Some problems on even factors in graphs

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In this note, we will state several problems on even factors in graphs and recall a related open question from [1]. All of these problems are inspired by the 1884 conjecture of P. G. Tait that 3-connected cubic planar graphs are hamiltonian. As is well known, the conjecture was disproved by Tutte [3] in 1946.

One way to view a Hamilton cycle in a cubic graph G is as a connected even factor F. Here, *factor* is used as a synonym for 'spanning subgraph', and F is *even* if the degree of each vertex in F is even. Thus, Hamilton cycles in a cubic graph G are precisely the even factors whose edge set intersects every minimal edge-cut in G.

Since there are non-hamiltonian 3-connected cubic planar graphs, we may ask whether they at least admit an even factor intersecting 'many' minimal edge-cuts. One option regarding the definition of 'many' is to require that the even factor intersects all minimal edge-cuts whose size belongs to a given set A. For certain choices of A, positive results are known — even for larger classes of graphs.

For instance, the assertion that every planar graph admits an even factor intersecting all minimal edge-cuts C with $|C| \in \{3, 5, 7, \ldots\}$ was shown in [1] to be equivalent to the Four Colour Theorem. The main result of [1] shows that every graph (planar or not) contains an even factor intersecting all minimal edge-cuts of size 3 and 4. On the other hand, the Petersen graph admits no even factor intersecting all minimal edge-cuts C with $|C| \in \{3, 5\}$: any such factor would have to be a 2-factor, and each 2-factor in the Petersen graph is edge-disjoint from a unique minimal 5-edge-cut. The following conjecture was made in [1]:

Conjecture 1 *Every graph which contains no subdivision of the Petersen graph admits an even factor intersecting all minimal edge-cuts of size 3 or 5.*

For planar graphs, Conjecture 1 holds by the Four Colour Theorem, while for cubic graphs, it is likewise implied by the corresponding special case of Tutte's 4-flow conjecture (see [2, Theorem 10.2]). In general, however, it is still open.

We now turn to the other problems referred to at the beginning of this note, which come from another interpretation of a Hamilton cycle in a cubic graph.

Given a subgraph H of a graph G, let G/H be the graph obtained by contracting each edge of H. (We allow parallel edges and loops in our graphs.)

If G is a cubic bridgeless graph, then a Hamilton cycle in G is easily seen to be the same thing as an even factor H such that G/H contains no circuits of length at least 2. An obvious relaxation of this property leads to the following:

Question 2 *Does every planar cubic graph G contain an even factor H such that G/H contains no cycle of length ≥ 3?*

Observe that any graph with no cycles of length at least 3 can be obtained from a forest by adding parallel edges and loops. It is not hard to see that any such graph is a plane dual of a *cactus forest*, i.e., of a disjoint union of graphs in which every edge is contained in at most one circuit.

What is a dual interpretation of an even factor H of G? Since H is just an element of the cycle space of G, and planar duality is well-known to map the cycle space of G to the cut space of the dual graph G^*, the set of edges of G^* corresponding to H is a *separation* — a set of the form

$$[X, G^* - X] = \{e \in E(G^*) : |e \cap X| = 1\}$$

for some set X of vertices of G^*. From this, it is not hard to see that the following is a dual form of Question 2:

Question 3 *Does every planar triangulation T admit a 2-colouring (A_1, A_2) of the vertices such that each induced subgraph $T[A_i]$ is a cactus forest?*

Returning to Question 2, one might ask if it holds for non-planar cubic graphs. While we do not know of any example showing the answer to be negative, it is perhaps more likely that it exists. Noting that any graph obtained from a forest by adding parallel edges is series-parallel, we propose the following question:

Question 4 *Does every cubic graph G contain an even factor H such that G/H is series-parallel?*

References

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[2] R. Thomas, Recent excluded minor theorems for graphs, in: Surveys in combinatorics, 1999 (Canterbury), vol. 267, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 1999, pp. 201–222.

[3] W. T. Tutte, On Hamiltonian circuits, J. London Math. Soc. s1-21 (1946), 98–101.