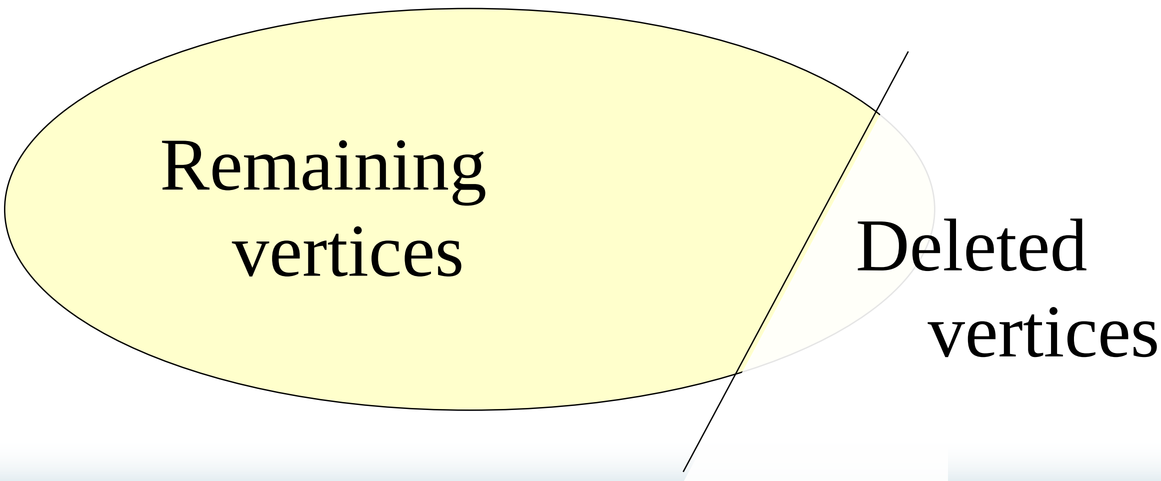
Then we want to show "$\exists$ Tutte cycles".

Why do we succeed for plane graphs and graphs on surfaces??

To show the existence of Tutte cycles

Then we want to show ``∃Tutte cycles”.

Why do we succeed for plane graphs and graphs on surfaces??



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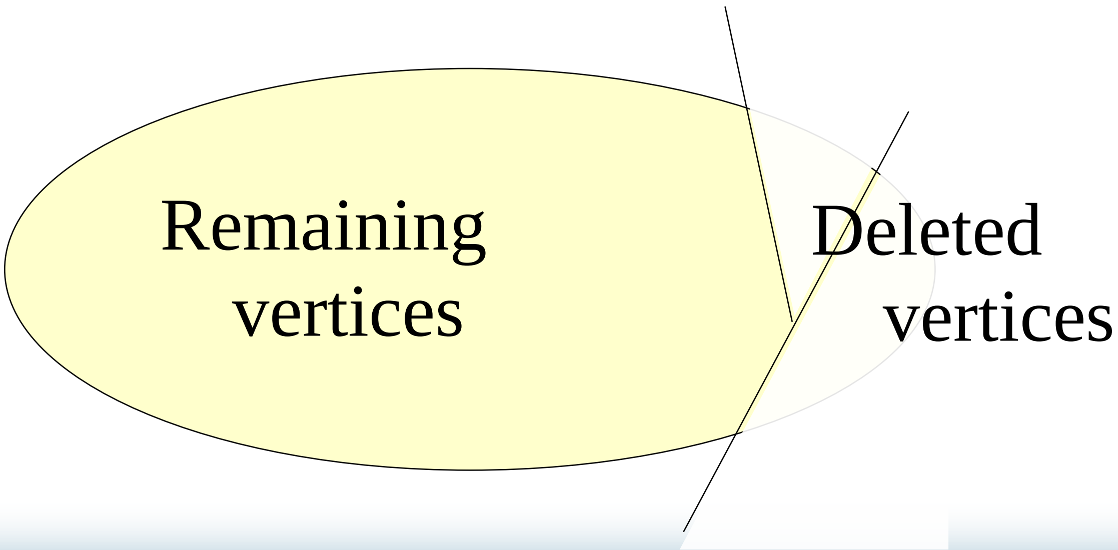

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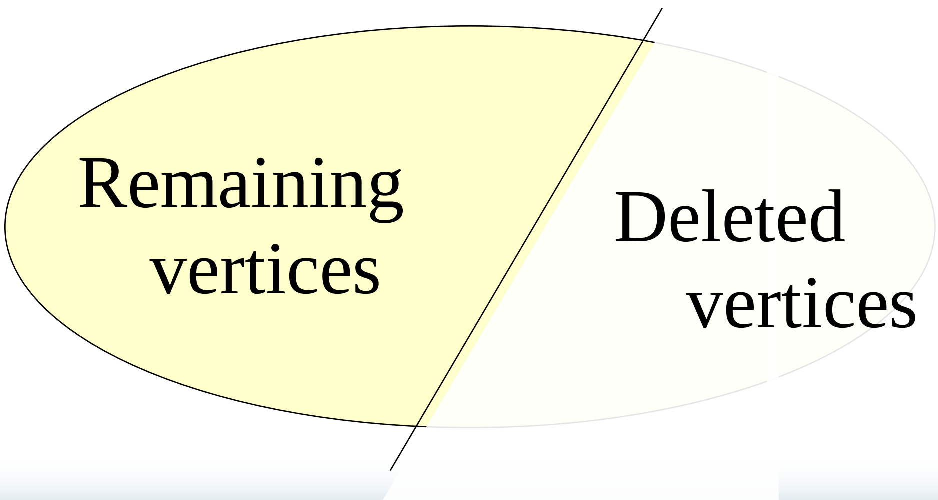

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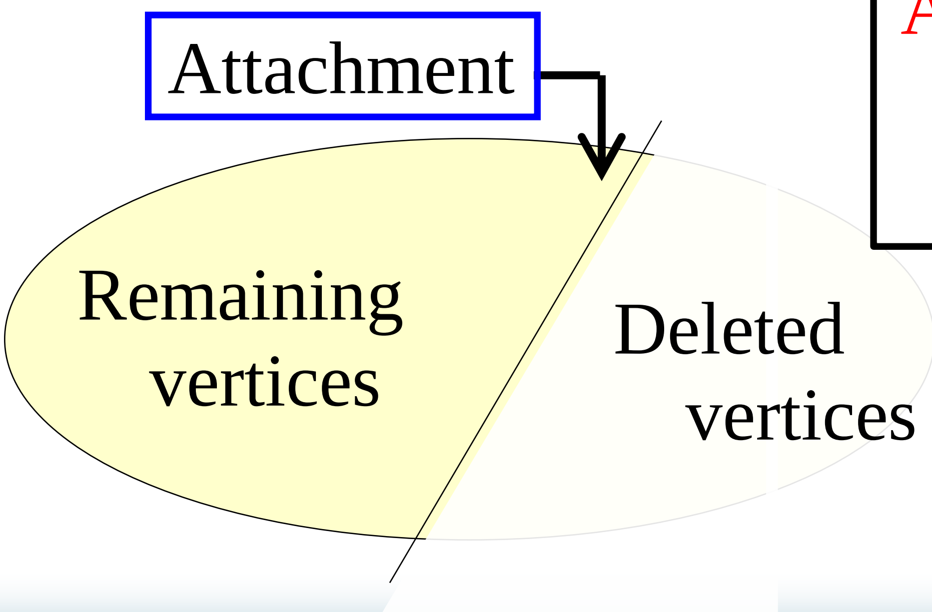


Attachments:
vertices having a neighbor in deleted vertices

For plane graphs,
we can delete vertices
s.t. ∀attachments appear
on the outer facial cycle

L-Tutte cycle

L : a subgraph of G (containing all attachments)

Usually L is the outer facial cycle

C : L-Tutte cycle in G

$\Leftrightarrow$

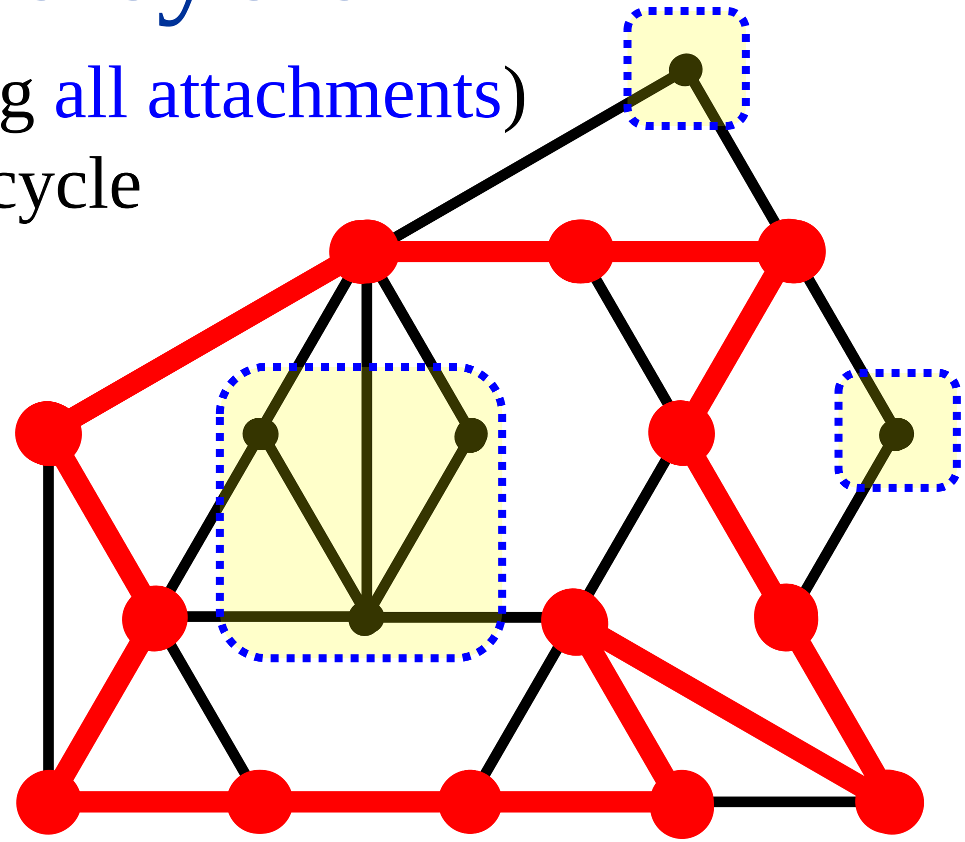
$\boxed{|C| \geq 4}$

$\forall$ compo. of $G - V(C)$

has ≤ 3 neighbors on C,

and ≤ 2 neighbors on C

if it has a vertex of L



L-Tutte cycle in G

L-Tutte cycle

Thm. G : 2-conn. plane graph

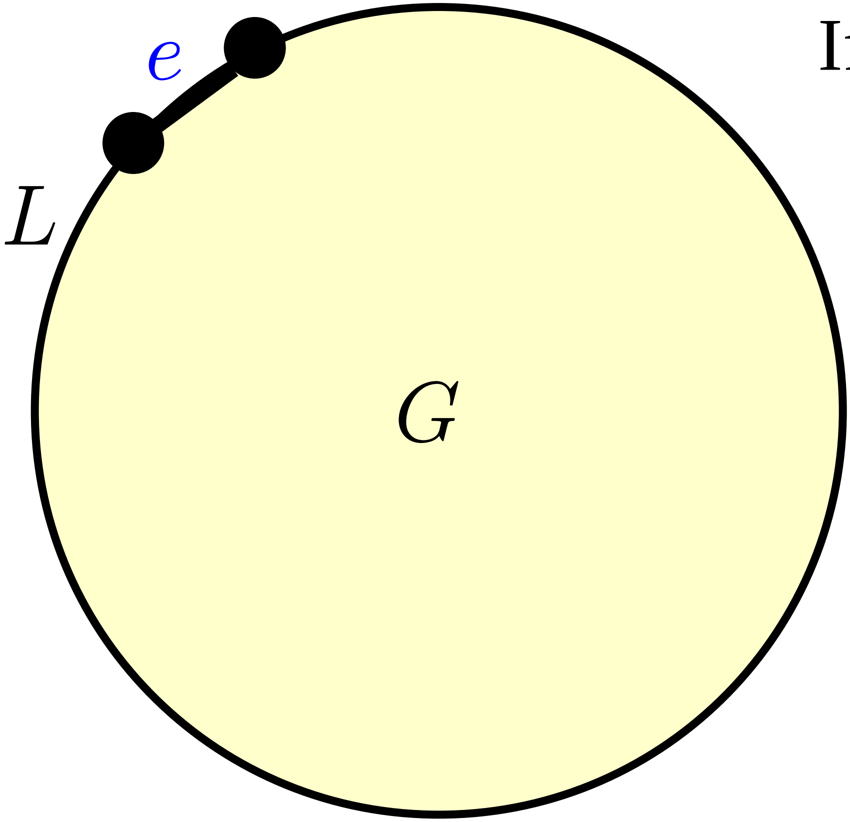
L : the outer facial cycle e : an edge of L

$\Rightarrow \exists\, C$: L-Tutte cycle containing e

Remark: Even for conn. graphs, using block decomposition,
we can show the same one, unless e is a bridge

Remark2: This thm. is NOT the one Tutte (and others) proved,
but to find the main ideas, it is enough and easier.

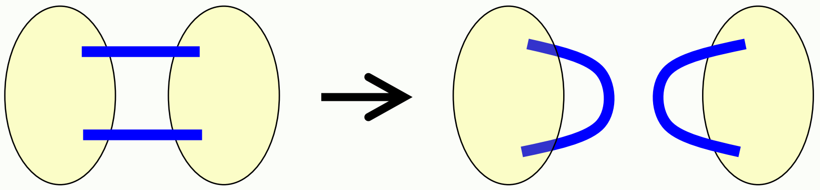
Idea of the proof for plane graphs



If G has a 2-cut,

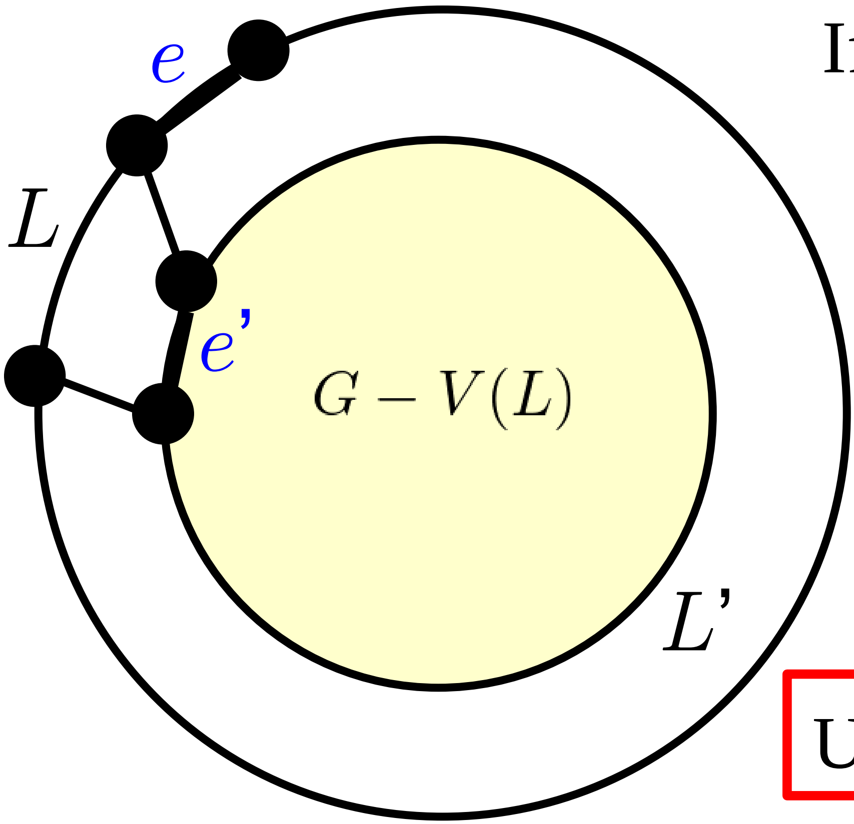
replace a compo. with an edge

c.f.



(ess.) 2-edge-cut

Idea of the proof for plane graphs



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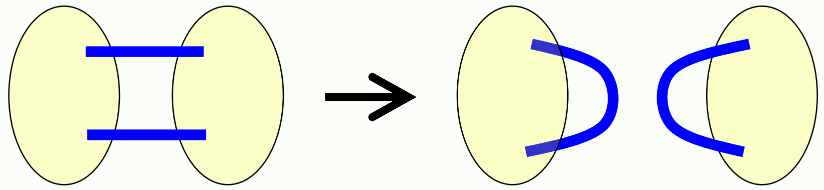
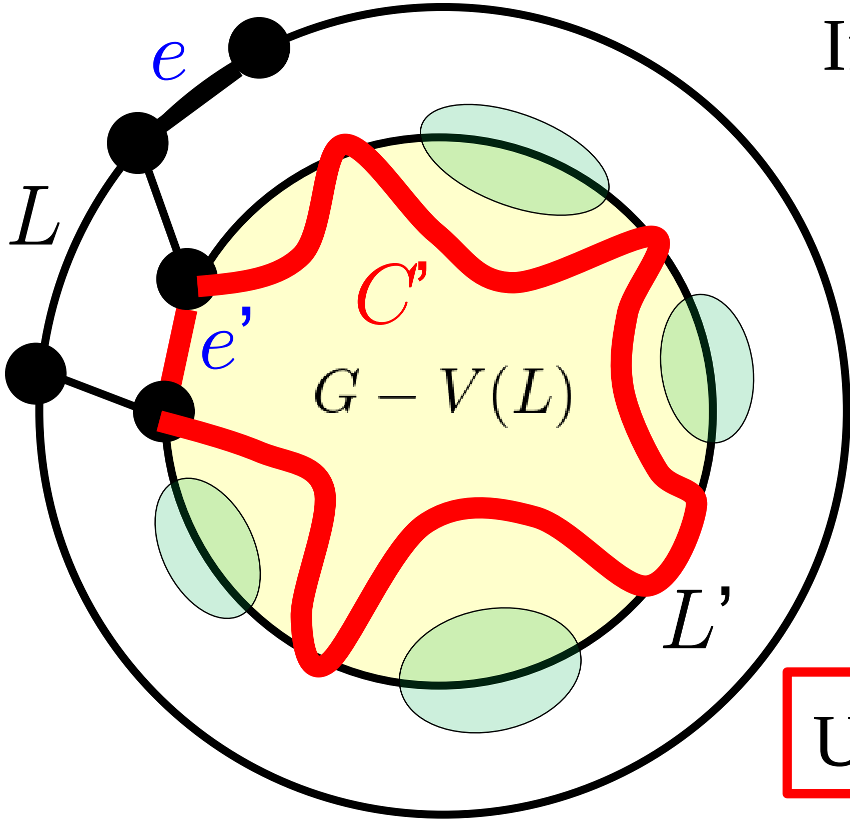


(ess.) 2-edge-cut

Use induction hypo. to $G - V(L)$

Idea of the proof for plane graphs



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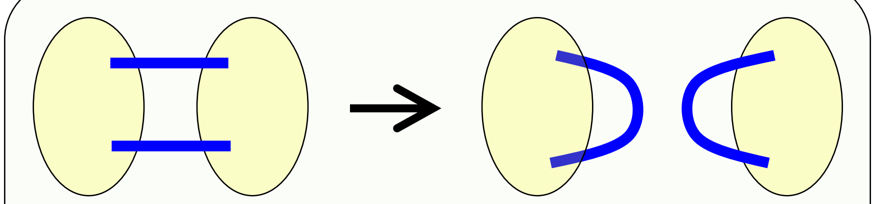
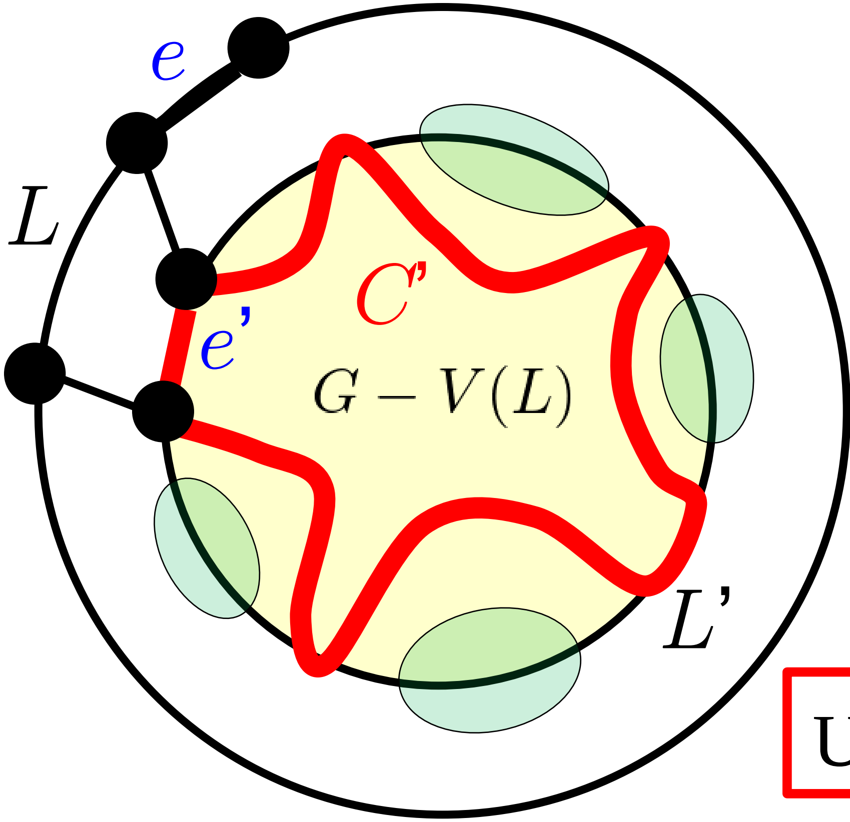


(ess.) 2-edge-cut

$\boxed{\text{Use induction hypo. to } G - V(L)}$

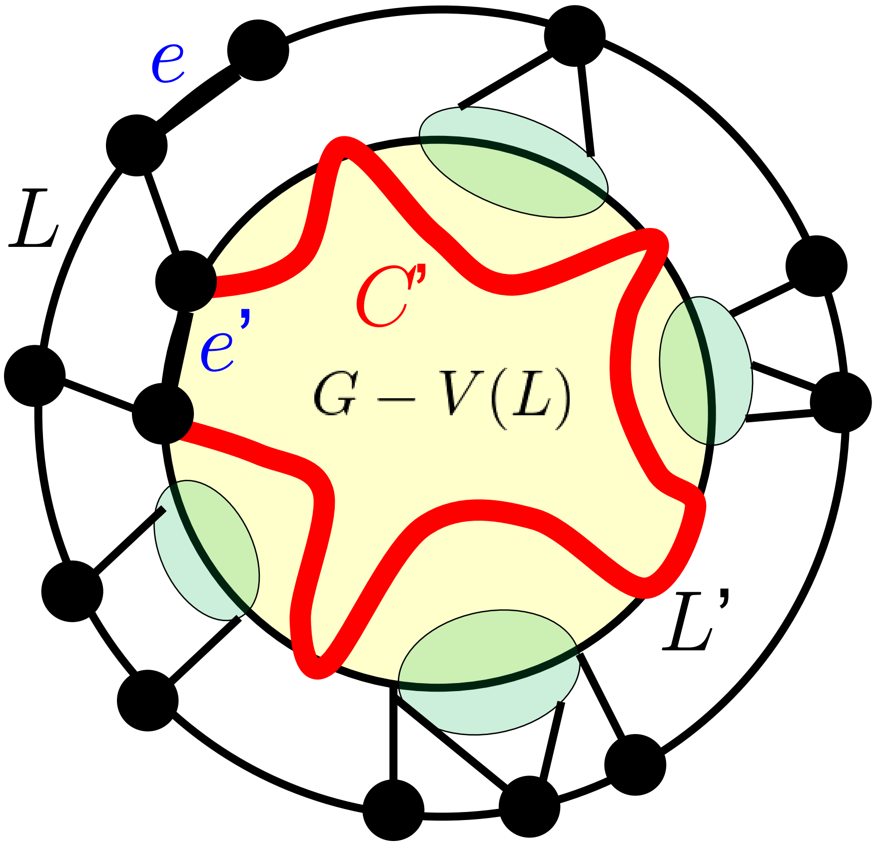
Idea of the proof for plane graphs



Extend C' along with L

Use induction hypo. to $G - V(L)$

Idea of the proof for plane graphs

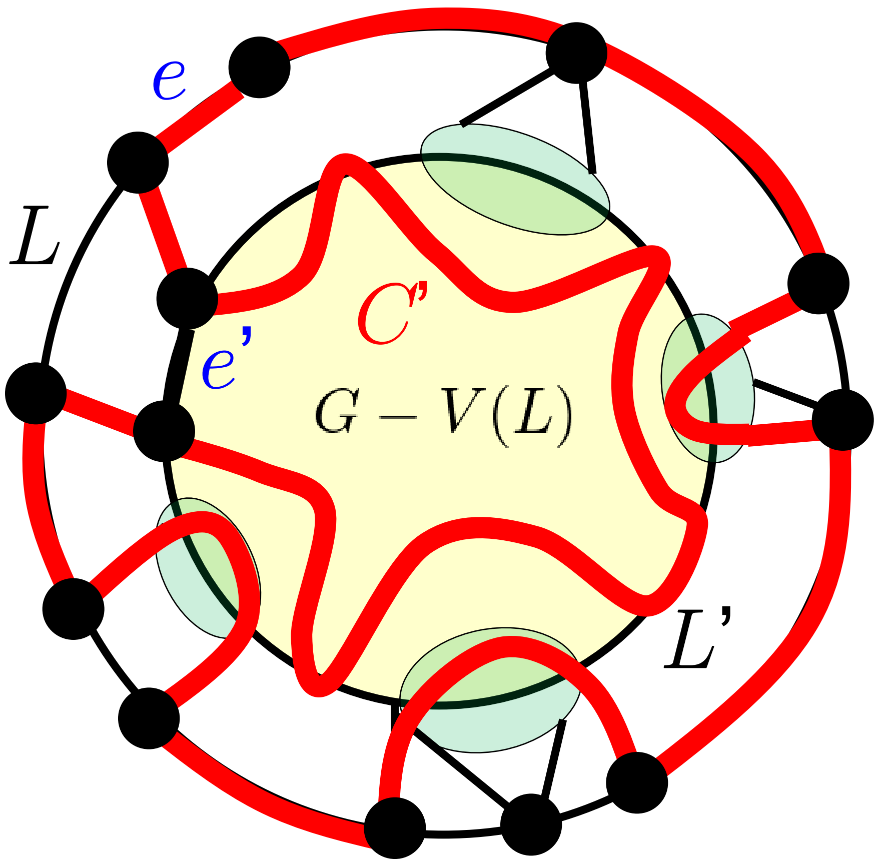


Extend C' along with L

Consider a comp. of that has a neighbor on L

(It has ≤ 2 neighbor on C', since it has a vertex in L')

Idea of the proof for plane graphs



Extend C' along with L

Consider a comp. of

that has a neighbor on L

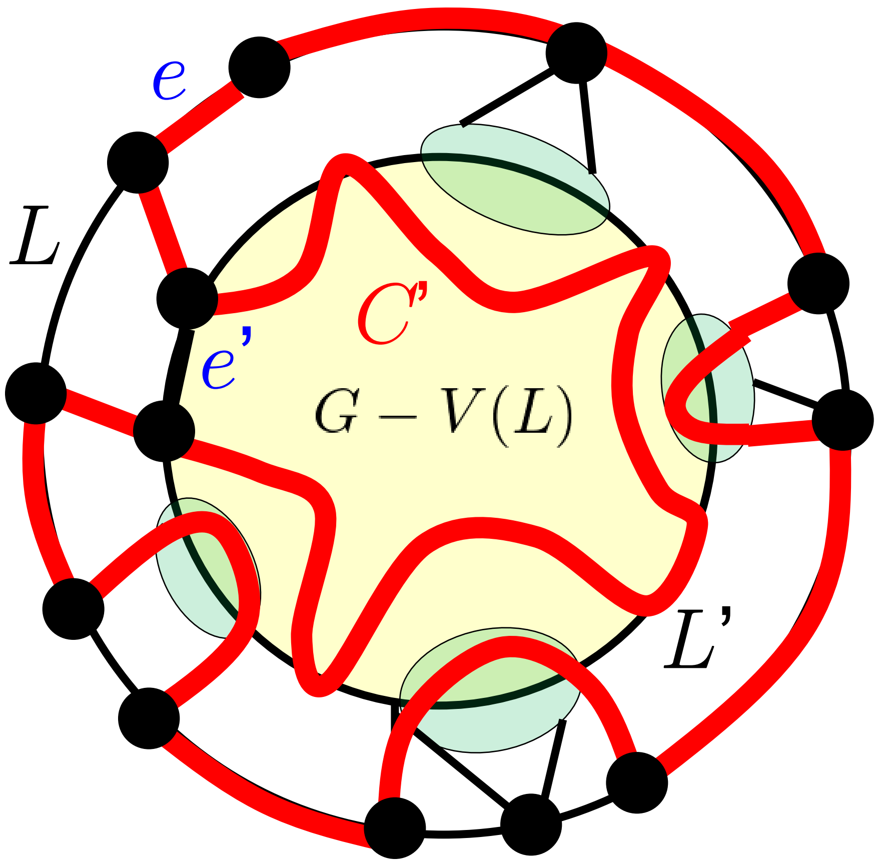
(It has ≤ 2 neighbor on C',

since it has a vertex in L')

If it has only one neighbor on L

$\Rightarrow$ Need to do nothing.

Idea of the proof for plane graphs



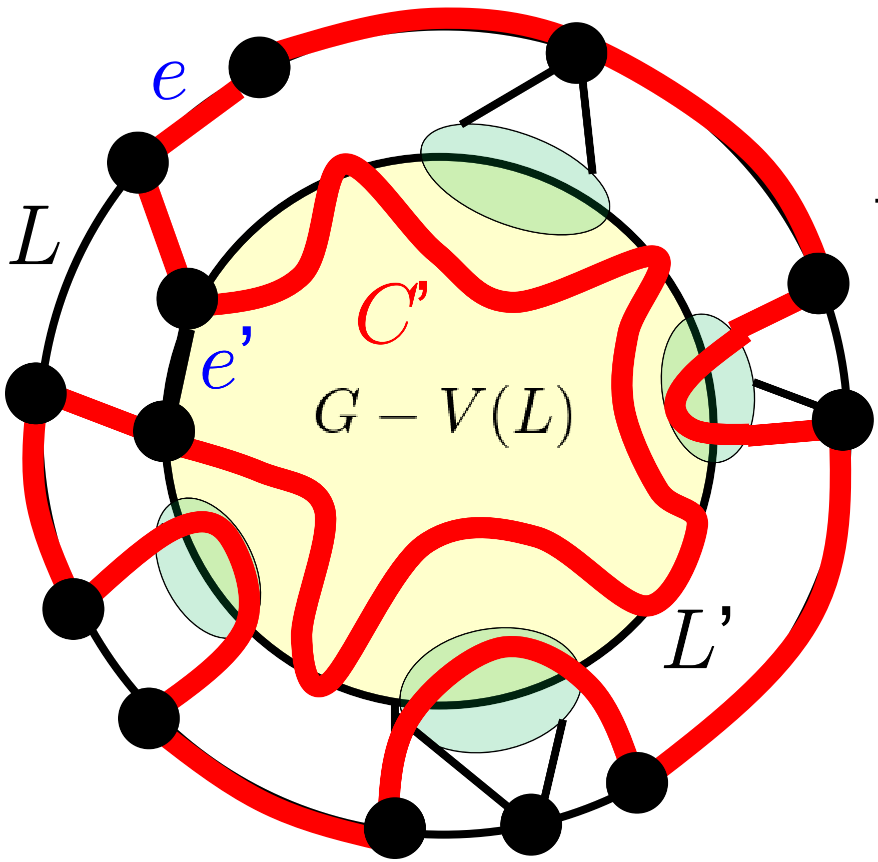
Extend C' along with L

Consider a comp. of
that has a neighbor on L
(It has ≤ 2 neighbor on C',
since it has a vertex in L')

If it has only one neighbor on L
$\Rightarrow$ Need to do nothing.

If it has ≥ 2 neighbor on L
$\Rightarrow$ Reroute L into the compo.

Idea of the proof for plane graphs

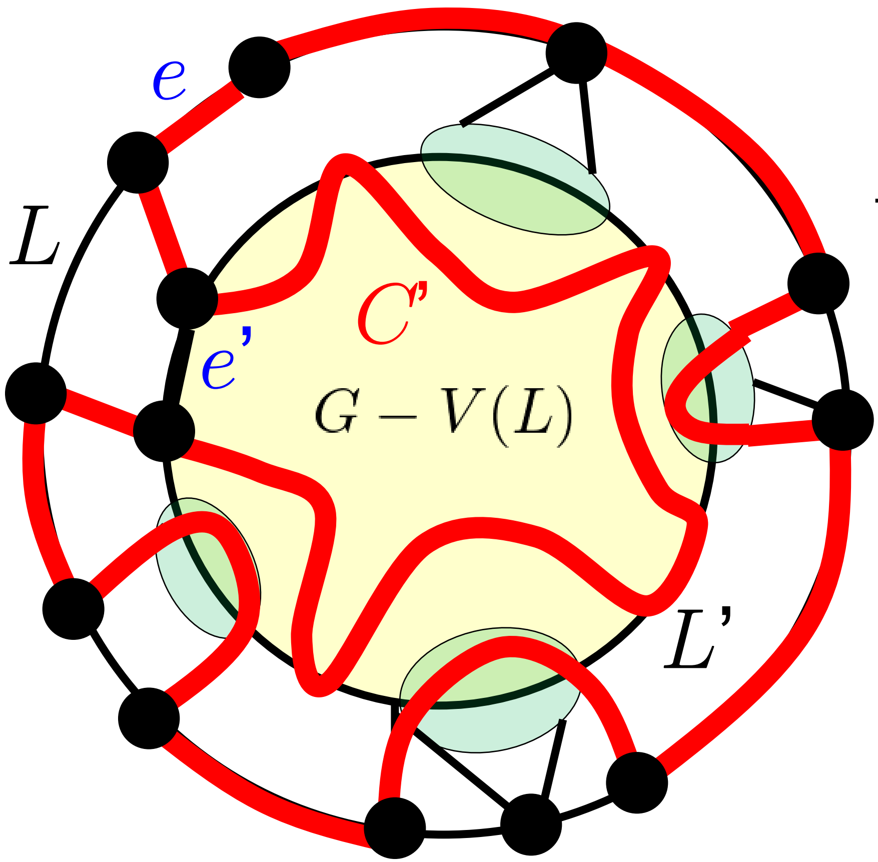


Extend C' along with L

Good properties of plane graphs

- ✓ ∀ attachments appear on the outer facial cycle
- ✓ ∀ compo.s of $G - V(L) - V(C')$ appear in order
- ✓ ∀ attachment for $G - V(L)$ is NOT one for G

Idea of the proof for plane graphs



Extend C' along with L

Good properties of plane graphs

- ✓ $\forall$ attachments appear on the outer facial cycle
- ✓ ~~$\forall$ compo.s of $G - V(L) - V(C')$ appear in order~~
- ✓ $\exists \leq 2$ compo.s of $G - V(L) - V(C')$ corresponding to ``the ends''
- ✓ $\forall$ attachment for $G - V(L)$ is NOT one for G

To show the existence of Tutte cycles

	General graphs	Plane graphs	Line graphs
H-cycle	**NP**-c (Karp `72)	**NP**-c (GJT `76)	**NP**-c (Bertossi `81)
H-conn	**NP**-c (Dean `93)	??	??
2-H 1-HC 2−e-HC	**NP**-c (KRV `09) **NP**-c (Oz, V `13+)	**P** (Thomas & Yu, `94) (Sanders, `96) (Oz. & Vrána, `13+)	??

Thm (Kužel, Ryjáček, Vrána, `12) At least one of the following is FALSE

- Thomassen's Conjecture
- ``The problem of deciding 2-e-HC (1HC) is NP-complete"
- **P** $\neq$ **NP** Conjecture

Main targets of this talk

The following 2 kinds of cycles
in line graphs (and claw-free graphs)

I. Tutte cycles

- ✓ Equivalent conj.s to Thomassen's conj.
- ✓ A result and problem on the Ryjáček closure
- ✓ Why are Tutte cycles good for the plane graphs?

II. Cycles of linear length with the # of vertices

- ✓ Linear length is enough for the case $\delta \geq 5$
- ✓ Relationship to Bondy's conjecture

Motivation

Thm. (Broersma, Kriesell, Ryjáček `01)
Thomassen's conj. is equiv. to the following

Conj.6 $\exists f$: sublinear function ($\lim_{n\to\infty} f(n)/n = 0$) s.t.
G :4-conn. line graph $\Rightarrow$ $\exists$ cycle of length $\geq |G| - f(|G|)$

What about a linear function?

Conj.7 $\exists c$: constant with $0 < c < 1$ s.t.
G :4-conn. line graph $\Rightarrow$ $\exists$ cycle of length $\geq c|G|$

Motivation

$\underline{\text{Conj.0 (Thomassen)}}$

G : 4-conn. line graph

$\Rightarrow \exists$ Hamiltonian cycle

$\downarrow$ Trivial $\quad\uparrow$ Problem

$\underline{\text{Conj.7}}$

$\exists\, c$: const. with $0 < c < 1$

s.t. G : 4-conn. line graph

$\Rightarrow \exists$ cycle of length $\geq c|G|$

Motivation

Conj.0 (Thomassen)

G : 4-conn. line graph

$\Rightarrow \exists$ Hamiltonian cycle

$\longrightarrow$ Trivial

$\longleftarrow$ Problem

Conj.8

G : 4-conn. line graph

$\delta(G) \geq 5$

$\Rightarrow \exists$ Hamiltonian cycle

$\downarrow$ Trivial $\uparrow$ Problem

$\downarrow$ Trivial $\uparrow$

Conj.7

$\exists\, c$: const. with $0 < c < 1$

s.t. G : 4-conn. line graph

$\Rightarrow \exists$ cycle of length $\geq c|G|$

$\longrightarrow$ Trivial

$\longleftarrow$ Problem

Conj.9

$\exists\, c$: const. with $0 < c < 1$

s.t. G : 4-conn. line graph

$\delta(G) \geq 5$

$\Rightarrow \exists$ cycle of length $\geq c|G|$

Motivation

Conj.0 (Thomassen)

G : 4-conn. line graph

$\Rightarrow \exists$ Hamiltonian cycle

$\longrightarrow$ Trivial

$\longleftarrow$ Problem

Conj.8

G : 4-conn. line graph

$\delta(G) \geq 5$

$\Rightarrow \exists$ Hamiltonian cycle

$\downarrow$ Trivial $\uparrow$ Problem

$\downarrow$ Trivial $\uparrow$ Our result

Conj.7

$\exists\, c$: const. with $0 < c < 1$

s.t. G : 4-conn. line graph

$\Rightarrow \exists$ cycle of length $\geq c|G|$

$\longrightarrow$ Trivial

$\longleftarrow$ Problem

Conj.9

$\exists\, c$: const. with $0 < c < 1$

s.t. G : 4-conn. line graph

$\delta(G) \geq 5$

$\Rightarrow \exists$ cycle of length $\geq c|G|$

Conj.8 (linear length) implies Conj. 9

Suppose

$\exists\, G$: 4-conn. line graph

$\delta(G) \geq 5$

non-Hamiltonian

Conj.8 (linear length) implies Conj. 9

Suppose

$\exists\, G$: 4-conn. line graph

$\delta(G) \geq 5$

non-Hamiltonian

$\downarrow$ Pre-image

$\exists\, H_0$: ess. 4-edge-conn.

min. edge deg. ≥ 5

$\nexists$ dominating CT

Conj.8 (linear length) implies Conj. 9

Suppose

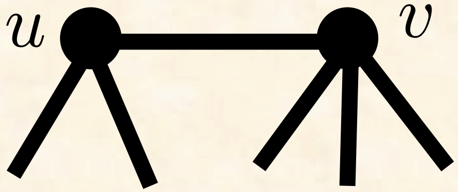
$\exists G$: 4-conn. line graph
$\delta(G) \geq 5$
non-Hamiltonian

↓ Pre-image

$\exists H_0$: ess. 4-edge-conn.
min. edge deg. ≥ 5
$\nexists$ dominating CT

$V_{\geq 4}(H)$: the set of vrt.s of deg. ≥ 4 in H

Since min. edge deg. ≥ 5
for $\forall uv \in E(H_0)$,
u or v has degree ≥ 4



If $\exists$CT visiting$\forall$vertices in $V_{\geq 4}(H_0)$
then it is dominating CT

$\longrightarrow$ $\nexists$ such a CT

Conj.8 (linear length) implies Conj. 9

Suppose

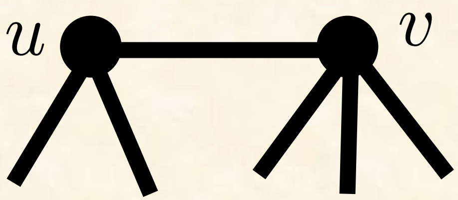
$\exists G$: 4-conn. line graph
$\delta(G) \geq 5$
non-Hamiltonian

$\downarrow$ Pre-image

$\exists H_0$: ess. 4-edge-conn.
min. edge deg. ≥ 5
$\nexists$ CT visiting
$\forall$ vertices in $V_{\geq 4}(H_0)$

$V_{\geq 4}(H)$: the set of vrt.s of deg. ≥ 4 in H

Since min. edge deg. ≥ 5
for $\forall uv \in E(H_0)$,
u or v has degree ≥ 4



If $\exists$CT visiting$\forall$vertices in $V_{\geq 4}(H_0)$
then it is dominating CT
$\longrightarrow \nexists$ such a CT

Conj.8 (linear length) implies Conj. 9

H_i : obtained from H_{i-1}
by identifying a copy of H_0
at a vertex in $V_{\geq 4}(H_0)$
to each vertex in $V_{\geq 4}(H_{i-1})$

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$l'(H)$: max. # of edges
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Claim $\lim_{i \to \infty} \dfrac{l'(H_i)}{|E(H_i)|} = 0$

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Proof : Calculation!!

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Idea : Let $h_0 := |V_{\geq 4}(H_0)|$
For each i, max. CT can visit
$\leq h_0 - 1$ vertices within h_0 new ones

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Proof : Calculation!!

Idea : Let $h_0 := |V_{\geq 4}(H_0)|$
For each i, max. CT can visit
$\leq h_0 - 1$ vertices within h_0 new ones

$\rightarrow$ $\leq |H|^{\log_{h_0}(h_0-1)}$ vertices can be visited

Conj.8 (linear length) implies Conj. 9

Conj.0 (Thomassen)

G : 4-conn. line graph

$\Rightarrow \exists$ Hamiltonian cycle

$\longrightarrow$ Trivial

$\longleftarrow$ Problem

Conj.8

G : 4-conn. line graph

$\delta(G) \geq 5$

$\Rightarrow \exists$ Hamiltonian cycle

$\downarrow$ Trivial $\quad \uparrow$ Problem

$\downarrow$ Trivial $\quad \uparrow$ Our result

Conj.7

$\exists\, c$: const. with $0 < c < 1$

s.t. G : 4-conn. line graph

$\Rightarrow \exists$ cycle of length $\geq c|G|$

$\longrightarrow$ Trivial

$\longleftarrow$ Problem

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Relationship to Bondy's conj.

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Recall Hajo's talk

Our result

Conj.10 (Bondy)

$\exists$ c : const. with $0 < c < 1$

s.t. H : cyc. 4-e-conn. cubic graph

$\Rightarrow \exists$ cycle of length

Relationship to Bondy's conj.

Conj.0 (Thomassen)

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$\Rightarrow \exists$ Hamiltonian cycle

↓ Trivial ↑ Problem

Conj.7

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Recall Hajo's talk ↑ Our result

Conj.10 (Bondy)

$\exists$ c : const. with $0 < c < 1$

s.t. H : cyc. 4-e-conn.cubic graph

$\Rightarrow \exists$ cycle of length

Let G : 4-conn. line graph.

↓ Pre-image

H : ess. 4-edge-conn. graph.

How to use Conj. 10?

→ Inflation!.. But two issues

1. High degree vertices

2. Pendant edges

Use Conj. 10 and get a const. c

→ We get a const. c for Conj. 7

Relationship to Bondy's conj.

Conj.0 (Thomassen)

G : 4-conn. line graph

$\Rightarrow \exists$ Hamiltonian cycle

Trivial

Problem

Conj.8

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Thank you for your attention

Idea of the proof

$\boxed{H : \text{(unique) exceptional comp.}}$

$\boxed{\text{Condition}}$ G : ~~4-conn.~~ 2-conn

$x, y \in V(G)$ $\boxed{C : \text{face containing } x}$

Want to find :

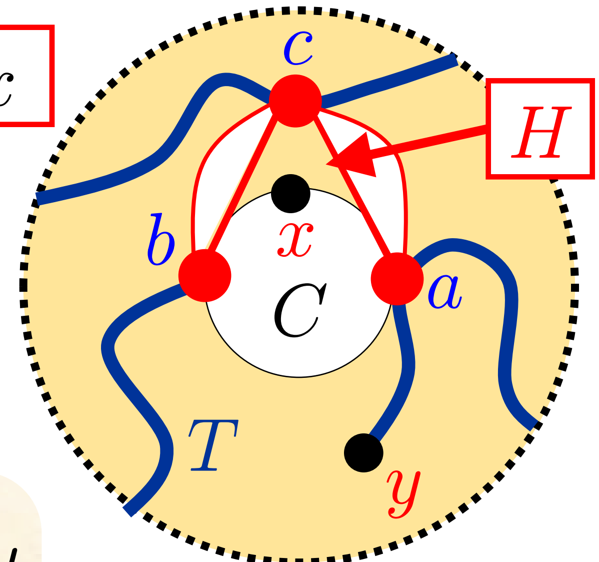
T : C-Tutte path between x, y

Or

$\exists \{a, b, c\}$: 3-cut of G separating x and y

$\exists H$: plane comp. of $G - \{a, b, c\}$ containing x (or $x = b$)

$\exists T$: C-Tutte path in $G - (H - \{a, b, c\})$ between b, y thr. a, c



Idea of the proof

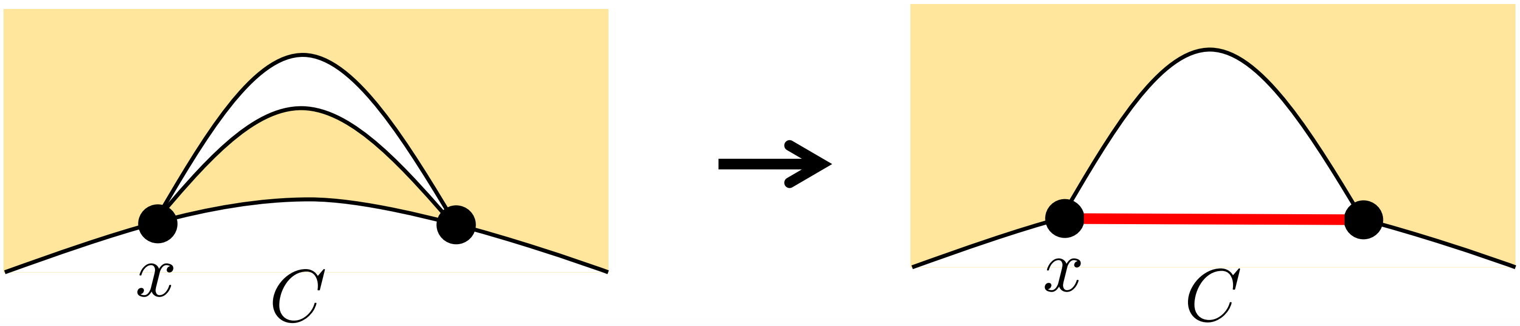
Induction on $|V(G)|$

Base case is the case where $\mathrm{rep}(G) = 2$

(In the ordinary method,

we need both cases where $\mathrm{rep}(G) = 1$ and =2)

Case I : $\exists S$: 2-cut with $x \in S$

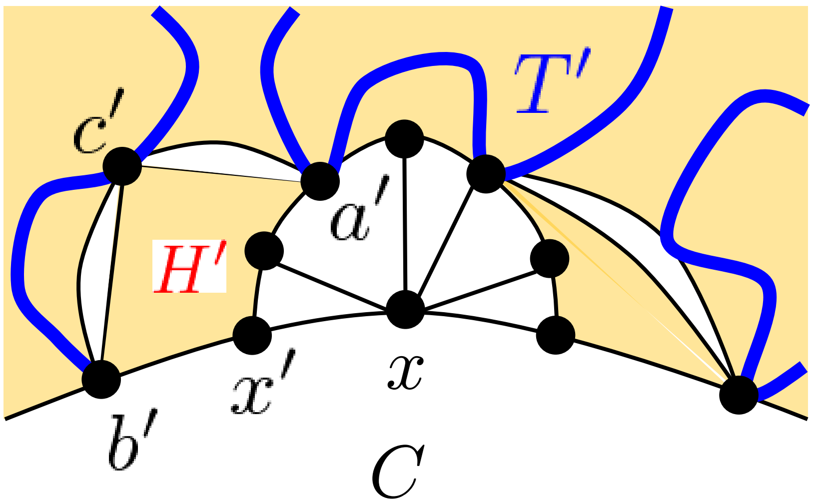


Idea of the proof

Case II : $\nexists S$: 2-cut with $x \in S$

Then $G - x$ is also 2-connected,

and use induction hypothesis to $G - x$



Idea of the proof

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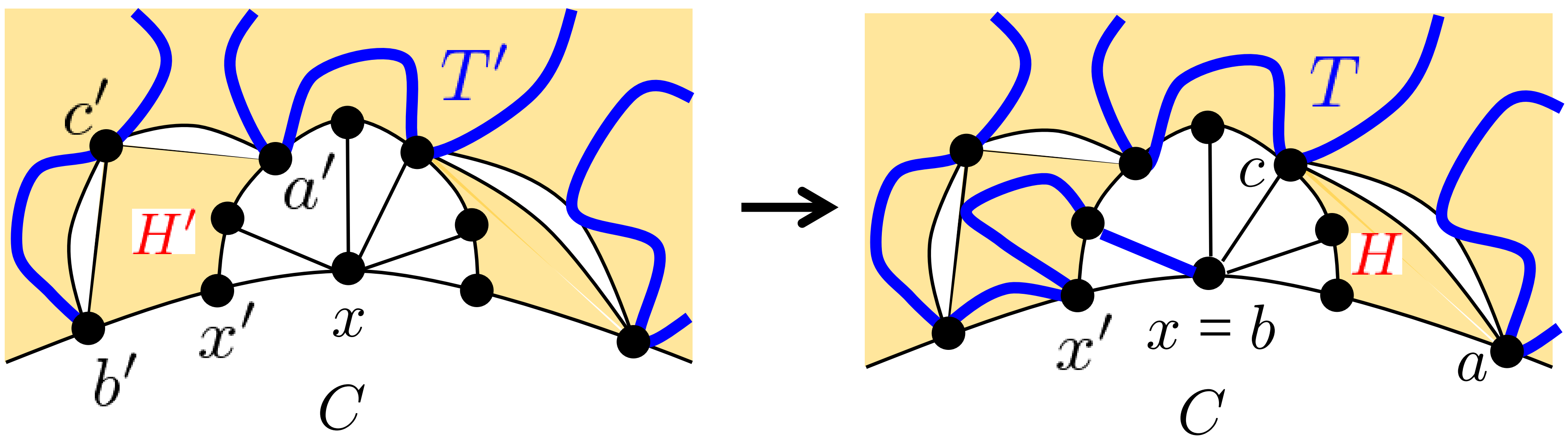

Idea of the proof for plane graphs

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by adding a copy of H_0

to each vertex in $V_{\geq 4}(H_{i-1})$

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Claim $\lim_{i \to \infty} \frac{l'(H_i)}{|E(H_i)|} = 0$

$$h_0 := |V_{\geq 4}(H_0)|$$

$$|V_{\geq 4}(H_i)| = h_0 |V_{\geq 4}(H_{i-1})| = h_0^{i+1}$$