Unions of perfect matchings in cubic graphs (Extended abstract)

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1 Introduction

A well-known conjecture of Fulkerson [3] states that every bridgeless cubic graph contains a family of six perfect matchings covering each edge precisely twice. The following weaker version of the conjecture is also open:

Conjecture 1. The edges of any bridgeless cubic graph can be covered by 5 perfect matchings.

We remark that even if the number 5 is replaced by any larger constant (independent of the graph $G$), the statement is apparently not known to be true.

In this paper, we investigate the maximum possible size of the union of a given number of perfect matchings in a cubic bridgeless graph. More precisely, we study, for $k \in \{2, 3\}$, the numbers

$$m(k) = \sup_{G} \max_{M_1, \ldots, M_k} \frac{\left| \bigcup_i M_i \right|}{|E(G)|},$$

where the supremum is taken over all bridgeless cubic graphs $G$, and $M_1, \ldots, M_k$ range over all perfect matchings of $G$.

We obtain the precise value of $m(2)$ and a reasonably tight estimate for $m(3)$.

Let us begin by considering the upper bounds. The Petersen graph $P_{10}$ has 15 edges and 6 distinct perfect matchings. It is well-known that any 2 perfect matchings of $P_{10}$ have precisely 3 edges in common, and that the intersection of
any 3 perfect matchings is of size 1. Simple counting then shows that $m(2) \leq 3/5$ and $m(3) \leq 4/5$.

Our result can be summarized as follows:

**Theorem 2** The value of $m(2)$ is $3/5$, and

$$0.771 \approx \frac{27}{35} \leq m(3) \leq \frac{4}{5}.$$  

Our main tool is the Perfect matching polytope theorem of Edmonds [2], reviewed in Section 2.

We use standard terminology and notation of graph theory, as found, e.g. in [1]. Supplementary information on the Fulkerson conjecture can be found in [5]. Schrijver [4] includes an extensive account of matching polytopes.

### 2 The perfect matching polytope

Let $G = (V, E)$ be a graph which may contain multiple edges. A *cut* in $G$ is any set $C \subseteq E$ such that $G \setminus C$ has more components than $G$ does, and $C$ is minimal with this property with respect to inclusion. A $k$-cut (where $k$ is an integer) is a cut of cardinality $k$. For a set $X \subseteq V$, we set $\partial X$ to be the set of edges with precisely one end in $X$. Note that $\partial \emptyset = \partial V = \emptyset$.

Let $w$ be a vector in $\mathbb{R}^E$. The entry of $w$ corresponding to $e \in E$ is denoted by $w(e)$, and for $A \subseteq E$, we define the *weight* $w(A)$ of $A$ as $\sum_{e \in A} w(e)$. The vector $w$ is a *fractional perfect matching* in $G$ if it has the following properties:

(i) for each $e \in E$, $0 \leq w(e) \leq 1$,

(ii) for each vertex $v \in V$, $w(\partial \{v\}) = 1$, and

(iii) for each $X \subseteq V$ of odd cardinality, $w(\partial X) \geq 1$.

Let $P(G)$ be the set of all fractional perfect matchings in $G$. For any set $F \subseteq E$, the characteristic vector $\chi^F$ of $F$ is defined by

$$\chi^F(e) = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

If $M$ is a perfect matching, then $\chi^M$ is clearly contained in $P(G)$. Furthermore, if $w_1, \ldots, w_n \in P(M)$, then any convex combination

$$\sum_{i=1}^n \alpha_i w_i$$

(where $\alpha_1, \ldots, \alpha_n \geq 0$ are positive reals summing up to 1) is also in $P(G)$. It follows that $P(G)$ contains the convex hull of the vectors $\chi^M$ with $M$ ranging over the perfect matchings of $G$. The Perfect matching polytope theorem asserts the converse:
Theorem 3 (Edmonds [2]) For any graph $G$, the set $P(G)$ coincides with the convex hull of the characteristic vectors of perfect matchings of $G$.

Naturally, $P(G)$ is called the perfect matching polytope of $G$.

Lemma 4 If $w$ is a fractional perfect matching in a graph $G = (V, E)$ and $c \in \mathbb{R}^E$, then $G$ has a perfect matching $M$ such that

$$c \cdot \chi^M \geq c \cdot w,$$

where · denotes the scalar product. Moreover, for any cut $C$ such that $w(C) = 1$, it holds that $|M \cap C| = 1$. □

3 Proof of Theorem 2

In this section, we prove Theorem 2. In view of the upper bounds presented in Section 1, it suffices to show that $m(2) \geq 3/5$ and $m(3) \geq 4/5$. Throughout the section, $G = (V, E)$ is an arbitrary cubic bridgeless graph.

Observation 5 If $X \subseteq V$ has odd size, then $|\partial X|$ is odd and larger than 1. □

Define $w_1 \in \mathbb{R}^E$ to have the value $1/3$ on all edges $e \in E$. Clearly, $w_1$ fulfills the conditions (i) and (ii) from the definition of a fractional perfect matching. Condition (iii) is also satisfied since by Observation 5, the sets $\partial X$ with $|X|$ odd have size at least 3. Thus, $w_1$ is a fractional perfect matching. Moreover, the weight $w_1(C)$ of each 3-cut $C$ equals 1, so by Lemma 4, there is a perfect matching $M_1$ intersecting each 3-cut in a single edge.

We now use $M_1$ to define a vector $w_2 \in \mathbb{R}^E$:

$$w_2(e) = \begin{cases} 
  1/5 & \text{if } e \in M_1, \\
  2/5 & \text{otherwise.}
\end{cases}$$

To show that $w_2$ is a fractional perfect matching, we need to verify condition (iii) of the definition. Let $X \subseteq V$ be a set of vertices of odd size. If $|\partial X| \geq 5$, then $w_2(\partial X)$ is clearly at least 1. Thus, we may assume that $|\partial X| = 3$. We can only have $w_2(\partial X) < 1$ if $M_1 \subseteq \partial X$, but this is impossible since $M_1$ intersects each 3-cut in one edge only. Hence, $w_2$ is indeed a fractional perfect matching.

For each $e \in E$, set $c_2(e) = 1 - \chi^{M_2}(e)$. By Lemma 4, there exists a perfect matching $M_2$ such that

$$c_2 \cdot \chi^{M_2} \geq c_2 \cdot w_2 = \frac{2}{5} \cdot \frac{2}{3} |E| = \frac{4}{15} |E|.$$ 

Since $c_2 \cdot \chi^{M_2}$ is just $|M_2 \setminus M_1|$, it follows that

$$|M_1 \cup M_2| = \left(\frac{1}{3} + \frac{4}{15}\right) \cdot |E| = \frac{3}{5} |E|.$$
We conclude that \( m(2) = 3/5 \).

It remains to establish a lower bound on \( m(3) \). To begin with, observe that by Lemma 4, if \( M_1 \) contains a 5-cut \( C \), then \( |C \cap M_2| = 1 \). We define \( w_3 \in \mathbb{R}^E \) by

\[
w_3(e) = \begin{cases} 
1/7 & \text{if } e \in M_1 \cap M_2, \\
2/7 & \text{if } e \in (M_1 \cup M_2) \setminus (M_1 \cap M_2), \\
3/7 & \text{otherwise}.
\end{cases}
\]

As before, it can be verified that \( w_3 \) is a fractional perfect matching. For the vector \( c_3 \in \mathbb{R}^E \) defined by \( c_3(e) = 1 - \chi_{M_1 \cup M_2}(e) \), Lemma 4 implies that a perfect matching \( M_3 \) of \( G \) satisfies

\[
|M_3 \setminus (M_1 \cup M_2)| = c_3 \cdot \chi_{M_3} \geq c_3 \cdot w_3 = \frac{3}{7} \cdot |E \setminus (M_1 \cup M_2)|.
\]

Consequently,

\[
|M_1 \cup M_2 \cup M_3| = |M_1 \cup M_2| + |M_3 \setminus (M_1 \cup M_2)| \\
\geq \frac{3}{5} |E| + \frac{3}{7} \cdot \frac{2}{5} |E| = \frac{27}{35} |E|,
\]

so \( m(3) \geq 27/35 \) as claimed. \( \Box \)

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References


