Nowhere-zero flows in Cartesian bundles of graphs

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Joint work with **Martin Škoviera** (Comenius University, Bratislava)

Nowhere-zero flow

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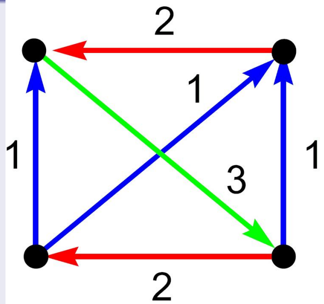
Definition

A couple (D, f) is *$\mathbb{Z}$-flow* of a graph G



Flow number: the smallest k for which a graph has a NZ k-flow

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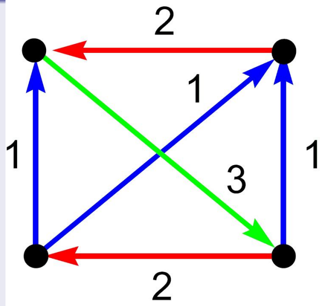

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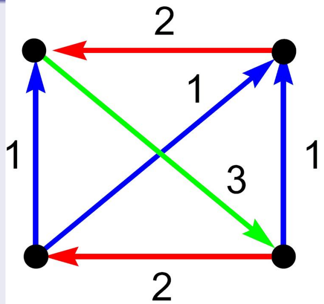

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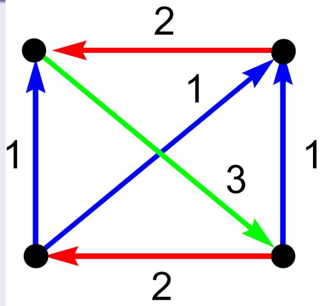

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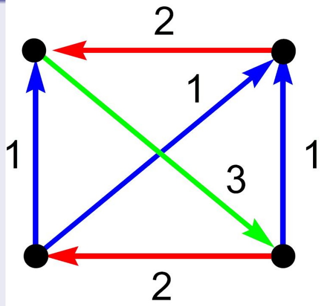
If $\forall x \in D(G)$: $0 < |f(x)| < k$ then $\mathbb{Z}$-flow is a *nowhere-zero k-flow*.



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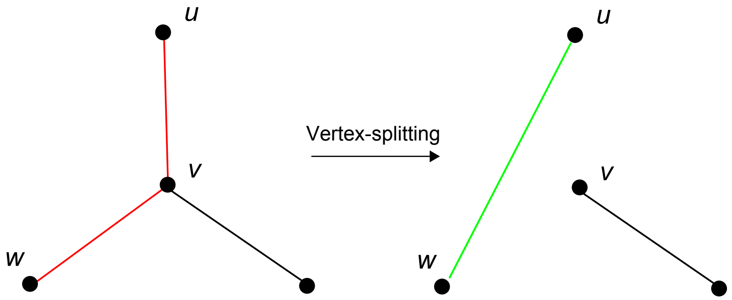
Vertex splitting

- vertex-splitting (V.-S.)



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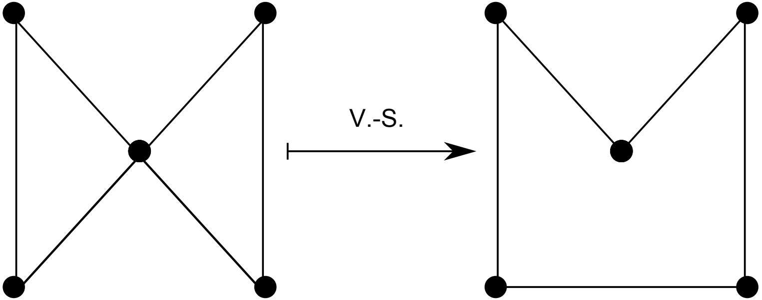
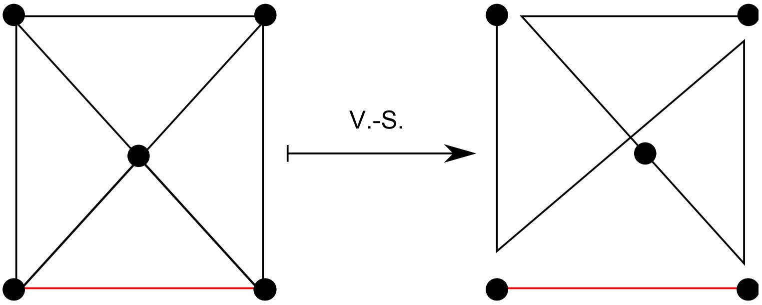


Vertex splitting

- vertex-splitting (V.-S.)
    - G – eulerian $\overset{V.-S.}{\mapsto}$ one circuit
    - G with $2m$ vertices of odd degree $\overset{V.-S.}{\mapsto}$ m disjoint paths



Cartesian product

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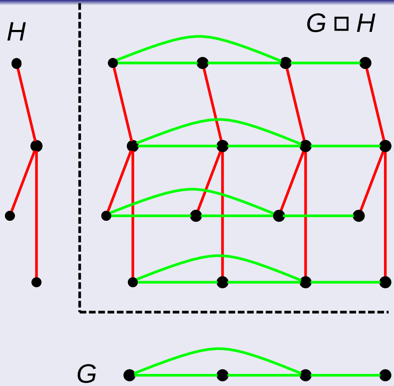
Definition

Cartesian product $G \square H$:

Vertices: $V = V(G) \times V(H)$,

Edges: $\{(x_g, x_h), (y_g, y_h)\} \in E$,
if either:



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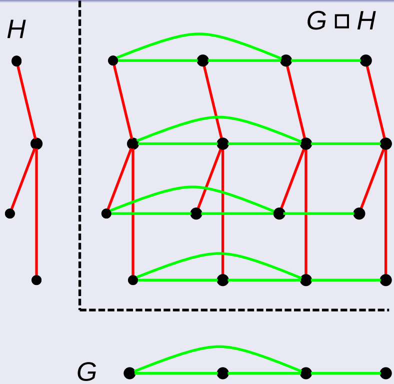

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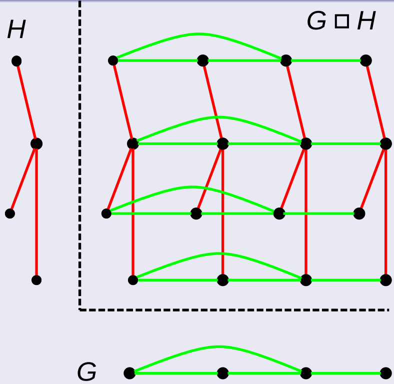

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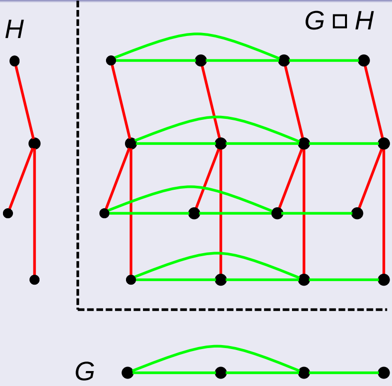
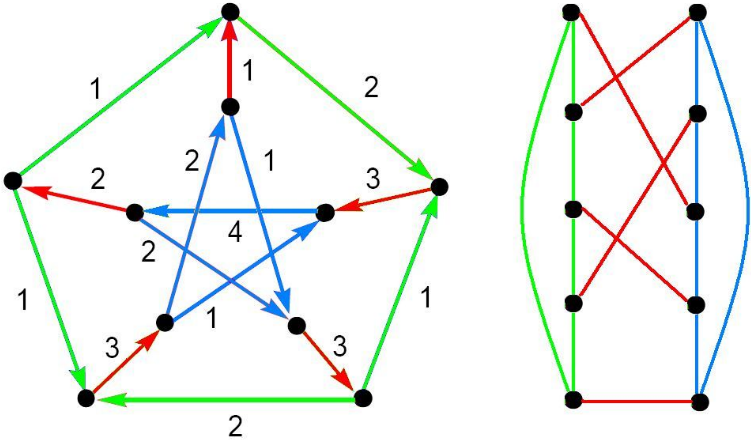


Theorem [Imrich & Škrekovski 2003]

Let G and H be connected nontrivial graphs.
Then the *Cartesian product* $G \square H$ has a nowhere-zero *4-flow*.

Random matching

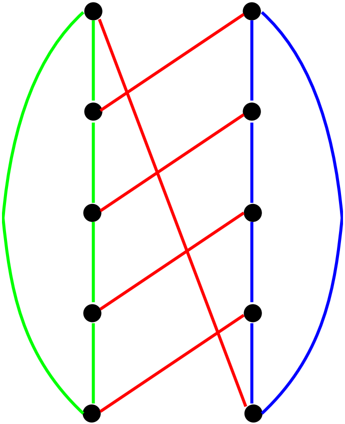


Cartesian bundle: matching given by automorphism

Automorphism:

bijection $\phi : V(G) \to V(G)$

s. t. $uv \in E(G) \Leftrightarrow \phi(u)\phi(v) \in E(G)$



REMARK: automorphism preserves the vertex degree

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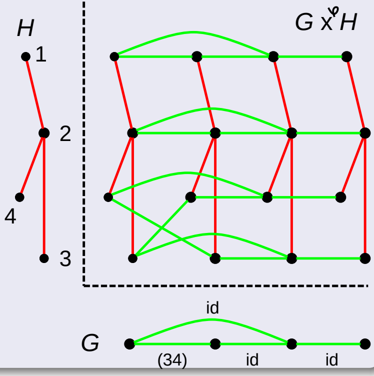

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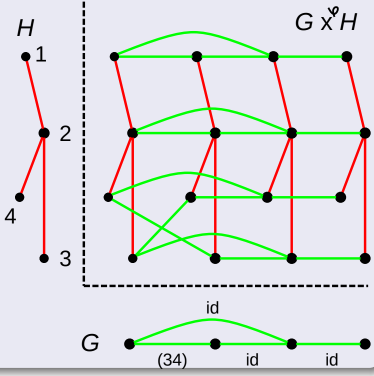

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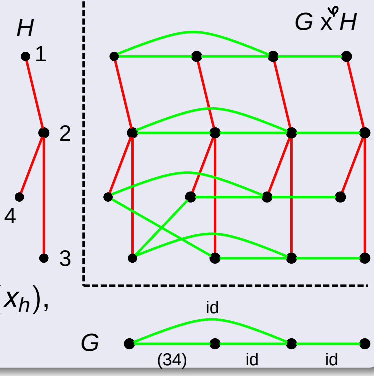
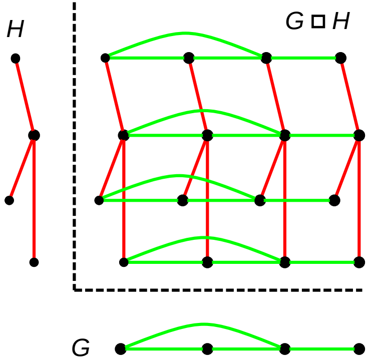
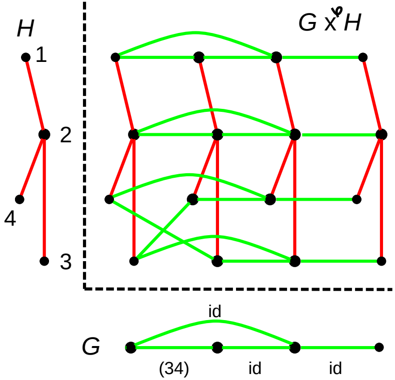
- $[x_g, y_g] \in E(G)$ and $y_h = \phi_{x_g y_g}(x_h)$,
  where $\phi \colon E(G) \to \mathrm{Aut(H)}$.



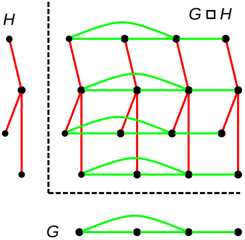
Cartesian product vs. Cartesian bundle

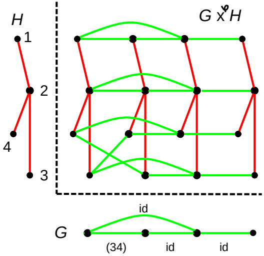




Relationship:

If $\phi_x = id_H, \forall x \in D(G)$ then $G \square^{\phi} H$ is the Cartesian product $G \square H$.

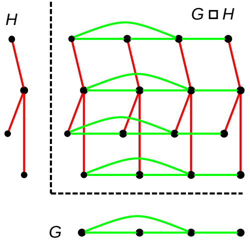


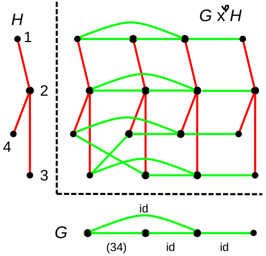


Relationship:

If $\phi_x = id_H, \forall x \in D(G)$ then $G \square^{\phi} H$ is the Cartesian product $G \square H$.

If G is a forest, then every Cartesian bundle $G \square^{\phi} H$ is the Cartesian product $G \square H$.





Cartesian bundle

Theorem [R. & Škoviera 2009]

Let G and H be graphs without isolated vertices. Then *every* *Cartesian bundle* $G\square^{\phi}H$ has a nowhere-zero *4-flow*.

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Let G and H be graphs without isolated vertices. Then *every Cartesian bundle* $G\Box^{\phi}H$ has a nowhere-zero *4-flow*.

1. - using series of vertex-splitting on G (but not on H!) to obtain G'
   - modify ϕ into ϕ':

$$\phi'_z = \begin{cases} \phi_z & \text{for } z \in D(G') \cap D(G), \\ \phi_{vw} \circ \phi_{uv} & \text{for } z = uw. \end{cases}$$

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     $\Rightarrow G \square^{\phi} H$ is eulerian

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- find a nowhere-zero flows of K and L
- combine them into a nowhere-zero flow of $G' \square^{\phi'} H$ and thus $G \square^{\phi} H$

Another proof

- nowhere-zero flow $\longrightarrow$ *group connectivity*
- flow methods $\longrightarrow$ *collapsibility*
- **J. Yan, S. Yao, H.-J. Lai, X. Gu: Group connectivity in products of graphs (2010)**

Lemma

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- Cartesian bundle $G \square^{\phi} H$ is 4-circuit connected.
- If G is eulerian, it contains a collapsible subgraph.

Thank you for your attention.