Perfect matchings with restricted intersection in cubic graphs

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Abstract

A conjecture of G. Fan and A. Raspaud asserts that every bridgeless cubic graph contains three perfect matchings with empty intersection. We suggest a possible approach to problems of this type, based on the concept of a balanced join in an embedded graph. We use this method to prove that bridgeless cubic graphs of oddness two have Fano colorings using only 5 lines of the Fano plane. This is a special case of a conjecture by E. Máčajová and M. Škoviera.

1 Introduction

A number of problems involving cubic graphs concerns the existence of perfect matchings whose intersection is small or empty. This is natural as the existence of two disjoint perfect matchings in a cubic graph G is equivalent to G being 3-edge-colorable, a fundamental property in the world of cubic graphs. Although there are cubic graphs (even without bridges) that do not have this property, the following has been conjectured by Fan and the second author [2] in 1994:

Conjecture 1. *Every bridgeless cubic graph contains perfect matchings M_1, M_2, M_3 such that*

$$M_1 \cap M_2 \cap M_3 = \emptyset.$$

We remark that Conjecture 1 would be implied by the celebrated Berge–Fulkerson conjecture [4] (see also [9]).

Another related set of problems, studied, e.g., in [6, 7], concerns so-called Fano colorings. Let G be a cubic graph. A *Fano coloring* of G is any assignment

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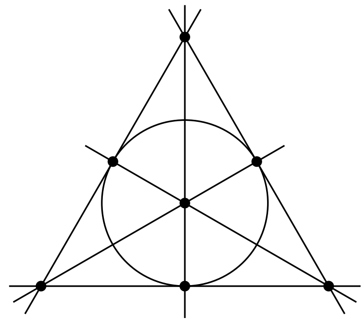

Figure 1: The Fano plane $\mathscr{F}_7$.

of points of the Fano plane $\mathscr{F}_7$ (see Figure 1) to the edges of G such that the three edges incident with any vertex of G are mapped to three distinct collinear points of $\mathscr{F}_7$. For $k \leq 7$, a *k-line Fano coloring* is a Fano coloring in which only at most k lines of $\mathscr{F}_7$ appear as color patterns at the vertices.

It is shown in [7] that every bridgeless cubic graph admits a 6-line Fano coloring. Furthermore, bridgeless cubic graphs are conjectured to admit 4-line Fano colorings, and it is observed that the conjecture is equivalent to Conjecture 1. A natural intermediate conjecture on 5-line Fano colorings is stated in [7] as Conjecture 5.1; we give it in a slightly less specific form:

Conjecture 2. *Every bridgeless cubic graph admits a 5-line Fano coloring.*

We will refer to this conjecture as the *5-line conjecture*. Before we present a (more or less well-known) equivalent formulation in the spirit of Conjecture 1, we need an important definition. A *join* in a graph G is a set $J \subset E(G)$ such that the degree of every vertex in G has the same parity as its degree in the graph $(V(G), J)$. (In the literature, the terms *postman join* or *parity subgraph* have essentially the same meaning.)

It can be shown that the following is equivalent to Conjecture 2:

Conjecture 3. *Every bridgeless cubic graph admits two perfect matchings M_1, M_2 and a join J such that*

$$M_1 \cap M_2 \cap J = \emptyset.$$

In the present paper, we suggest a possible approach to problems like Conjectures 1 and 3, based on what we call ‘balanced joins’ in embedded graphs (see Section 3). The technique was inspired by that of [7] and may be regarded as its refinement. In Section 4, we relate balanced joins in an embedded graph to independent sets in its dual.

Our main result is a special case of the 5-line conjecture. Recall that the *oddness* of a cubic graph G is the minimum number of odd circuits in a 2-factor of G. In Section 6 we prove:

Theorem 4. *Conjecture 2 (and Conjecture 3) is true for bridgeless cubic graphs of oddness two.*

A major tool in the proof of Theorem 4 is a splitting lemma (Lemma 13) proved in Section 5. In the concluding section, we give several intriguing open problems.

2 Preliminaries

Our graphs may contain loops and parallel edges. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Each edge is viewed as composed of two *half-edges* (that are *associated* to each other) and we let $E(v)$ denote the set of half-edges incident with a vertex v. If h is a half-edge, we use $h^\sharp$ to denote the edge containing h. For a set H of half-edges, we write $H^\sharp = \left\{h^\sharp : h \in H\right\}$.

The *degree* of a vertex $v \in V(G)$ is denoted by $d_G(v)$. A loop at v contributes 2 to $d_G(v)$. For $X \subset V(G)$, we let $\partial_G(X)$ be the set of edges with exactly one endvertex in X.

As usual, e.g., in the theory of nowhere-zero flows, we define a *cycle* in a graph G to be any subgraph $H \subset G$ such that each vertex of H has even degree in H. (Thus, a cycle need not be connected.)

Observation 5. *A subgraph $H \subset G$ is a cycle in G if and only if $E(G) - E(H)$ is a join.*

An *edge-cut* (or just *cut*) in G is a set $C \subset E(G)$ such that $G - C$ has more components than G, and C is inclusionwise minimal with this property. A *bridge* is a cut of size 1. A graph is *bridgeless* if it contains no bridge (note that with this definition, a bridgeless graph may be disconnected). A *k-edge-connected* graph (where $k \geq 1$) is one that is connected and contains no cut of size at most $k - 1$.

A cut C in G is *odd* if $|C|$ is odd. The *odd edge-connectivity* $\lambda^{\mathrm{odd}}(G)$ of G is the size of a smallest odd cut in G, or ∞ if no such cut exists.

For terms not defined here, we refer the reader to any standard textbook of graph theory such as [1].

3 Balanced joins in embedded graphs

We begin by recalling a combinatorial representation of graphs embedded on surfaces, developed in the works of Heffter, Edmonds and Ringel (see, e.g., [8, Chapter 3]). A *rotation system* for H is a mapping assigning to every vertex $v \in V(H)$ a cyclic ordering of the half-edges in $E(v)$ (a *rotation at* v). A rotation system for H corresponds to an embedding of H in some orientable surface. Accordingly, we will speak of a graph H with a given rotation system as an *embedded graph*. For half-edges h and h' incident with a vertex v, we may say that h immediately *precedes* or *follows* h' at v, or that h and h' are *consecutive* at v, always referring to the rotation order.

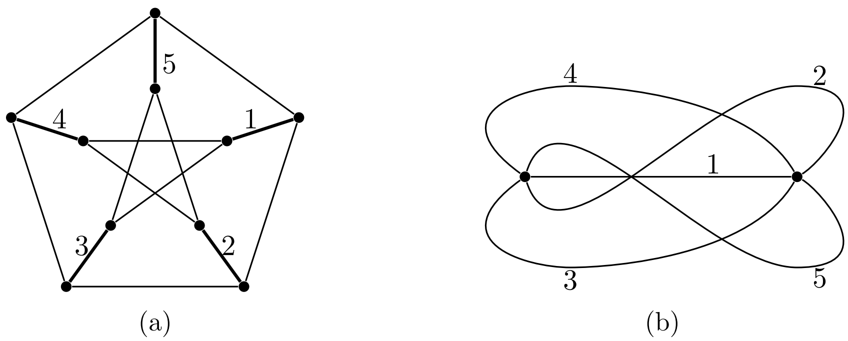

Figure 2: (a) The Petersen graph $\mathscr{P}$ with a perfect matching M (bold). (b) An M-contraction $\tilde{\mathscr{P}}$ of $\mathscr{P}$; the clockwise ordering of half-edges at vertices specifies the rotation system.

Cubic graphs and embedded graphs are related in the following way. Let G be a cubic graph and M a perfect matching in G, and let o be an orientation of the complementary 2-factor $\overline{M}$ (that is, o determines a direction for each circuit of $\overline{M}$). Contract each circuit C of $\overline{M}$ to a vertex v_C. The resulting graph $G_{M,o}$ can be viewed as an embedded graph: the rotation at v_C is given by the order in which the corresponding half-edges meet the circuit C in G. A graph of the form $G_{M,o}$ for some orientation o will be called an *M-contraction* of G. We identify the edges of $G_{M,o}$ with the corresponding edges of G.

As an example, consider the Petersen graph $\mathscr{P}$ and the perfect matching M shown in Figure 2a. The result, $\tilde{\mathscr{P}}$, of the above process (for a suitable orientation o) is shown in Figure 2b. Note that we get essentially the same embedded graph for any other choice of M and o. An embedding of $\tilde{\mathscr{P}}$ corresponding to its rotation system (in an orientable surface of genus 2, i.e., the double torus) is shown in Figure 3.

Let H be an arbitrary embedded graph. For $z \in V(H)$ and $h, h' \in E(z)$, we define $[h, h']$ as the sequence of half-edges at z starting with h, containing all the subsequent half-edges encountered as one passes from h to h' in the direction given by the rotation at v, and terminated by the occurence of h'. If $h = h'$, then $[h, h']$ is terminated by the second occurence of h.

For $J \subset E(H)$, a *J-segment (at z)* is any sequence of the form $[h, h']$, where h and h' are half-edges incident with z and $[h, h']^\sharp \cap J = \{h, h'\}^\sharp$. In addition, if $J \cap E(z)^\sharp = \emptyset$, we consider $E(z)$ as a J-segment. The *length* or *size* of a J-segment is its length as a sequence. (Note that in a J-segment of the form $[h, h]$, the half-edge h is counted twice.)

We define a *balanced join* in H to be a set $J \subset E(H)$ such that for each vertex v, every J-segment at v has even length. The choice of the term is partly justified by the following observation:

Observation 6. *Every balanced join in an embedded graph is a join.*

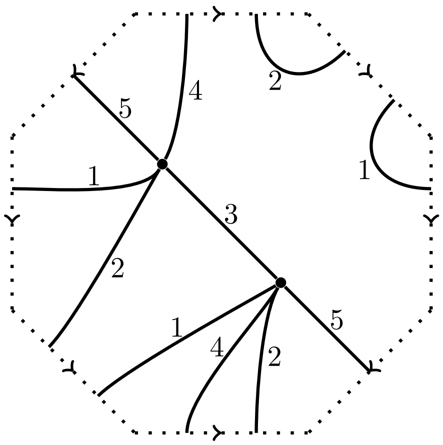

Figure 3: An embedding of the graph $\tilde{\mathscr{P}}$ of Figure 2b in the double torus determined by its rotation system. The opposite edges of the bounding octagon are identified (in such a way that the arrows match).

Proof. Let J be a balanced join in H and $v \in V(H)$. Summing the lengths of all the J-segments at v, we count each half-edge h with $h^\sharp \in J$ twice and the other half-edges once; thus, the sum equals $d_H(v) + d_J(v)$. Since the lengths of J-segments are even, so is the sum. Consequently, $d_H(v)$ and $d_J(v)$ have the same parity as claimed. □

The most important reason for us to consider balanced joins, however, is a close relation to perfect matchings:

Lemma 7. *Let $\tilde{G}$ be an M-contraction of a cubic graph G, where M is a perfect matching of G. A set $J \subset E(\tilde{G})$ is a balanced join if and only if there is a perfect matching M' in G such that*

$$M \cap M' = J.$$

Proof. Let v_C be a vertex of $\tilde{G}$ and $J \subset E(\tilde{G})$. Let $(h_1, \ldots, h_k)$ be a J-segment at v. The lemma follows from the observation that k is even if and only if one can match the vertices of G incident with $h_2, \ldots, h_{k-1}$ using edges of $G - M$. □

4 Balanced joins and the dual graph

There is a relation between balanced joins in an embedded graph H and independent sets in the dual graph H^*. Although this relation is not our major concern in the present paper, it is worth a brief investigation which we give in this section.

Recall that if H is an embedded graph, the dual graph H^* contains one edge joining faces F, F' (of the given embedding of H) for each edge of H with the face F on one side and F' on the other. In particular, if H contains an edge with the same face on both sides, then H^* contains a loop.

We call a set of faces of an H *independent* if it corresponds to an independent set of vertices in H^*. An edge will be said to be *belong* to a face F if it is contained in the boundary of F. Some balanced joins can be obtained from independent sets of faces:

Proposition 8. *Let I be an independent set of faces in an M-contraction H of a cubic graph G. The edges of H that do not belong to any face in I form a balanced join in H.*

Proof. Let J be the set of edges of H not adjacent to any face in I. We need to show that every J-segment has even length. Consider first a J-segment

$$[h_1, h_k] = (h_1, \dots, h_k) \tag{1}$$

of size k at a vertex v. We may assume that $k > 2$. Let F_i be the face of H delimited by $h_i^\sharp$ and $h_{i+1}^\sharp$ at v, where $i = 1, \dots, k-1$. Note that $h_1^\sharp \in J$, so by the definition of J, $F_1 \notin I$. Now $F_2 \in I$, because $h_2^\sharp \notin J$. By the assumption that I is independent, $F_3 \notin I$. Continuing in this way, we conclude that $F_i \in I$ if and only if i is even. Since $F_{k-1} \notin I$, k is even as claimed.

It remains to consider a segment consisting of all the half-edges incident with a vertex v. As above, we observe that the faces in I and not in I must alternate as we traverse around v, so there must be an even number of them. Consequently, $|E(v)|$ is even and the proof is complete. □

It is important to note that not all balanced joins in an embedded graph H correspond to independent sets in H^*. In our example from Figure 2, the embedding of $\tilde{\mathscr{P}}$ shown in Figure 3 has a single face. Thus, the dual graph consists of a single vertex with 5 loops, and the only independent set in $\tilde{\mathscr{P}}^*$ is $\emptyset$. This corresponds to the balanced join $E(\tilde{\mathscr{P}})$. However, $\tilde{\mathscr{P}}$ also contains 5 balanced joins of size 1.

Given Proposition 8, it is not surprising that non-intersecting balanced joins are related to face colorings:

Proposition 9. *Let $\tilde{G}$ be an M-contraction of a cubic graph G. If the faces of $\tilde{G}$ can be properly colored in k colors, then G has k perfect matchings with empty intersection.*

Proof. Consider a coloring of the faces of $\tilde{G}$ in k colors with color classes $C_1, \dots, C_k$. Each C_i is an independent set of faces. Let M'_i be the set of edges not adjacent to any face in C_i. Since C_k is independent, each edge of $\tilde{G}$ is adjacent to a face in at least one of the sets C_i with $i \le k-1$. Consequently,

$$M'_1 \cap \dots \cap M'_{k-1} = \emptyset.$$

By Proposition 8, each M'_i is a balanced join, so by Lemma 7, there is a perfect matching M_i in G such that $M_i \cap M = M'_i$. Since

$$M_1 \cap \dots \cap M_{k-1} \cap M = M'_1 \cap \dots \cap M'_{k-1},$$

we see that the k perfect matchings $M_1, \ldots, M_{k-1}, M$ have empty intersection. □

For $k = 3$, one can actually get a stronger result:

Theorem 10. *Let G be a cubic graph with a perfect matching M. If the faces of some M-contraction of G can be properly colored with 3 colors, then G is 3-edge-colorable.*

The proof is similar to that of Proposition 9 and is omitted. Alternatively, at least under the assumption that every face of an M-contraction $\tilde{G}$ is bounded by a circuit, Theorem 10 follows from standard facts on nowhere-zero flows and cycle covers (see, e.g., [9]).

As a curiosity, let us mention the following (probably known) consequence of Theorem 10 and the Grötzsch theorem [5]:

Proposition 11. *Let G be a cyclically 4-edge-connected cubic plane graph with a perfect matching M such that each circuit of $\overline{M}$ bounds a face. Then G is 3-edge-colorable.*

Proof. Consider an M-contraction $\tilde{G}$ of G. Since $\tilde{G}$ is 4-edge-connected, its dual $\tilde{G}^*$ is triangle-free and hence 3-colorable by the Grötzsch theorem. By Theorem 10, G is 3-edge-colorable. □

5 A splitting lemma

A basic tool we use in the proof of Theorem 4 is splitting. Let G be a graph and h a half-edge incident with a vertex $v \in V(G)$. For convenience, we define the *target* of h to be the vertex incident with the half-edge associated to h. Let h' be another half-edge incident with v. Furthermore, let w and w' be the targets of h and h', respectively.

Splitting off h and h' is the operation of removing the edges containing h and h', and adding an edge between w and w' (a loop if $w = w'$). The resulting graph is denoted by $G(v; h, h')$.

The following statement is an equivalent form of the well-known splitting lemma of Fleischner [3] (see also [9, Theorem A.5.2]):

Lemma 12. *Let v be a vertex of a 2-edge-connected graph G with $d_G(v) \geq 4$. Let $h_0, h_1, h_2 \in E(v)$. If both $G(v; h_0, h_1)$ and $G(v; h_0, h_2)$ fail to be 2-edge-connected, then $\{h_0, h_1, h_2\}^\sharp$ is a 3-cut in G.*

Our purpose requires a slightly different kind of a splitting lemma which we derive next. Let $\mathscr{G}_2$ be the class of all bridgeless graphs with exactly two vertices of odd degree.

Lemma 13. *Let G be a graph in $\mathscr{G}_2$ and let $h_0, \dots, h_4$ be distinct half-edges incident with a vertex z of even degree. Let w_i ($i = 0, \dots, 4$) denote the target of h_i and assume that w_0 has odd degree. If both $G(z; h_1, h_2)$ and $G(z; h_3, h_4)$ contain bridges, then $\{h_0, h_1, h_2\}^\sharp$ or $\{h_0, h_3, h_4\}^\sharp$ is a 3-cut in G.*

Proof. For $i = 0, \dots, 4$, let $e_i = h_i^\sharp$. Let y_1 and y_2 denote the two odd degree vertices in G, where $y_1 = w_0$. Choose a bridge b' in the graph $G' := G(z; h_1, h_2)$ and a bridge b'' in $G'' := G(z; h_3, h_4)$. Since $G \in \mathscr{G}_2$, each component of $G' - b'$ and $G'' - b''$ contains exactly one of y_1 and y_2.

The bridge b' is distinct from the edge created by splitting off h_1 and h_2 (otherwise G would contain a bridge). Thus, b' (and similarly b'') is an edge of G.

Claim 1. *The sets*

$$C' = \{e_1, e_2, b'\},$$
$$C'' = \{e_3, e_4, b''\}$$

are 3-cuts in G.

Consider C' first. Clearly, $G - C'$ is disconnected, so all we need to prove is that C' is inclusionwise minimal with this property. For the sake of a contradiction, suppose that $C \subset C'$ is a 2-cut and let $C' - C = \{e\}$. Then e is contained in a component K of $G - C$. We distinguish two cases: $e = b'$ and $e \neq b'$.

Assume that $e = b'$. Since b' is not a bridge in G, it is contained in a cycle. This cycle contains either both or none of the edges e_1, e_2, and hence corresponds to a cycle in G' containing b', which is a contradiction since b' is a bridge in G'.

Thus, we may assume that $e = e_1$. Let x be the endvertex of b' in K. We show that there exists a path P from x to w_1 in K not using the edge e_1. Indeed, let P_1 and P_2 be edge-disjoint paths from x to w_1 in G. If one of them is contained in K and avoids e_1, we are done. Thus, we may assume that P_1 ends with e_1 and P_2 contains b' and e_2. However, combining the part of P_1 from x to z with the part of P_2 from z to w_1, we obtain a trail from which we can select the desired path P.

Since P extends to a cycle containing $\{e_1, e_2, b'\}$, b' is contained in a cycle in G', a contradiction which proves that C' is indeed a 3-cut. A similar argument shows that C'' is a 3-cut and completes the proof of the claim.

Claim 2. *The edge e_0 is either b' or b''.*

For $i \in \{1, 2\}$, we write A'_i (A''_i) for the vertex set of the component of $G' - b'$ ($G'' - b''$, respectively) that contains y_i, and we set

$$A_{ij} = A'_i \cap A''_j \quad \text{and} \quad a_{ij} = |\partial(A_{ij})|,$$

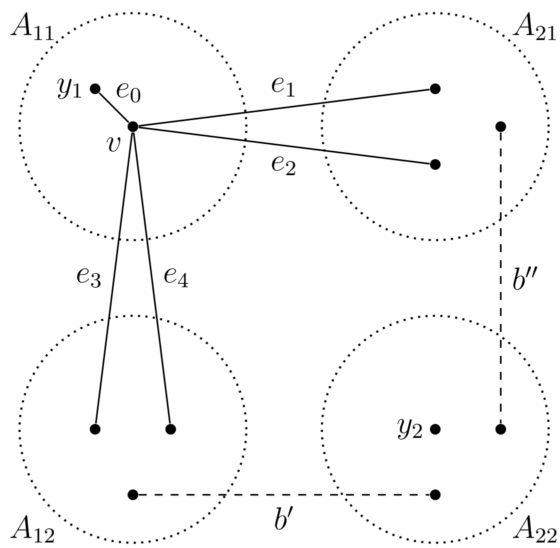

Figure 4: An illustration of the notation in the proof of Lemma 13.

where $i, j \in \{1, 2\}$ and ∂_G is abbreviated to ∂. (Figure 4 illustrates the situation.) Thus, $y_1 \in A_{11}$ and $y_2 \in A_{22}$. Hence each of $\partial(A_{11})$ and $\partial(A_{22})$ has odd cardinality, and since G is bridgeless, the cardinality is at least 3. It follows that

$$a_{11} \geq 3 \text{ and } a_{22} \geq 3. \tag{2}$$

On the other hand, if we let d denote the number of edges with one end in A_{12} and the other in A_{21}, then

$$|C'| + |C''| = a_{11} + a_{22} + 2d. \tag{3}$$

Since $|C'| = |C''| = 3$ and $a_{11}, a_{22} \geq 3$ by (2), we conclude that $a_{11} = a_{22} = 3$ and $d = 0$.

This implies that $z \notin A_{11}$, for otherwise we would have $\{e_1, e_2, e_3, e_4\} \subset \partial(A_{11})$, contradicting the fact that $a_{11} = 3$. A similar argument shows that $z \notin A_{22}$. By symmetry, we may therefore assume that $z \in A_{21}$, so e_0 has its ends in A_{11} and A_{21}. But then $e_0 \in C'$ and hence $e_0 = b'$. This finishes the proof of the claim.

The lemma follows directly from Claims 1 and 2. $\square$

6 The 5-line conjecture for graphs of oddness two

Let us begin with a notion that can be used to reformulate Conjecture 3. We will say that a set $X \subset E(G)$ is *sparse* if X contains no odd cut.

Observation 14. *A set $X \subset E(G)$ is sparse if and only if there exists a join J with $X \cap J = \emptyset$.*

In this section, we prove that every embedded graph from the class $\mathscr{G}_2$ admits a sparse balanced join. At the end of the section, we derive Theorem 4 as a corollary.

Let G be an embedded graph, $z \in V(G)$ and $h \in E(z)$. An unordered pair $\{f, f'\}$ of half-edges is *near h at z* if

(1) $h \notin \{f, f'\}$,

(2) f and f' are consecutive at z and one of them immediately follows or precedes h at z.

Lemma 15. *Let h be a half-edge of an embedded graph G with endvertex z. Let G' be a bridgeless (embedded) graph obtained by splitting off a pair of half-edges near h at z. If G' has a sparse balanced join containing $h^\sharp$, then so does G.*

Proof. Assume we split off the half-edges f_1 and f_2 immediately following h at z (in this order) to obtain G'. Let J' be a sparse balanced join in G' with $h^\sharp \in J'$, and let J be the corresponding join in G. Note that $h^\sharp \in J$. It is easy to compare the sizes of the J'-segments at z in G' and the J-segments in G. They are the same, except that:

- if $f_1^\sharp, f_2^\sharp \notin J$, then the J-segment at z in G containing f_1 and f_2 is longer by 2 than the corresponding J'-segment in G',
- otherwise, there are two extra J-segments at z in G whose size is 2 (namely (h, f_1) and (f_1, f_2)).

In either case, the lengths of all the J-segments are even, which means that J is balanced.

Assume that J contains an odd cut C. Necessarily, $\{f_1, f_2\}^\sharp \subset J$. Let $C' = C - \{f_1, f_2\}^\sharp$. Since C' is an odd cut in G', we have $C' \not\subset J'$ and hence $C \not\subset J$, a contradiction. It follows that J is sparse. The proof is finished. $\square$

Theorem 16. *Let $G \in \mathscr{G}_2$ be a (bridgeless) embedded graph and $e \in E(G)$ a non-loop edge incident with an odd degree vertex. Then G admits a sparse balanced join J such that $e \in E(J)$.*

Proof. Let G be a counterexample with as few edges as possible. Since G is bridgeless and $G \in \mathscr{G}_2$, it has at least 3 edges. If G has exactly 3 edges, it consists of two vertices joined by three parallel edges, and the sparse balanced join $\{e\}$ provides a contradiction.

We may thus assume that $|E(G)| > 3$ and that the assertion holds for graphs with fewer edges. We may also assume that the minimum degree in G is at least 3 and that e has only one endvertex (u, say) of odd degree. Let z be the other endvertex of e, h_0 the half-edge of e incident with z, and let v denote the remaining vertex of odd degree in G.

By Lemma 15 and the minimality of G, it is not possible to split off any pair of half-edges near h_0 at z so as to obtain a bridgeless graph. Enumerate the half-edges incident with z, starting with h_0 and proceeding in the rotation order, as $h_0, \dots, h_{k-1}$. By the above, the degree $k = d_G(z)$ of z is even and not equal to 2.

Assume now that $k = 4$. By Lemma 12, either $G(z; h_1, h_2)$ or $G(z; h_2, h_3)$ is a bridgeless graph. Since both $\{h_1, h_2\}$ and $\{h_2, h_3\}$ are near e at z, we get a contradiction. Thus, $k \geq 6$.

Lemma 13 implies that at least one of the sets $\{h_0, h_1, h_2\}^\sharp$ or $\{h_0, h_{k-1}, h_{k-2}\}^\sharp$ is a 3-cut. By symmetry, we may assume that $\{h_0, h_1, h_2\}^\sharp$ is a 3-cut. Since $d_G(z) \geq 6$, z is a cutvertex. Moreover, there are exactly three edges joining z to the component X of $G - z$ containing u. Since $|\partial_G(V(X))|$ is odd, v (the other vertex of odd degree in G) is not contained in X.

Set $G' = G - V(X)$. Note that the degree of z in G' is odd (namely, $k - 3$) and that $G' \in \mathscr{G}_2$. By the minimality of G, there is a balanced join J' in G' such that $h_3^\sharp \in J'$ and J' contains no odd cut in G'. Clearly,

$$J = J' \cup \{e\}$$

is a balanced join in G. Furthermore, since every odd cut in G is contained either in $G - V(X)$ or in $G[V(X) \cup \{z\}]$, J is sparse. Thus, J has the properties stated in the theorem, a contradiction. $\square$

We can now derive Theorem 4 as a corollary of Theorem 16.

Proof of Theorem 4. Let G be a cubic bridgeless graph of oddness two. Let M_1 be a perfect matching such that the 2-factor $\overline{M_1}$ has exactly two odd components. Choose an orientation o of $\overline{M_1}$ and consider the M_1-contraction $H = G_{M_1,o}$. Note that $H \in \mathscr{G}_2$. By Theorem 16, H admits a sparse balanced join J_2.

Observation 7 implies that J_2 can be extended to a perfect matching M_2 in G such that $M_1 \cap M_2 = J_2$. Since J_2 is sparse in H, $M_1 \cap M_2$ is sparse in G. By Observation 14, there is a join J in G such that $M_1 \cap M_2 \cap J = \emptyset$. This completes the proof. $\square$

7 Concluding remarks

There are a number of natural questions about balanced joins. The first of them concerns a possible extension of Theorem 16 to embedded graphs with more than two odd degree vertices. The statement of the theorem does not hold for all bridgeless graphs: for instance, it is easy to see that the Petersen graph $\mathscr{P}$, together with an arbitrary rotation system, has no sparse join. However, the following may be true:

Conjecture 17. *Let H be an embedded graph with $\lambda^{odd}(H) \geq 5$. Then H contains a sparse balanced join.*

A natural approach to the Fan–Raspaud conjecture (Conjecture 1) along the lines of our proof of Theorem 4 would require finding two disjoint balanced joins in an embedded graph. Again, the Petersen graph $\mathscr{P}$ does not even have two disjoint joins, but we find it plausible that higher odd edge-connectivity helps.

Problem 18. *Is there an integer k such that every embedded graph H of odd edge-connectivity $\lambda^{odd}(H) \geq k$ contains two disjoint balanced joins?*

Observe that if the balanced joins in Problem 18 exist, they are necessarily sparse.

We may go one step further and ask about partitions into balanced joins. For instance, if G is a 3-edge-colorable cubic graph, then (by Lemma 7) the edges of any M-contraction of G can be partitioned into three balanced joins.

On the other hand, consider 'the' M-contraction $\tilde{\mathscr{P}}$ of the Petersen graph in Figure 2b. Since $E(\tilde{\mathscr{P}})$ and any set consisting of a single edge are balanced joins, the edge set of $\tilde{\mathscr{P}}$ can be partitioned into 1 or 5 balanced joins. However, there is no partition of $E(\tilde{\mathscr{P}})$ into 3 balanced joins.

Problem 19. *Is there a function f such that for every odd ℓ, the edge set of each embedded graph H with $\lambda^{odd}(H) \geq f(\ell)$ can be partitioned into ℓ balanced joins?*

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