Some problems on matchings and toughness

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We present a couple of problems related to generalizations of the famous 1-factor theorem of Tutte [6]. For a graph H, let $\omega_{\text{odd}}(H)$ denote the number of components of H of odd order, and let $\omega(H)$ be the total number of components of H. Tutte's theorem says:

Theorem 1 (Tutte) *A graph G has a 1-factor if and only if for all $X \subset V(G)$,*

$$\omega_{\text{odd}}(G - X) \leq |X| \, .$$

There are several extensions of Theorem 1 dealing with systems of disjoint paths. The first such result was proved by Gallai [1]. Let (G, T) be a *graft*; that is, let G be a graph and $T \subset V(G)$ with $|T|$ even. A *T-path* is defined to be a path with ends in T.

Theorem 2 (Gallai) *The maximum number of vertex-disjoint T-paths equals*

$$\min_{X \subset V(G)} \Big(|X| + \sum_K \Big\lfloor \frac{|K \cap T|}{2} \Big\rfloor \Big),$$

where K ranges over components of $G - X$.

Recent work on [3] motivated us to examine systems of disjoint T-paths spanning all of T, which we called *T-path coverings* (in a given graft (G, T)). Specifically, we needed to show that if G is cubic and 3-connected, then every edge is contained in a T-path covering. This is somewhat reminiscent of a theorem of Plesník [5] for 1-factors:

Theorem 3 (Plesník) *Every edge of a k-regular $(k-1)$-edge-connected graph is contained in a 1-factor.*

However, simple examples show that the direct analogue of Theorem 3 is not true for T-path coverings. On the other hand, we established [2] the following:

Theorem 4 *Every edge of a k-regular k-edge-connected graph is contained in a T-path covering.*

We obtained Theorem 4 as a corollary of Theorem 5 below, a result on tough graphs. Recall that a graph G is *tough* if for all $X \subset V(G)$,

$$\omega(G - X) \leq |X| \,.$$

Note that by a simple counting argument, every k-regular k-edge-connected graphs is tough.

Theorem 5 *Let (G, T) be a graft with G tough. An edge $uv \in E(G)$ is not contained in a T-path covering if and only if there is a set $X \subset V(G)$ such that:*

(i) $\{u, v\} \subset X \subset T$,

(ii) $G - X$ *has precisely* $|X|$ *components, and*

(iii) each of these components contains an odd number of vertices in T.

It is natural to ask whether the above characterization could be extended to *pairs* of edges:

Problem 6 *For a graft (G, T) with G tough, characterize the pairs of edges that are not contained in a T-path covering.*

The line of proof of Theorem 5 does not seem to work for Problem 6. More specifically, the proof involves a reduction to the following result of Mader [4] that generalizes Theorem 2. Given a graph G and a system $\mathcal{S}$ of disjoint subsets of $V(G)$, an $\mathcal{S}$-*path* is defined to be a path with ends in distinct sets in $\mathcal{S}$.

Theorem 7 (Mader) *The maximum number of vertex-disjoint $\mathcal{S}$-paths equals*

$$\min_{X,F}\Big(|X| + \sum_{K} \Big\lfloor \frac{|K \cap (T \cup V(F))|}{2} \Big\rfloor\Big),$$

where the minimum is taken over all $X \subset V(G)$ and all $F \subset E(G-X)$ containing no $\mathcal{S}$-paths, and K ranges over components of $G - X - F$.

Problem 6, however, cannot be directly reduced to Mader's theorem. Instead, a result involving the structure of the $\mathcal{S}$-paths, in the following sense, would be useful. Let us define a subgraph of G to be $\mathcal{S}$-*acyclic* if the corresponding subgraph in $G/\mathcal{S}$ is a forest, where $G/\mathcal{S}$ is the graph obtained by contracting each set in $\mathcal{S}$ to a vertex.

Problem 8 *Is there a relation, similar to the one in Theorem 7, for the maximum number of disjoint $\mathcal{S}$-paths whose union is $\mathcal{S}$-acyclic?*

We conclude with a problem concerning a possible generalization of Theorem 1 along different lines:

Problem 9 *Is there a sufficient condition in the spirit of Tutte's theorem for the existence of two edge-disjoint 1-factors? How about the simplest candidate:*

$$\omega_{\rm odd}(G - X) \leq \frac{|X|}{C},$$

for all $X \subset V(G)$, where C is a large constant?

References

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