Eulerian colorings and the Bipartizing matchings conjecture of Fleischner (Extended abstract)

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Abstract

A trigraph is a multigraph with half-edges colored in three colors. We introduce the notion of an eulerian coloring of a trigraph and show that the existence of two orthogonal eulerian colorings in a special class of trigraphs is closely related to the Bipartizing matchings conjecture of Fleischner, and hence to the Cycle double cover conjecture and Tutte's 5-flow conjecture. We prove that every trigraph has an eulerian coloring and that a rainbow cubic trigraph has a pair of orthogonal eulerian colorings if and only if it has a perfect matching.

1 Introduction

Fleischner [2] has proposed a novel approach to several deep graph-theoretic conjectures, such as the Cycle double cover conjecture [6, 7] or the 5-flow conjecture of Tutte [8]. A key ingredient in this approach is the Bipartizing matchings conjecture (Conjecture 1.2 below, abbreviated BMC). We introduce the notion of an eulerian coloring of a trigraph, providing a setting that allows a generalization of the Bipartizing matchings conjecture and inspires new questions. Extending a result from [4], we show that every trigraph has an eulerian coloring. We investigate the existence of two orthogonal eulerian colorings in cubic trigraphs.

We describe in detail the connection between BMC and the Cycle double cover conjecture which asserts that any bridgeless graph has a *cycle double cover (CDC)*, i.e., a (multi)set of circuits which covers each edge exactly twice. It is well known [9, Theorem 7.2.4] that it is enough to prove the CDC conjecture for 3-connected cubic graphs, in fact, even only for so-called snarks.

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It is easy to find a CDC of a hamiltonian cubic graph. Although no snark has a Hamilton cycle, they are all conjectured to be rather close to being hamiltonian:

Conjecture 1.1 (Dominating cycle conjecture [1]) *Every snark has a dominating cycle, i.e., a cycle incident with every edge.*

If a cubic graph has a dominating cycle, does it help in finding a CDC? Not in an obvious way. However, with the help of another conjecture of Fleischner [2] (Conjecture 1.2 below), it is indeed possible to construct a CDC. Let C be a dominating cycle in a cubic graph G. A *bipartizing matching* in G (with respect to C) is a matching M which is edge-disjoint from C, such that M covers all vertices not on C and $G - M$ is a subdivision of a cubic bipartite graph. Fig. 1 shows a bipartizing matching M in the Petersen graph with respect to the "outside" dominating cycle. Indeed, the removal of M yields a subdivision of $K_{3,3}$.

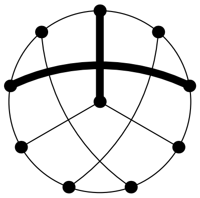


Figure 1: A bipartizing matching in the Petersen graph (bold).

Conjecture 1.2 (Bipartizing matchings conjecture [2]) *Every snark with a dominating cycle C has a pair of (edge-)disjoint bipartizing matchings with respect to C.*

Fleischner [2, 3] showed that Conjecture 1.2 implies the existence of a CDC and a nowhere-zero 5-flow for any snark with a dominating cycle. Thus, if Conjectures 1.1 and 1.2 are true, then the CDC conjecture and the 5-flow conjecture follow. Speaking about two bipartizing matchings, it is natural to ask whether at least one bipartizing matching (with respect to a given dominating cycle) always exists. Indeed, Fleischner and Stiebitz [4] showed that this is the case.

Conjecture 1.2 holds for any cubic graph with a dominating cycle C which is in fact a Hamilton cycle [2]. An alternative proof of this, which is the starting point of our work, is as follows. Two chords of a Hamilton cycle C in G *intersect* if one of them separates the end-vertices of the other. Let us define the *intersection graph* $I(G)$ of G to be the graph whose vertices are the chords of C in G, with edges joining intersecting pairs of chords. Throughout this paper, an *eulerian* graph is a graph that has even vertex degrees and may or may not be connected.

Proposition 1.3 *Let C be a Hamilton cycle in the cubic graph G. Then a matching $M \subset E(G) - E(C)$ is bipartizing if and only if $I(G) - M$ is an eulerian graph.*

Thus, there are two disjoint bipartizing matchings in G if and only if $I(G)$ can be partitioned into two eulerian induced subgraphs. Not only is the latter condition always true; somewhat surprisingly, the following stronger assertion holds for all graphs. The theorem appears as Problem 5.17(a) in [5]:

Theorem 1.4 *The vertices of any graph H can be partitioned into two sets, each of which induces an eulerian subgraph of H.*

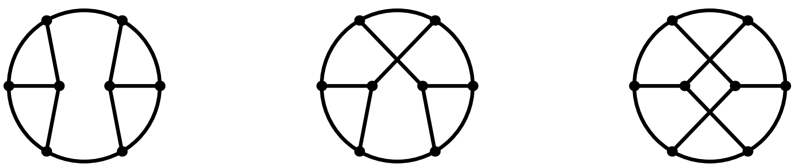
By Proposition 1.3, the partition in Theorem 1.4 yields a partition of the chords of C in G into two bipartizing matchings, thus providing a new proof of Conjecture 1.2 in the hamiltonian case. In what follows, we modify the definition of the intersection graph for the case that C is just a dominating cycle.

2 Eulerian colorings of trigraphs

In this section, we present our own results in the direction indicated by Proposition 1.3. The key approach is to transfer questions about bipartizing matchings to a more general level of eulerian colorings of trigraphs (defined below).

Let C be a dominating cycle of a cubic graph G. We can assume that C has no chords; this is harmless, since each chord can be replaced by an internal vertex in a way preserving all the relevant properties. Thus, every edge not on C is incident with a vertex not on C (an *internal vertex* as we shall call it).

Let us define an analogue of the intersection graph from Section 1. In contrast to the previous case, there are essentially 3 possible configurations of two internal vertices x and y, as shown in Fig. 2. To tell them apart, one can form a cyclic sequence σ that contains the symbol x for each neighbor of x, and the symbol y for each neighbor of y, as they appear in the clockwise order around C. Then, internal vertices x and y form a *disjoint* pair if $\sigma = xxxyyy$, a *crossing* pair if $\sigma = xxyxyy$ and an *alternating* pair if $\sigma = xyxyxy$.



A disjoint pair. A crossing pair. An alternating pair.

Figure 2: The three configurations of two internal vertices.

Now, a *trigraph*, an analogue of the intersection graph, is a loopless multigraph whose edges are viewed as composed of two half-edges, and each half-edge is colored in one of the colors $1, 2, 3$. If an edge e joins vertices x and y, then the color of its half-edge incident with x is also referred to as the *color of e at x*.

The *trigraph* $T(G)$ *of* G (with respect to C) is defined as follows. Its vertices are the internal vertices of G. Two vertices are joined by 0, 1 or 3 edges according to whether they form a disjoint, crossing or alternating pair, respectively.

The coloring of the half-edges of $T(G)$ reflects the special role of certain edges, e.g., in a crossing pair of internal vertices. Choose a labelling ℓ of $V(C)$ by 1, 2 and 3 such that the neighbors of each internal vertex get all three labels. For a pair of vertices $x, y \in V(T(G))$, let G' be the graph obtained from G by removing all internal vertices except x and y, and suppressing all vertices of degree 2 this creates. Thus, G' has 8 vertices, and is isomorphic to one of the graphs in Fig. 2.

If x and y form a crossing pair, let x_1 and y_1 be the unique neighbors of x and y, respectively, that are not contained in any triangle in G'. The (single) edge xy in $T(G)$ will be assigned the color $\ell(x_1)$ at the vertex x, and the color $\ell(y_1)$ at y.

If x and y form an alternating pair, then for each neighbor x' of x, there is a unique vertex, y', such that x' and y' are not contained in any 4-cycle in G'. Correspondingly, one of the 3 edges joining x to y in $T(G)$ will have the color $\ell(x')$ at x and $\ell(y')$ at y. The other choices of x' yield the colorings of the remaining two edges. This completes the definition of $T(G)$.

Not all trigraphs arise as $T(G)$ for a cubic graph G. A trivial obstruction is the presence of more than 3 parallel edges joining two vertices. It is, however, not difficult to construct such examples without any multiple edges as well.

The notion of a bipartizing matching translates nicely to trigraphs. Matchings covering all internal vertices (let us call them *full matchings*) are clearly in 1-1 correspondence with (not necessarily proper) vertex colorings of $T(G)$ by the colors $1, 2, 3$. For a trigraph T and a coloring $f : V(T) \to \{1, 2, 3\}$, let T_f be the graph on $V(T)$ that consists of all edges e such that the color of e at x differs from $f(x)$, and the color of e at y differs from $f(y)$. The coloring f is *eulerian* if T_f is an eulerian graph. Eulerian colorings are related to bipartizing matchings:

Proposition 2.1 *Let M be a full matching in a cubic graph G with a dominating cycle C, and let f_M be the corresponding vertex coloring of $T(G)$. M is a bipartizing matching if and only if f_M is an eulerian coloring.*

The following generalization of a result from [4] gives some hope that trigraphs provide a reasonable setting for the study of Conjecture 1.2:

Theorem 2.2 *Every trigraph has an eulerian coloring.*

A pair of disjoint bipartizing matchings in G corresponds to a pair of eulerian colorings of $T(G)$ that differ at each vertex; we call such colorings *orthogonal*. There are trigraphs with no pair of orthogonal eulerian colorings, since there are examples of cubic graphs with no pair of disjoint bipartizing matchings for a given dominating cycle. For instance, an example due to Fleischner yields the trigraph shown in Fig. 3(a). There are also 2-connected such trigraphs, see Fig. 3(b). We

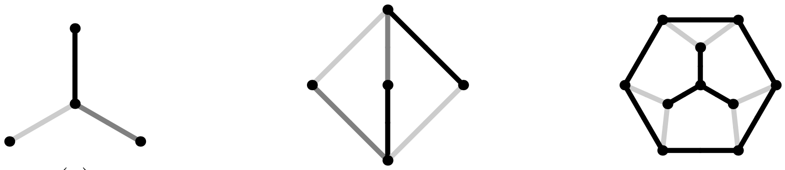


Figure 3: Trigraphs with no orthogonal pair of eulerian colorings (the colors $1, 2, 3$ are represented by grey shades).

have investigated in detail the existence of two orthogonal eulerian colorings in *cubic* trigraphs, i.e., ones in which each vertex has degree 3. Such a trigraph is *rainbow* if the three half-edges incident with any vertex have all three colors. The existence of a perfect matching in such a trigraph plays a key role:

Theorem 2.3 *A rainbow cubic trigraph T has a pair of orthogonal eulerian colorings if and only if T has a perfect matching.*

For non-rainbow cubic trigraphs, the equivalence in Theorem 2.3 fails to hold, see Fig. 3(c).

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