Sets of cycles whose union dominates all edges *

Shuya Chiba (Tokyo University of Science)†

Most of the results in this talk are motivated by Matthews and Sumner's conjecture [4] (every 4-connected claw-free graph is Hamiltonian) and Thomassen's conjecture [6] (every 4-connected line graph is Hamiltonian). Since every line graph is claw-free, Thomessen's conjecture is a special case of Matthews and Sumner's conjecture. However it is known that a result on closures introduced by Ryjáček [5] implies that the above conjectures are equivalent.

On the other hand, as a possible approach to solve Matthews and Sumner's conjecture and Thomassen's conjecture, in 2008, Kaiser and Škrekovski [3] studied the existence of vertex-disjoint closed trails intersecting all edge-cut sets of prescribed size in a graph. An *edge-cut set* in a graph G is an inclusionwise minimal set of edges whose removal increases the number of components of G. Let $\mathbb{N}$ be the set of positive integers and $A \subseteq \mathbb{N}$. We say that a set $\mathcal{T}$ of vertex-disjoint closed trails with at least two vertices in a graph G is an A-*covering set* if for every edge-cut set X in G with $|X| \in A$, there exists a closed trail T in $\mathcal{T}$ such that T intersects X, i.e., $E(T) \cap X \neq \emptyset$. A graph G is A-*coverable* if G has an A-covering set. For $k \in \mathbb{N}$, let $\mathbb{N} + k = \{n + k \mid n \in \mathbb{N}\}$. By the definition, it is easy to see that Thomassen's conjecture is equivalent with the following statement (see [1, 3]):

($*$) *Every cyclically 4-edge-connected cubic graph is* $(\mathbb{N} + 3)$*-coverable.*

In this research, we concentrate on statement ($*$), and we consider the weaker version of it. Now let l be an integer with $l \geq 0$. A set $\mathcal{T}$ of vertex-disjoint cycles of a graph G is called a D_l*-set* of G if each component of $G - \bigcup_{T\in\mathcal{T}} V(T)$ has at most l vertices. If $|\mathcal{T}| = 1$, say $\mathcal{T} = \{T\}$, then we call T a D_l*-cycle*. Then we can obtain the following proposition.

Proposition 1 *Let k be an integer with $k \geq 4$, and let H be a cyclically 4-edge-connected cubic graph. If H is $\{4, 5, \cdots, k\}$-coverable, then one of the following holds.*

(i) *H has a D_1-cycle, i.e., H is $(\mathbb{N} + 3)$-coverable.*

(ii) *H has a D_1-set of cardinality at least 2 in which each cycle has at least $k + 1$ vertices.*

---

*This is a joint-work with Roman Čada, Kenta Ozeki and Kiyoshi Yoshimoto

†E-mail: sh.chiba@gmail.com

By Proposition 1, we have that if a cyclically 4-edge-connected cubic graph is $\{4, 5, \cdots, k\}$-coverable but not $(\mathbb{N}+3)$-coverable, then the graph has a set of vertex-disjoint long cycles whose union dominates all edges. On the other hand, at the moment, we do not know the converse also holds or not. However, we can show the existence of such a set in 3-edge-connected cubic graph, which is our main theorem (Theorem 2).

Concerning the existence of a set of vertex-disjoint long cycles dominating all edges, in 2009, Jackson and Yoshimoto [2] proved the following theorem.

Theorem A (Jackson and Yoshimoto [2]) *Every 3-edge-connected cubic graph H has a D_0-set in which each cycle has at least $\min\{5, |V(H)|\}$ vertices.*

It is also known that there exist infinitely many 3-edge-connected cubic graphs in which every D_0-set in the graph has a cycle with at most 5 vetices (see [2]). In this research, we consider a "D_1-set version" of Theorem A as follows.

Theorem 2 *If H is a 3-edge-connected cubic graph, then one of the following holds.*

(i) *H has a D_1-cycle, i.e., H is $(\mathbb{N}+3)$-coverable.*

(ii) *H has a D_1-set of cardinality at least 2 in which each cycle has at least 6 vertices.*

We do not know the lower bound on the lengths of cycles in Theorem 2 is best possible or not. At the moment, we can construct infinitely many 3-edge-connected cubic graphs in which every D_1-set in the graph has a cycle with at most 9 vetices.

References

[1] R. Čada, S. Chiba, K. Ozeki, P. Vrána and K. Yoshimoto, $\{4, 5\}$ is not coverable: a counterexample to a conjecture of Kaiser and Škrekovski, SIAM J. Discrete Math. **27** (2013), 141–144.

[2] B. Jackson and K. Yoshimoto, Spanning even subgraphs of 3-edge-connected graphs, J. Graph Theory **62** (2009), 37–47.

[3] T. Kaiser and R. Škrekovski, Cycles intersecting edge-cuts of prescribed sizes, SIAM J. Discrete Math. **22** (2008), 861–874.

[4] M.M. Matthews and D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory **8** (1984), 139–146.

[5] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B **70** (1997), 217–224.

[6] C. Thomassen, Reflections on graph theory, J. Graph Theory **10** (1986), 309–324.