Unions of perfect matchings in cubic graphs (Extended abstract)

Tomáš Kaiser[1]
Daniel Král'[2]
Sergei Norine[3]

1 Introduction

A well-known conjecture of Fulkerson [3] states that every bridgeless cubic graph contains a family of six perfect matchings covering each edge precisely twice. The following weaker version of the conjecture is also open:

Conjecture 1 *The edges of any bridgeless cubic graph can be covered by* 5 *perfect matchings.*

We remark that even if the number 5 is replaced by any larger constant (independent of the graph G), the statement is apparently not known to be true. In this paper, we investigate the maximum possible size of the union of a given number of perfect matchings in a cubic bridgeless graph. More precisely, we study, for $k \in \{2, 3\}$, the numbers

$$m(k) = \sup_G \max_{M_1,\ldots,M_k} \frac{|\bigcup_i M_i|}{|E(G)|},$$

where the supremum is taken over all bridgeless cubic graphs G, and $M_1, \ldots, M_k$ range over all perfect matchings of G.

We obtain the precise value of $m(2)$ and a reasonably tight estimate for $m(3)$. Let us begin by considering the upper bounds. The Petersen graph P_{10} has 15 edges and 6 distinct perfect matchings. It is well-known that any 2 perfect matchings of P_{10} have precisely 3 edges in common, and that the intersection of

[1]Department of Mathematics and Institute for Theoretical Computer Science, University of West Bohemia, Univerzitní 8, 306 14 Plzeň, Czech Republic. E-mail: kaisert@kma.zcu.cz.
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[2]TBD
[3]TBD

any 3 perfect matchings is of size 1. Simple counting then shows that $m(2) \leq 3/5$ and $m(3) \leq 4/5$.

Our result can be summarized as follows:

Theorem 2 *The value of* $m(2)$ *is* $3/5$, *and*

$$0.771 \approx \frac{27}{35} \leq m(3) \leq \frac{4}{5}.$$

Our main tool is the Perfect matching polytope theorem of Edmonds [2], reviewed in Section 2.

We use standard terminology and notation of graph theory, as found, e.g. in [1]. Supplementary information on the Fulkerson conjecture can be found in [5]. Schrijver [4] includes an extensive account of matching polytopes.

2 The perfect matching polytope

Let $G = (V, E)$ be a graph which may contain multiple edges. A *cut* in G is any set $C \subseteq E$ such that $G \setminus C$ has more components than G does, and C is minimal with this property with respect to inclusion. A k-cut (where k is an integer) is a cut of cardinality k. For a set $X \subseteq V$, we set ∂X to be the set of edges with precisely one end in X. Note that $\partial\emptyset = \partial V = \emptyset$.

Let w be a vector in $\mathbb{R}^E$. The entry of w corresponding to $e \in E$ is denoted by $w(e)$, and for $A \subseteq E$, we define the *weight* $w(A)$ of A as $\sum_{e \in A} w(e)$. The vector w is a *fractional perfect matching* in G if it has the following properties:

(i) for each $e \in E$, $0 \leq w(e) \leq 1$,

(ii) for each vertex $v \in V$, $w(\partial \{v\}) = 1$, and

(iii) for each $X \subseteq V$ of odd cardinality, $w(\partial X) \geq 1$.

Let $P(G)$ be the set of all fractional perfect matchings in G. For any set $F \subseteq E$, the characteristic vector χ^F of F is defined by

$$\chi^F(e) = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{otherwise.} \end{cases}$$

If M is a perfect matching, then χ^M is clearly contained in $P(G)$. Furthermore, if $w_1, \dots, w_n \in P(M)$, then any convex combination

$$\sum_{i=1}^{n} \alpha_i w_i$$

(where $\alpha_1, \dots, \alpha_n \geq 0$ are positive reals summing up to 1) is also in $P(G)$. It follows that $P(G)$ contains the convex hull of the vectors χ^M with M ranging over the perfect matchings of G. The Perfect matching polytope theorem asserts the converse:

Theorem 3 (Edmonds [2]) *For any graph G, the set $P(G)$ coincides with the convex hull of the characteristic vectors of perfect matchings of G.*

Naturally, $P(G)$ is called the *perfect matching polytope* of G.

Lemma 4 *If w is a fractional perfect matching in a graph $G = (V, E)$ and $c \in \mathbb{R}^E$, then G has a perfect matching M such that*

$$c \cdot \chi^M \geq c \cdot w,$$

where $\cdot$ denotes the scalar product. Moreover, for any cut C such that $w(C) = 1$, it holds that $|M \cap C| = 1$. □

3 Proof of Theorem 2

In this section, we prove Theorem 2. In view of the upper bounds presented in Section 1, it suffices to show that $m(2) \geq 3/5$ and $m(3) \geq 4/5$. Throughout the section, $G = (V, E)$ is an arbitrary cubic bridgeless graph.

Observation 5 *If $X \subseteq V$ has odd size, then $|\partial X|$ is odd and larger than 1.* □

Define $w_1 \in \mathbb{R}^E$ to have the value $1/3$ on all edges $e \in E$. Clearly, w_1 fulfills the conditions (i) and (ii) from the definition of a fractional perfect matching. Condition (iii) is also satisfied since by Observation 5, the sets ∂X with $|X|$ odd have size at least 3. Thus, w_1 is a fractional perfect matching. Moreover, the weight $w_1(C)$ of each 3-cut C equals 1, so by Lemma 4, there is a perfect matching M_1 intersecting each 3-cut in a single edge.

We now use M_1 to define a vector $w_2 \in \mathbb{R}^E$:

$$w_2(e) = \begin{cases} 1/5 & \text{if } e \in M_1, \\ 2/5 & \text{otherwise.} \end{cases}$$

To show that w_2 is a fractional perfect matching, we need to verify condition (iii) of the definition. Let $X \subseteq V$ be a set of vertices of odd size. If $|\partial X| \geq 5$, then $w_2(\partial X)$ is clearly at least 1. Thus, we may assume that $|\partial X| = 3$. We can only have $w_2(\partial X) < 1$ if $M_1 \subseteq \partial X$, but this is impossible since M_1 intersects each 3-cut in one edge only. Hence, w_2 is indeed a fractional perfect matching.

For each $e \in E$, set $c_2(e) = 1 - \chi^{M_1}(e)$. By Lemma 4, there exists a perfect matching M_2 such that

$$c_2 \cdot \chi^{M_2} \geq c_2 \cdot w_2 = \frac{2}{5} \cdot \frac{2}{3} |E| = \frac{4}{15} |E| .$$

Since $c_2 \cdot \chi^{M_2}$ is just $|M_2 \setminus M_1|$, it follows that

$$|M_1 \cup M_2| = (\frac{1}{3} + \frac{4}{15}) \cdot |E| = \frac{3}{5} |E| .$$

We conclude that $m(2) = 3/5$.

It remains to establish a lower bound on $m(3)$. To begin with, observe that by Lemma 4, if M_1 contains a 5-cut C, then $|C \cap M_2| = 1$. We define $w_3 \in \mathbb{R}^E$ by

$$w_3(e) = \begin{cases} 1/7 & \text{if } e \in M_1 \cap M_2, \\ 2/7 & \text{if } e \in (M_1 \cup M_2) \setminus (M_1 \cap M_2), \\ 3/7 & \text{otherwise.} \end{cases}$$

As before, it can be verified that w_3 is a fractional perfect matching. For the vector $c_3 \in \mathbb{R}^E$ defined by $c_3(e) = 1 - \chi^{M_1 \cup M_2}(e)$, Lemma 4 implies that a perfect matching M_3 of G satisfies

$$|M_3 \setminus (M_1 \cup M_2)| = c_3 \cdot \chi^{M_3} \geq c_3 \cdot w_3 = \frac{3}{7} \cdot |E \setminus (M_1 \cup M_2)| \, .$$

Consequently,

$$\begin{aligned} |M_1 \cup M_2 \cup M_3| &= |M_1 \cup M_2| + |M_3 \setminus (M_1 \cup M_2)| \\ &\geq \frac{3}{5} |E| + \frac{3}{7} \cdot \frac{2}{5} |E| = \frac{27}{35} |E| \, , \end{aligned}$$

so $m(3) \geq 27/35$ as claimed. □

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