Eulerian colorings and the Bipartizing Matchings Conjecture of Fleischner

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1 Introduction

Fleischner [3] has proposed a novel approach to several deep graph-theoretic conjectures, such as the Cycle Double Cover Conjecture [9,10] or the 5-Flow Conjecture of Tutte [12]. A key ingredient in this approach is the Bipartizing Matchings Conjecture (Conjecture 1.2 below, abbreviated BM Conjecture). In this paper, we introduce the notion of an Eulerian coloring of a trigraph, providing a setting that allows a generalization of the Bipartizing Matchings Conjecture and inspires new questions. We also prove the counterparts of major known facts about bipartizing matchings in the more general context.

Let us describe in detail the connection between the BM Conjecture and the Cycle Double Cover Conjecture which asserts that any bridgeless graph has a *cycle double cover (CDC)*, i.e., a (multi)set of circuits which covers each edge exactly twice. It is well known [13, Theorem 7.2.4] that it is enough to prove the CDC Conjecture for 3-connected cubic graphs, in fact, even only for snarks. (Recall that a *snark* is a cubic cyclically 4-edge-connected graph of girth at least 5 which is not 3-edge-colorable. See [6] or [11] for more information.)

It is easy to find a CDC of a hamiltonian cubic graph. Although no snark has a Hamilton cycle, they are all conjectured to be rather close to being hamiltonian:

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Conjecture 1.1 (Dominating cycle conjecture [2]) *Every snark has a* dominating cycle*, i.e., a cycle incident with every edge.*

If a cubic graph has a dominating cycle, does it help in finding a CDC? Not in an obvious way. However, with the help of the BM Conjecture of Fleischner [3] (Conjecture 1.2 below), it is indeed possible to construct a CDC. Let C be a dominating cycle in a cubic graph G. A *bipartizing matching* in G (with respect to the dominating cycle C) is a matching M that is edge-disjoint from C, covers all the vertices not on C, and has the property that $G - M$ is a subdivision of a cubic bipartite graph. Figure 1 shows a bipartizing matching M in the Petersen graph with respect to the "outside" dominating cycle. Indeed, the removal of M yields a subdivision of $K_{3,3}$.

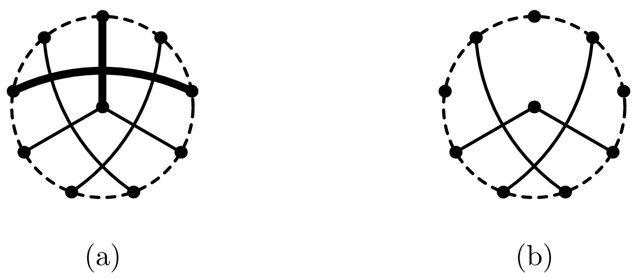


Fig. 1. (a) A bipartizing matching M (bold) in the Petersen graph P. The dominating cycle is shown dashed. (b) The graph $P - M$ is a subdivision of $K_{3,3}$.

Conjecture 1.2 (Bipartizing Matchings Conjecture [3]) *Given any dominating cycle C in a snark, there is a pair of (edge-)disjoint bipartizing matchings with respect to C.*

Fleischner [3,4] showed that Conjecture 1.2 implies the existence of a CDC and a nowhere-zero 5-flow for any snark with a dominating cycle. (For information on nowhere-zero flows, consult [8].) Thus, if Conjectures 1.1 and 1.2 hold, then the CDC Conjecture and the 5-Flow Conjecture follow. Speaking about two disjoint bipartizing matchings, it is natural to ask whether at least one bipartizing matching (with respect to a given dominating cycle) always exists. Fleischner and Stiebitz [5] showed that this is indeed the case:

Theorem 1.3 *Any cubic graph has a bipartizing matching with respect to any dominating cycle.*

As shown in [3], an analogue of Conjecture 1.2 holds for hamiltonian cubic graphs. More precisely, any cubic graph with a Hamilton cycle C has a pair of disjoint bipartizing matchings with respect to C. The following alternative proof of this fact is the starting point of our work.

Two chords of a Hamilton cycle C in G *intersect* if one of them separates the end-vertices of the other. Let us define the *circle graph* $\mathrm{CG}(G)$ of G (with respect to C) to be the graph whose vertices are the chords of the cycle C in G, and whose edges join intersecting pairs of chords. An example of a circle graph is given in Figure 2.

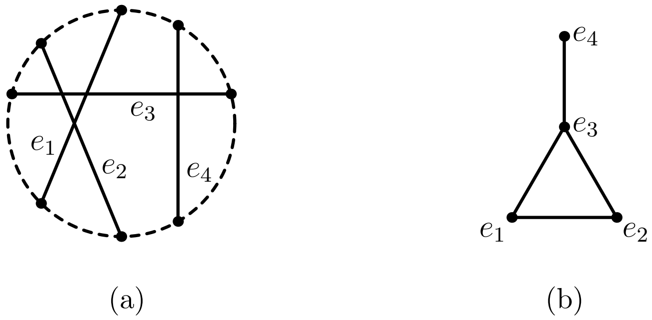


Fig. 2. (a) A graph G with a Hamilton cycle C (dashed). (b) The circle graph $\mathrm{CG}(G)$ of G with respect to C.

Throughout this paper, an *Eulerian* graph is a graph all of whose vertices have even degrees (the graph may or may not be connected).

Proposition 1.4 *Let C be a Hamilton cycle in a cubic graph G. A matching $M \subset E(G) - E(C)$ is bipartizing if and only if the set $V(\mathrm{CG}(G)) - M$ induces an Eulerian graph.*

Proof. Consider a matching $M \subset E(G) - E(C)$. For each chord $f \notin M$ of the cycle C, fix a cycle $C_f \subset G - M$ consisting of f and one of the two parts of C delimited by the endvertices of f. We prove the following:

Claim *The degree of any chord f, as a vertex of $\mathrm{CG}(G) - M$, has the same parity as the number of vertices on C_f whose degree in $G - M$ is 3.*

A vertex of C_f has degree 3 in $G - M$ if and only if it is an endvertex of a chord not in M. A chord $h \notin M$ contributes 1 such vertex if it crosses f, and it contributes 0 or 2 such vertices if it does not cross f. Since the degree of f in $\mathrm{CG}(G) - M$ is precisely the number of chords $h \notin M$ crossing f, the claim follows.

To prove the equivalence in the proposition, note that M is bipartizing if and only if each cycle in $G - M$ contains an even number of degree 3 vertices. In particular, if M is bipartizing, then each cycle C_f contains an even number of degree 3 vertices. By the claim, the degree of each $f \in V(\mathrm{CG}(G))$ is even.

For the converse, let G' be the graph obtained from $G - M$ by suppressing all degree 2 vertices, and let C' and C'_f denote the cycles corresponding to C and C_f in G', respectively. By the claim, all of these cycles are of even length. If the number of chords $f \notin M$ of C is t, then the set

$$\{C'\} \cup \{\, C'_f \,:\, f \notin M \text{ is a chord of } C\,\}$$

is a set of $t+1$ independent vectors in the cycle space $Z(G')$ of G' (see, e.g., [1, p. 52]). The graph G' has $2t$ vertices and $3t$ edges, so the dimension of $Z(G')$ is $3t-2t+1 = t+1$. Hence, we have a basis of $Z(G')$ consisting of even cycles. Clearly, G' is bipartite, and so M is bipartizing. □

Thus, G has two disjoint bipartizing matchings with respect to the Hamilton cycle C if and only if $\mathrm{CG}(G)$ contains two induced Eulerian subgraphs H_1 and H_2, such that $V(\mathrm{CG}(G)) = V(H_1) \cup V(H_2)$. Not only is the latter condition always true. Somewhat surprisingly, there is a much stronger result due to T. Gallai (see Problem 5.17(a) in [7]). We include a proof since we use similar ideas later, in the more complicated proof of Theorem 4.1. As usual, the induced subgraph of a graph H on a set $X \subset V(H)$ is denoted by $H[X]$.

Theorem 1.5 *The vertices of any graph H can be* partitioned *into two sets, each of which induces an Eulerian subgraph of H.*

Proof. If H itself is Eulerian, then the trivial partition into $V(H)$ and $\emptyset$ does the job. Otherwise, let v be a vertex of odd degree and let its neighborhood be denoted by N. Remove v and perform the *complementation on* N (delete all edges on N and replace all former non-edges on N with edges) to obtain a graph H'. Since H was not the one-vertex graph, H' is nonempty. By induction, $V(H')$ has a partition $V_1' \cup V_2'$ into sets inducing Eulerian subgraphs. Since $|N|$ is odd, $|V_i' \cap N|$ is even for precisely one i, say $i = 1$. If we reverse the complementation on N, the degree of each $w \in V_1' \cap N$ in $H[V_1']$ changes from even to odd, while all other degrees in $H[V_1']$ and all degrees in $H[V_2']$ stay even. Thus, the partition of $V(H)$ into $V_1 = V_1' \cup \{v\}$ and $V_2 = V_2'$ has the desired property. □

By Proposition 1.4, the partition in Theorem 1.5 yields a partition of the chords of C in G into two bipartizing matchings, thus providing a new proof of an analogue of Conjecture 1.2 in the hamiltonian case. This brings up the following question: is there a generalization of the circle graph that can be used if C is just a dominating cycle? We describe such a generalization in the following section.

One more comment on the proof of Theorem 1.5 is in order. An alternative argument involves a system of linear equations over $\mathbb{Z}_2$ where each $v \in V(H)$ has its own variable x_v and defines the following equation:

$$\sum_{w \in N(v)} (x_v + x_w + 1) = 0.$$

This system of equations always has a solution, and the solutions are in one-to-one correspondence with the partitions from Theorem 1.5. The interesting thing is that the two arguments, despite their apparent dissimilarity, are in fact very close to each other. Roughly speaking, the complementation on the neighborhood of a vertex (as in the proof of Theorem 1.5) corresponds to a step of the Gauss elimination on the above system of equations.

2 Trigraphs

We shall now define trigraphs and Eulerian colorings. These concepts will enable us to transfer questions about bipartizing matchings to a more general level, in the direction indicated by Proposition 1.4.

Throughout, C will always be a dominating cycle of a cubic graph G. Any vertex not on C will be called *internal*. For our purposes, it may be assumed that C has no chords (so every edge not on C is adjacent to an internal vertex). This is ensured by the following *chord elimination* process (originally from [3]) whose outcome is illustrated in Figure 3. Let c_1c_2 be a chord of C, with the vertex c_1 adjacent to a vertex d on C. Subdivide the edges c_1c_2 and c_1d, and join the two new vertices by an edge. The resulting graph G' is cubic and the dominating cycle C', arising from C, has one chord fewer. Furthermore, G' has a bipartizing matching with respect to C' if and only if G has one with respect to C. A similar claim holds for a pair of disjoint bipartizing matchings.

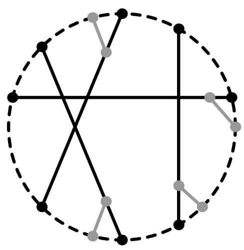
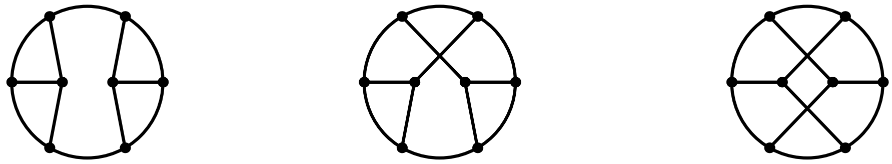


Fig. 3. A graph obtained by repeated chord elimination from the graph in Figure 2(a). The added vertices and edges are shown in grey.

We now define an analogue of the circle graph from Section 1. In contrast to the previous case, there are essentially 3 possible configurations of two internal vertices x and y, as shown in Figure 4. To tell them apart, one can form a cyclic sequence σ that contains the symbol x for each neighbor of x, and the symbol y for each neighbor of y, as they appear in the clockwise order around C. Then, internal vertices x and y form a *disjoint* pair if $\sigma = xxxyyy$ (as in Figure 4(a)), a *crossing* pair if $\sigma = xxyxyy$ (Figure 4(b)), or an *alternating* pair if $\sigma = xyxyxy$ (Figure 4(c)).

We now define our analogue of the circle graph. A *trigraph* is a pair (T, c),



(a) A disjoint pair. (b) A crossing pair. (c) An alternating pair.

Fig. 4. The three configurations of two internal vertices.

where T is a loopless multigraph whose edges are viewed as composed of two half-edges, and c is a coloring of the half-edges by the colors $1, 2, 3$. If an edge e joins vertices x and y, then the half-edge incident with x is denoted by e_x. The color $c(e_x)$ of e_x will be referred to as the *color of* e *at* x. In cases where the coloring c is clear from the context, we just speak of a trigraph T.

The *trigraph* $\mathrm{TG}(G, C)$ *of* a cubic graph G, with respect to a dominating cycle C, is defined as follows. Its vertices are the internal vertices of G. Two vertices are joined by 0, 1 or 3 edges according to whether they form a disjoint, crossing or alternating pair, respectively.

The coloring of the half-edges of $\mathrm{TG}(G, C)$ reflects the special role of certain edges, e.g., in a crossing pair of internal vertices. Choose a labelling ℓ of $V(C)$ by 1, 2 and 3 such that the neighbors of each internal vertex get all three labels. For a pair of vertices $x, y \in V(\mathrm{TG}(G, C))$, let G_{xy} be the graph obtained from G by removing all internal vertices except x and y, and suppressing all vertices of degree 2 this creates. Thus, G_{xy} has 8 vertices, and is isomorphic to one of the graphs in Figure 4. We shall now define the coloring in each of the three cases. At the same time, we shall define the notion of a *related pair* of vertices.

For a disjoint pair x, y, there is no edge between x and y to color, and there will also be no related neighbors of x and y.

If x and y form a crossing pair, let x' and y' be the unique neighbors of x and y, respectively, that are not contained in any triangle in G_{xy}. The (single) edge xy in $\mathrm{TG}(G, C)$ will be assigned the color $\ell(x')$ at the vertex x, and the color $\ell(y')$ at y. The vertices x' and y' form a related pair (written as $x' \sim y'$), and there are no other related pairs among the neighbors of x and y.

If x and y form an alternating pair, then for each neighbor x' of x, there is a unique vertex, y', such that x' and y' are not contained in any 4-cycle in G_{xy}. Correspondingly, one of the 3 edges joining x to y in $\mathrm{TG}(G, C)$ will have the color $\ell(x')$ at x and $\ell(y')$ at y. The other choices of x' yield the colorings of the remaining two edges joining x and y in $\mathrm{TG}(G, C)$. We have 3 pairs of related vertices in this case: each x' is related to the associated y'. This completes both the definition of $\mathrm{TG}(G, C)$ and of the related vertices.

Note that $\mathrm{TG}(G, C)$ depends on the choice of the labelling ℓ. However, trigraphs obtained for different choices of ℓ are equivalent in the following sense: trigraphs T_1 and T_2 are *isomorphic* if there is an isomorphism $\varphi : T_1 \to T_2$ of uncolored multigraphs, and for each vertex $v \in V(T_1)$ and edges e, e' incident with v, the colors of e, e' at v are the same if and only if the colors of $\varphi(e), \varphi(e')$ at $\varphi(v)$ are the same. Thus, isomorphic trigraphs only differ by color permutations at each vertex. Their combinatorial properties are the same, and we shall regard such trigraphs as identical. With this provision, $\mathrm{TG}(G, C)$ is well-defined.

An example of a cubic graph G with a dominating cycle C, along with the corresponding trigraph $\mathrm{TG}(G, C)$, is shown in Figure 5. We represent the colors $1, 2, 3$ by shades of grey (from lighter to darker).

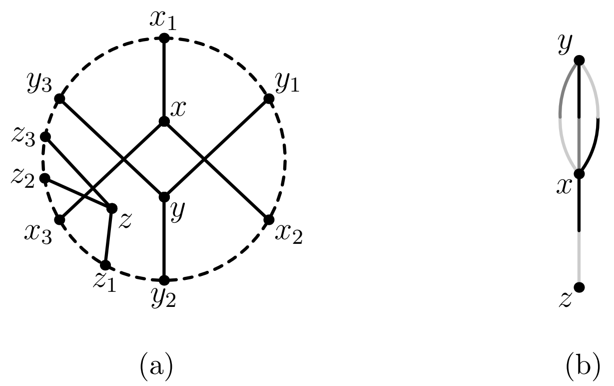


Fig. 5. (a) A cubic graph G. (b) The trigraph $\mathrm{TG}(G, C)$ with respect to the dashed dominating cycle C of G.

We remark that not all trigraphs arise as $\mathrm{TG}(G, C)$ for a cubic graph G and a dominating cycle C. A trivial obstruction is the presence of more than 3 parallel edges joining two vertices. It is, however, not difficult to construct such examples without any multiple edges as well.

3 Eulerian colorings of trigraphs

Fix a cubic graph G and a labelling ℓ of the vertices of the dominating cycle C by $\{1, 2, 3\}$. Let x_i denote the neighbor of an internal vertex x with $\ell(x_i) = i$. Define the *(x, i)-segment* $S_{x,i}$ to be the arc of C delimited by the neighbors of x different from x_i. More formally, $S_{x,i}$ is the component of $C - \{x_1, x_2, x_3\}$ containing no neighbor of x_i. Further, let $C_{x,i}$ be the cycle formed by the segment $S_{x,i}$, the two neighbors of x adjacent to it, and the vertex x itself.

The notion of a bipartizing matching has a natural counterpart in the trigraph setting. Let a *full matching* in G be a matching disjoint from C and covering all internal vertices. Since we have assumed that C has no chords, all bipartizing matchings are full. Any full matching M induces a (not necessarily proper) vertex coloring f_M of $\mathrm{TG}(G, C)$ by the colors $1, 2, 3$: a vertex x of $\mathrm{TG}(G, C)$ gets color i in f_M if the edge xx_i is contained in M. In fact, this defines a bijective correspondence between full matchings in G and vertex colorings of $\mathrm{TG}(G, C)$ by $1, 2, 3$. For a trigraph T and a coloring $f : V(T) \to \{1, 2, 3\}$, T_f is the multigraph on $V(T)$ obtained from T by the removal of all the edges e such that for at least one endvertex of e, $c(e_x) = f(x)$. The coloring f is *Eulerian* if T_f is an Eulerian graph. Eulerian colorings are related to bipartizing matchings by the following analogue of Proposition 1.4.

Proposition 3.1 *Let M be a full matching in a cubic graph G with a dominating cycle C, and let f_M be the corresponding vertex coloring of* $\mathrm{TG}(G, C)$*. The matching M is bipartizing if and only if f_M is an Eulerian coloring.*

Proof. Let x be an internal vertex of G with neighbors x_1, x_2, x_3 and assume that $xx_1 \in M$. For convenience, we shall write T for $\mathrm{TG}(G, C)$ and f for f_M.

Let us call a vertex $z \in V(C)$ *surviving* if z is not incident with an edge in M (and hence is not suppressed after the removal of M). We show:

Claim *The number s of surviving vertices in $S_{x,1}$ has the same parity as the degree $d(x)$ of x in T_f.*

To prove this claim, we consider each internal vertex $y \in V(T)$ in turn and evaluate its contribution to s and $d(x)$ based on the type of the pair (x, y).

If y forms a disjoint pair with x, then all the y_i reside in a single (x, j)-segment, and exactly 2 of them are surviving. Thus, the contribution of y to s is 0 or 2. This is fine because there are no edges between y and x in T_f.

Next, assume y forms a crossing pair with x. If any neighbor of y is related to x_1, then no y_i is contained in $S_{x,1}$ — a zero contribution to the number s. On the other hand, the contribution to $d(x)$ is also zero, since any edge colored 1 at x gets deleted on the way to T_f. If some y_i is related to x_2, then the neighbors of y other than y_i are either both in $S_{x,1}$ or both in $S_{x,3}$, and y_i is in the other one of these two segments. It follows that the contribution to s is 0 or 2 if $yy_i \in M$, and 1 otherwise. Now y contributes 0 to $d(x)$ if $yy_i \in M$ (because edges colored i at y are deleted), and 1 if $yy_i \notin M$. Hence, the contribution to $d(x)$ is as required. The case that some y_i is related to x_3 is symmetric.

Finally, consider the case that x and y form an alternating pair. The only neighbor of y in $S_{x,1}$ (say, y_j) is related to x_1. If y_j is surviving, then the contribution of y to s is 1. So is the contribution to $d(x)$, because the edge colored j at y gets deleted because of its color at x, exactly one other edge will be deleted because of its color at y, and one edge remains. Thus, assume that y_j is not surviving, i.e. $yy_j \in M$. Then two of the three edges between x and y are in T_f, so y contributes an even number to $d(x)$ and 0 to s as desired.

We have shown that the number s of surviving vertices in $S_{x,1}$ is the same, modulo 2, as $d(x)$. Note that the number of surviving vertices in $C_{x,i}$ is $s+2$ (because x is not surviving, but two of its neighbors are). This implies that if M is bipartizing, then f_M is Eulerian.

To see the converse, proceed exactly as in the proof of Proposition 1.4. Assume that f_M is Eulerian and that there are a total of t internal vertices. Let G' be the graph obtained from $G - M$ by suppressing all degree 2 vertices, and let C' and $C'_{x,i}$ denote the cycles corresponding to C and $C_{x,i}$ in G'. These $t+1$ even cycles form a basis of the cycle space $Z(G')$ of G'. Hence, G' is bipartite and M is bipartizing. $\square$

4 The existence of Eulerian colorings

The main result of this section is a generalization of Theorem 1.3 which justifies the hope that trigraphs provide a reasonable setting for the study of Conjecture 1.2:

Theorem 4.1 *Every trigraph has an Eulerian coloring.*

For the time being, we postpone the proof of Theorem 4.1, and observe first that it suffices to prove the theorem for a special class of trigraphs. To begin with, we may restrict ourselves to *rainbow* trigraphs, i.e. trigraphs in which no two edges e, e' with the same endvertices x and y have the same color at x. Suppose a trigraph T contains such edges and form a trigraph T' as follows. If $c(e_y) = c(e'_y)$, then remove e and e' from T. Otherwise, replace e and e' with a single edge e'' colored so that $c(e''_x) = c(e_x)$ and $c(e_y) \neq c(e''_y) \neq c(e'_y)$. In either case, it is easy to check that any Eulerian coloring of T is an Eulerian coloring of T' and vice versa. Since the new trigraph T' has fewer edges than T, repeating the above process yields a rainbow trigraph. The result is well-defined. Note that rainbow trigraphs contain at most 3 edges between any pair of vertices.

Next, we eliminate all the triple edges. Let x, y be vertices of T joined by 3 parallel edges e_1, e_2, e_3. Permute the colors of all half-edges incident with y so as to make each e_i have the same color at x and y. (Note that this permutation applies to all edges incident with y, not just the edges e_i.) Now the trigraph T'' is obtained by replacing the edges e_i with 2 parallel edges h, h', colored as follows:

$$c(h_x) = c(h'_y) = 1 \quad \text{and} \quad c(h_y) = c(h'_x) = 2.$$

Observe that if f is any coloring of the vertices of T (with $1, 2, 3$), then the number of edges between x and y in T_f has the same parity as in T''_f. It follows that T has an Eulerian coloring if and only if T'' does. Eventually, this process yields a rainbow trigraph with no triple edges. However, since the result of this reduction depends on the order in which triple edges are eliminated, we prefer to allow triple edges in the following definitions.

We are now going to define the 'flip', an operation that corresponds to the neighborhood complementation of Section 1. Fix a vertex x of a rainbow trigraph (T, c) and a color $k \in \{1, 2, 3\}$. Let $X_k = X_{x,k}$ be the set of vertices which are joined to x by at least one edge whose color at x differs from k. For each vertex $z \in X_k$, we define the *distinguished color* $\gamma_k(z) \in \{1, 2, 3\}$ as the unique γ such that the number of edges e between x and z with $c(e_x) \neq k$ and $c(e_z) \neq \gamma$ is even. It is easy to check that the choice is indeed unique in each of the possible situations:

- if x and z are joined by a single edge e with $c(e_x) \neq k$, set $\gamma_k(z) = c(e_z)$,
- if x and z are joined by two edges e, e' with $c(e_x) = k$, set $\gamma_k(z) = c(e'_z)$,
- if x and z are joined by two edges e, e' with $k \notin \{c(e_x), c(e'_x)\}$, then $\gamma_k(z)$ is the color different from both $c(e_z)$ and $c(e'_z)$,
- if x and z are joined by three edges e, e', e'' with $c(e_x) = k$, set $\gamma_k(z) = c(e_z)$.

The (x, k)-*flip* is an operation whose result is a rainbow trigraph $T^*_{x,k}$ on $V(T) - \{x\}$. To perform the (x, k)-flip in T, start with the trigraph $T - x$, add a new edge between every two vertices in X_k and set the color of each new edge at an endvertex z to be $\gamma_k(z)$. Finally, reduce the result to a rainbow trigraph.

For an example, consider the $(x, 1)$-flip in the trigraph T shown in Figure 6(a). The edges added to $T - x$ are shown in Figure 6(b), and the resulting rainbow trigraph is in Figure 6(c). Recall that color 1 corresponds to the lightest shade of grey. We have $X_1 = \{z, w, t\}$. The distinguished colors are $\gamma_1(z) = 1$ and $\gamma_1(w) = \gamma_1(t) = 2$. Note how the addition of a new edge between w and t (colored 2 on both ends) and the subsequent reduction to a rainbow trigraph eliminates an edge of T.

Fix a coloring f of the trigraph T and $k \in \{1, 2, 3\}$. Set

$$X_k(f) = \{\, z \in X_k \;:\; f(z) \neq \gamma_k(z) \,\}.$$

We make several observations used later in the proof of Theorem 4.1.

Lemma 4.2 *With respect to degree parity, the (x, k)-flip behaves like the complementation on $X_k(f)$. More precisely, let w and w' be two vertices of $T - x$.*

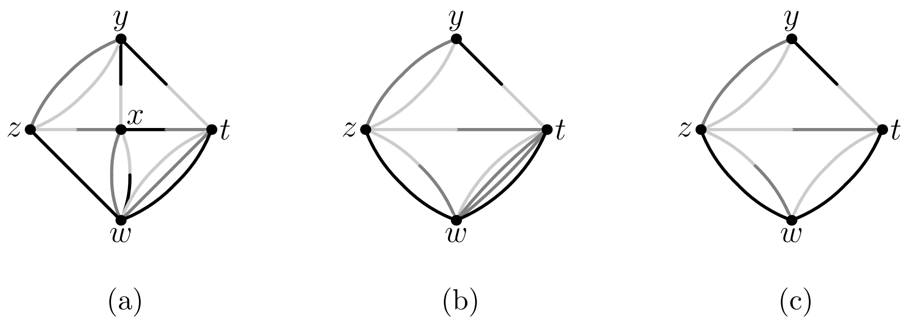


Fig. 6. (a) A trigraph T. (b) The intermediate step of the $(x, 1)$-flip in T. (c) The result of the $(x, 1)$-flip.

If m denotes the number of edges between w and w' in T_f and m^* denotes the number of edges between them in $(T^*_{x,k})_f$, then

$$m \not\equiv m^* \pmod 2 \quad \text{if and only if both } w \text{ and } w' \text{ are in } X_k(f).$$

Proof. Performing the (x, k)-flip, one begins with $T - x$ and adds a suitably colored edge between each pair of vertices in X_k. Let us call the resulting intermediate trigraph $\tilde{T}$. Assume that $w, w' \in V(T - x)$ and let $\tilde{m}$ denote the number of edges between w and w' in $\tilde{T}_f$. We prove the following:

Claim *The numbers m and $\tilde{m}$ have different parity if and only if both w and w' are in $X_k(f)$.*

If either of the two vertices is not contained in X_k, then the claim is trivially true, since the edges between w and w' in $\tilde{T}$ are, in this case, the same as in $T - x$. If both w and w' are in X_k, then $\tilde{T}$ contains a new edge between them, colored $\gamma_k(w)$ at w and $\gamma_k(w')$ at w'. This edge is in $\tilde{T}_f$ if and only if $f(w) \neq \gamma_k(w)$ and $f(w') \neq \gamma_k(w')$, i.e., if and only if $w, w' \in X_k(f)$. The claim follows.

To obtain $T^*_{x,k}$, one finally reduces $\tilde{T}$ to a rainbow trigraph. It is easy to check this does not change the parity in question, i.e., that $\tilde{m} \equiv m^* \pmod 2$. Thus, the statement of the lemma is implied by the claim. $\square$

For a coloring f of $T - x$, let $f^{x \mapsto k}$ be the *extension* of f to T obtained by assigning color k to x. We have the following easy observation.

Lemma 4.3 *If f is a coloring of $T - x$, then $f^{x \mapsto k}$ is an Eulerian coloring of T if and only if all of the following conditions hold:*

(i) all $z \notin X_k(f)$ have even degree in $(T - x)_f$,
(ii) all $z \in X_k(f)$ have odd degree in $(T - x)_f$, and
(iii) the size of $X_k(f)$ is even. $\square$

Another observation shows that the behavior of colorings under flips is somewhat similar as in Lemma 4.3. In particular, condition (i) is the same in both statements.

Lemma 4.4 *A coloring f of $T - x$ is Eulerian for $T^*_{x,k}$ if and only if all of the following hold:*

(i) all $z \notin X_k(f)$ have even degree in $(T - x)_f$, and
(ii) all $z \in X_k(f)$ have degree (in $(T - x)_f$) different from $|X_k(f)|$ modulo 2. $\square$

Proof of Theorem 4.1. Let T be a trigraph. As noted above, T may be assumed to be rainbow. We proceed similarly as in the Fleischner–Stiebitz proof [5] of Theorem 1.3. Let $x \in V(T)$. We show:

Claim

$$e(T) \equiv e(T^*_{x,1}) + e(T^*_{x,2}) + e(T^*_{x,3}) \pmod 2. \tag{1}$$

We now prove this congruence. Fix a coloring f of $T - x$. We show that the number A of Eulerian extensions of this coloring f to T has the same parity as the number B of distinct i such that f is an Eulerian coloring of $T^*_{x,i}$. We may assume that $A \neq B$ (otherwise there is nothing to prove).

If k is a color such that $f^{x \mapsto k}$ is Eulerian, then we claim that f is Eulerian for $T^*_{x,k}$. To begin with, conditions (i)–(iii) of Lemma 4.3 are satisfied. These conditions imply the two conditions of Lemma 4.4. Indeed, condition (i) is identical in these two lemmas, while (ii) of Lemma 4.4 follows from (ii) and (iii) of Lemma 4.3. Thus, Lemma 4.4 shows that f is Eulerian for $T^*_{x,k}$ as claimed.

By our assumption that $A \neq B$, there must be some j such that f is Eulerian for $T^*_{x,j}$, but $f^{x \mapsto j}$ is not Eulerian. By Lemma 4.4, all $z \notin X_j(f)$ have even degrees (all degrees are taken in $(T - x)_f$ in this paragraph) and all $z \in X_j(f)$ have degrees different from $|X_j(f)|$ modulo 2. On the other hand (since $f^{x \mapsto i}$ is not Eulerian), either (ii) or (iii) of Lemma 4.3 fail to hold. Thus, either some $z \in X_j(f)$ has even degree, or $|X_j(f)|$ is odd. Each of these possibilities leads to the same conclusion: all $z \in X_j(f)$ have even degrees, and $|X_j(f)|$ is odd.

An important consequence is that f must be Eulerian for $T - x$. To prove this, recall that f is Eulerian for $T^*_{x,j}$ and note that the degree of any $z \notin X_j(f)$ is the same in $(T^*_{x,j})_f$ as in $(T - x)_f$, i.e., even. Since we have shown that all $z \in X_j(f)$ also have even degrees in $(T - x)_f$, it follows that $(T - x)_f$ is Eulerian.

This has implications for colors $i \neq j$. Consider any color $i \in \{1, 2, 3\}$. By Lemmas 4.3 and 4.4,

$$f \text{ is Eulerian for } T^*_{x,i} \text{ if and only if either } X_i(f) = \emptyset \text{ or } |X_i(f)| \text{ is odd, while } f^{x \mapsto i} \text{ is Eulerian if and only if } X_i(f) = \emptyset. \tag{2}$$

In particular, the difference $B - A$ is the number of colors i such that $|X_i(f)|$ is odd.

It is not hard to see from the definition of the $X_i(f)$ (checking the several cases that may occur) that each neighbor of x is contained in 0 or 2 of the sets $X_i(f)$. Thus $X_3(f) = X_1(f) \oplus X_2(f)$, where $\oplus$ is the symmetric difference. Hence, either 0 or 2 of the sets $X_i(f)$ are of odd size. By the above, it follows that $B - A$ is even. Thus, we managed to show that $A \equiv B \pmod 2$ and to prove the congruence (1).

With (1) established, we prove, by induction on $|V(T)|$, that the number $e(T)$ of Eulerian colorings of T is odd. This is clear if T has a single vertex. Otherwise, the induction hypothesis implies that each of $e(T^*_{x,i})$ (where $i = 1, 2, 3$) is odd. By (1), $e(T)$ is odd as well. The proof is finished. $\square$

5 Orthogonal Eulerian colorings

Two Eulerian colorings of a trigraph T are *orthogonal* if they differ at each vertex. Observe that a pair of disjoint bipartizing matchings in a cubic graph G (with a dominating cycle C) corresponds to a pair of orthogonal Eulerian colorings of $\mathrm{TG}(G, C)$. In view of Conjecture 1.2, one might ask for conditions implying that a trigraph has a pair of orthogonal Eulerian colorings.

There are trigraphs with no pair of orthogonal Eulerian colorings, since there are examples of cubic graphs with no pair of disjoint bipartizing matchings (for a given dominating cycle). For instance, an example due to Fleischner yields the trigraph shown in Figure 7(a). There are also 2-connected such trigraphs, see Figures 7(b) and 7(c).

We have investigated in detail the existence of two orthogonal Eulerian colorings in *cubic* trigraphs, i.e., ones in which each vertex has degree 3. (We define the *degree* of a vertex x in a multigraph to be the total number of edges incident with x; thus, any multiple edges are counted with their multiplicities.) A cubic trigraph is *balanced* if the three half-edges incident with any vertex have all three colors. Rather unexpectedly, it turns out that the existence of a perfect matching in such a trigraph plays a key role:

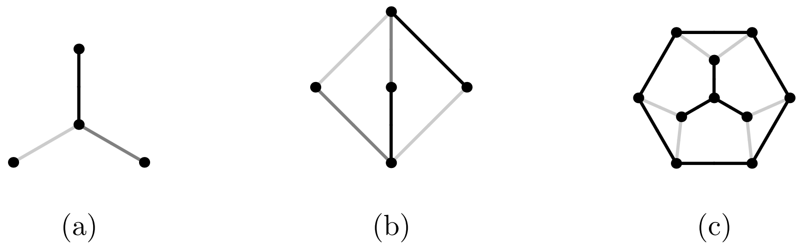


Fig. 7. Trigraphs with no orthogonal pair of Eulerian colorings.

Theorem 5.1 *A balanced cubic trigraph T has a pair of orthogonal Eulerian colorings if and only if T has a perfect matching.*

Proof. We begin with a reformulation of the problem. A *suitable* coloring of a cubic multigraph H is a coloring of its half-edges by colors from $\{B, Y, G\}$ (blue, yellow and green) with the property that

(1) the 3 edges incident with any vertex get all 3 colors,
(2) the edges e whose color on both ends differs from B form an Eulerian subgraph, and
(3) the same property holds for Y in place of B, i.e., the edges e whose color on both ends differs from Y form an Eulerian subgraph.

One way to interpret this definition is to regard the half-edges of color G as being 'blue' and 'yellow' at the same time. Property (2), for instance, then states that each vertex is incident with an even number of edges with both ends 'yellow'.

A balanced cubic trigraph T has a pair of orthogonal Eulerian colorings if and only if its underlying multigraph H has a suitable coloring. This can be seen as follows. Given the Eulerian colorings f_1 and f_2 of T, assign color B to all half-edges h of H satisfying $c(h_y) = f_1(y)$, where y is the endvertex of h. Similarly, color h with Y if $c(h_y) = f_2(y)$, and assign color G to the remaining half-edges. The result is a suitable coloring of H, and the correspondence is bijective.

We define the *type* of an edge to be the unordered pair of the colors of its ends. Assume now that H has a suitable coloring a. Consider a vertex $x \in V(H)$ and the possible color assignments on edges incident with x. We shall use the notation $a(e_x)$ just like in the case of trigraphs. Let $a_{x,G}$ denote the color of the edge e with $a(e_x) = G$ on the end opposite to x. Define $a_{x,B}$ and $a_{x,Y}$ in the analogous way. The properties (1)–(3) above restrict the possible values of $a_{x,G}$, $a_{x,B}$ and $a_{x,Y}$. In fact, it is straightforward to check that the following is the complete list of *admissible configurations*:

(a) $a_{x,G} = G$, $a_{x,Y} \neq B$ and $a_{x,B} \neq Y$,

(b) $a_{x,G} = B$, $a_{x,Y} = B$ and $a_{x,B} \neq Y$,
(c) $a_{x,G} = Y$, $a_{x,Y} \neq B$ and $a_{x,B} = Y$.

Conversely, let a coloring a of the half-edges of H in $\{B, Y, G\}$ be given. Assume that the configuration at each vertex x is admissible. Then it is easy to see that a is suitable.

Note that in each of the 3 cases, exactly one edge incident with each x is of type GG or BY. Consequently, the edges of these two types form a perfect matching in H. This proves one implication from the theorem.

We now prove the other implication. Let F be a perfect matching in H. Then $\overline{F} = H - E(F)$ is a 2-factor of H. Contract each component of $\overline{F}$ to a single point (preserving any multiple edges but discarding loops) to obtain a multigraph H'. We claim that H' has an acyclic spanning subgraph P such that for each vertex x, the degree of x in H' has the same parity as its degree in P. (Such a subgraph is usually called a *parity subgraph* of H'.) Indeed, to get P, it suffices to remove cycles (more precisely, their edge sets) from H', as long as there are any.

Let us return to the multigraph H. Each edge of P may be naturally identified with an edge of $F \subset H$; let us call these *parity edges*. Furthermore, other edges of F correspond to edges in $E(H') - E(P)$; these will be called *ordinary edges*.

We now construct an auxiliary coloring a of the half-edges of H which will later be modified to a suitable coloring. (See Figure 8.) Assign color G to both ends of each ordinary edge of H. Consider a cycle Z of $\overline{F}$. We have the following requirements on the coloring of Z:

(i) each edge of Z is of type BB or YY,
(ii) if two incident edges of Z share an endvertex of an ordinary edge, then their types differ, while if they share an endvertex of a parity edge, their types are the same.

Conditions (i) and (ii) can be ensured because by the construction of P, the cycle Z is incident with an even number of ordinary edges. Subject to these restrictions, any prescribed coloring of a half-edge on Z can be uniquely extended to Z.

The freedom in the choice of the coloring of Z enables us to impose another requirement for each parity edge e:

(iii) if one end of a parity edge is incident with edges of type YY, then the other end is incident with edges of type BB.

This can be achieved as follows. Consider a component R of P. Choose a root vertex r of the tree R and fix a coloring satisfying (i) and (ii) on the cycle of $\overline{F}$ corresponding to r (the *root cycle* for R). Traverse R from the root to the leaves; at each vertex, fix the (unique) coloring of the corresponding cycle of $\overline{F}$ satisfying (i), (ii) and (iii). Performed for all components R, this operation produces a coloring meeting all of our restrictions. Furthermore, color the half-edges of each parity edge by B or Y so as to mismatch, at each end, the color of the incident half-edges.

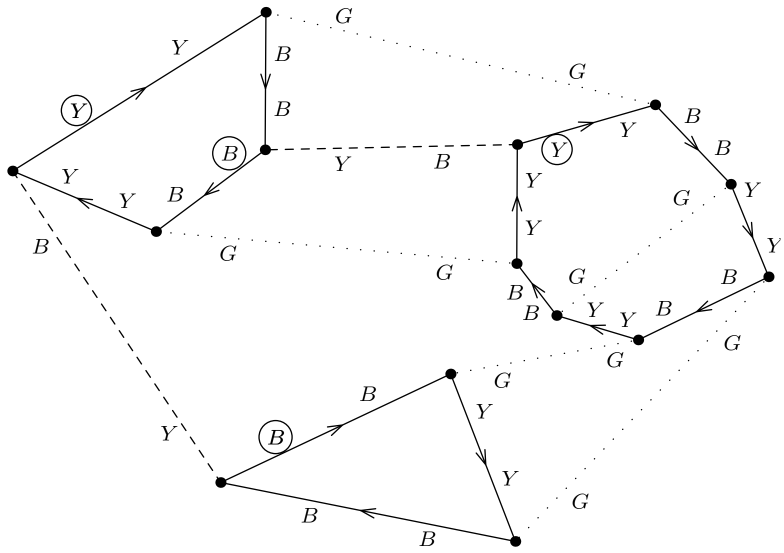


Fig. 8. An auxiliary half-edge coloring a. The solid edges form the 2-factor $\overline{F}$, the ordinary edges are dotted and the parity edges are dashed. The root cycle of $\overline{F}$ is the cycle of length 4. Changing the circled entries to G, one obtains a suitable coloring.

The resulting coloring a is certainly not suitable: at each end of a parity edge, the two incident edges are either both of type BB or both of type YY. However, this can be fixed. Orient each cycle Z of $\overline{F}$. If a half-edge h on Z is situated clockwise from the endvertex of a parity edge, then change the color of h to G. After this modification, each vertex is incident with half-edges of all 3 colors.

Let us determine the configurations of colors around a vertex x of H, to see if they are admissible (i.e., if they match the set of admissible configurations (a)–(c)). The symbols $a_{x,G}, a_{x,R}$ and $a_{x,Y}$ have the same meaning as before. The values are as in the following table:

$a_{x,G}$	$a_{x,Y}$	$a_{x,B}$	condition
G	Y/G	B/G	if x is an endvertex of an ordinary edge,
B	B	B/G	if x is incident with the 'Y' end of a parity edge,
Y	Y/G	Y	if x is incident with the 'B' end of a parity edge.

A comparison with the list of admissible configurations shows that indeed, all of these configurations are admissible. This means that we have constructed a suitable coloring. □

As shown by the trigraph in Figure 7(c), Theorem 5.1 cannot be extended to imbalanced cubic trigraphs. We conclude this paper with an open problem. Call a vertex x of a trigraph *bad* if all edges incident with x, except for one edge e, have the same color at x, and e has a different color. Each of the examples in Figure 7 contains a bad vertex. This suggests the following question. It can be shown that an affirmative answer would imply Conjecture 1.2.

Problem 5.2 *Does every 2-connected rainbow trigraph with no bad vertices admit a pair of orthogonal Eulerian colorings?*

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