Unions of perfect matchings in cubic graphs

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Abstract

We show that any cubic bridgeless graph with m edges contains two perfect matchings that cover at least $3m/5$ edges, and three perfect matchings that cover at least $27m/35$ edges.

1 Introduction

A well-known conjecture of Berge and Fulkerson states that every bridgeless cubic graph contains a family of six perfect matchings covering each edge exactly twice:

Conjecture 1. *Every cubic bridgeless graph G contains six perfect matching $M_1, \ldots, M_6$ such that each edge of G is contained in precisely two of the matchings.*

Conjecture 1 is attributed to Berge in [4], but it was first published in [3]. Closely related is the Cycle Double Conjecture (in particular, Conjecture 1 implies that every graph contains a family of cycles containing each edge four times). Note also that Conjecture 1 trivially holds for cubic graphs G that are 3-edge-colorable.

The following weaker version of Conjecture 1 due to Berge is also open:

Conjecture 2. *Every cubic bridgeless graph G contains five perfect matchings $M_1, \ldots, M_5$ such that each edge of G is contained in at least one of the matchings.*

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We remark that even if the number 5 in Conjecture 2 is replaced by any larger constant (independent of G), the statement is not known to be true. In this paper, we investigate the maximum possible size of the union of a given number of perfect matchings in a cubic bridgeless graph. More precisely, we study, for $k \in \{2,3\}$, the numbers

$$m_k = \inf_G \max_{M_1,\dots,M_k} \frac{|\bigcup_i M_i|}{|E(G)|},$$

where the infimum is taken over all bridgeless cubic graphs G, and $M_1, \dots, M_k$ range over all perfect matchings of G. Note that Conjecture 2 asserts that $m_5 = 1$.

We determine the precise value of m_2 and provide a non-trivial lower bound on m_3. Let us begin by considering the upper bounds. The Petersen graph P_{10} has 15 edges and 6 distinct perfect matchings. It can be checked that up to automorphism, the mutual position of any two (three) distinct perfect matchings in P_{10} is as shown in Figure 1a (Figure 1b, respectively). It follows that any two distinct perfect matchings in P_{10} have precisely one edge in common and the intersection of any three distinct perfect matchings is empty. Simple counting then shows that $m_2 \leq 3/5$ and $m_3 \leq 4/5$.

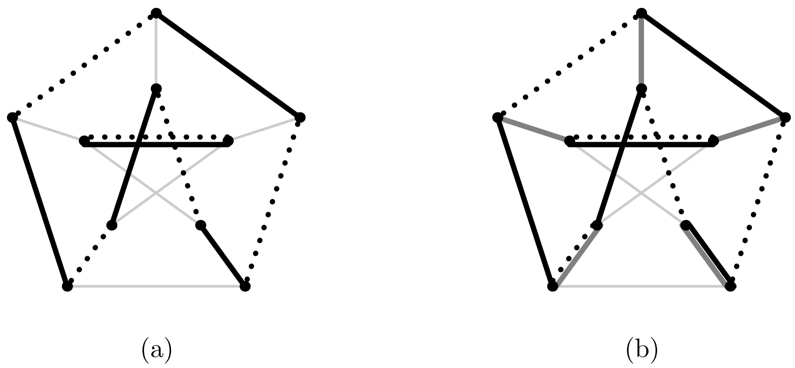

Figure 1: (a) Two perfect matchings in the Petersen graph P_{10} (solid and dotted edges). (b) Three perfect matchings in P_{10} (solid, dotted and thick grey edges).

Our contribution can be summarized as follows:

Theorem 1. *The value of* m_2 *is* 3/5, *and* $0.771 \approx 27/35 \leq m_3 \leq 4/5$.

Our main tool is the Perfect Matching Polytope Theorem of Edmonds [2] which we review in Section 2. Throughout the text, we use standard terminology and notation of graph theory as it can be found, e.g., in [1]. Supplementary information on Conjecture 1 can be found in [6], and a more detailed introduction to the theory of matching polytopes can be found in the recent monograph by Schrijver [5].

2 The perfect matching polytope

Let $G = (V, E)$ be a graph which may contain multiple edges. A *cut* in G is any set $C \subseteq E$ such that $G \setminus C$ has more components than G does, and C is inclusion-wise minimal with this property. A k-cut (where k is an integer) is a cut comprised of k edges. For a subset $X \subseteq V$ of the vertices of G, we set ∂X to be the set of edges with precisely one end in X. Note that $\partial\emptyset = \partial V = \emptyset$. Finally, a cut C is said to be *odd* if there exists a subset $X \subseteq V$ of odd cardinality such that $C = \partial X$.

Let w be a vector in $\mathbb{R}^E$. The entry of w corresponding to $e \in E$ is denoted by $w(e)$, and for $A \subseteq E$, we define the *weight* $w(A)$ of A as $\sum_{e\in A} w(e)$. The vector w is said to be a *fractional perfect matching* of G if it satisfies the following:

(i) $0 \leq w(e) \leq 1$ for each $e \in E$,

(ii) $w(\partial\{v\}) = 1$ for each vertex $v \in V$, and

(iii) $w(\partial X) \geq 1$ for each $X \subseteq V$ of odd cardinality.

$P(G)$ denotes the set of all fractional perfect matchings of G.

If M is a perfect matching, then the characteristic vector $\chi^M \in \mathbb{R}^E$ of M is contained in $P(G)$. Furthermore, if $w_1, \ldots, w_n \in P(G)$, then any convex combination $\sum_{i=1}^n \alpha_i w_i$ of $w_1, \ldots, w_n$ (where $\alpha_1, \ldots, \alpha_n \geq 0$ are positive reals summing up to 1) also belongs to $P(G)$. It follows that $P(G)$ contains the convex hull of all the vectors χ^M where M is a perfect matching of G. The Perfect Matching Polytope Theorem asserts that the converse inclusion also holds:

Theorem 2 (Edmonds [2]). *For any graph G, the set $P(G)$ coincides with the convex hull of the characteristic vectors of all perfect matchings of G.*

Rather naturally, $P(G)$ is called the *perfect matching polytope* of a graph G. Another fact that will be useful in our considerations is the following:

Lemma 3. *If w is a fractional perfect matching in a graph $G = (V, E)$ and $c \in \mathbb{R}^E$, then G has a perfect matching M such that*

$$c \cdot \chi^M \geq c \cdot w,$$

where $\cdot$ denotes the scalar product. Moreover, there exists such a perfect matching M that contains exactly one edge of each odd cut C with $w(C) = 1$.

Proof. Since w is a fractional perfect matching, it can be expressed by Theorem 2 as a convex combination of characteristic vectors of perfect matchings $M_1, \ldots, M_k$ of G. Since the cut C is odd, $\chi^{M_i}(C) \geq 1$ for every $i = 1, \ldots, k$. On the other hand, since w) is a convex combination of $\chi^{M_1}, \ldots, \chi^{M_k}$ and $w(C) = 1$, it must hold that $\chi^{M_i}(C) = 1$ for every $i = 1, \ldots, k$. Since $c \cdot w$ is a convex combination of $c \cdot \chi^{M_i}$, it holds that $c \cdot \chi_i^M \geq c \cdot w$ for at least one of the perfect matchings. $\square$

3 Proof of Theorem 1

In this section, we prove Theorem 1. By our discussion in Section 1, it suffices to show that $m_2 \geq 3/5$ and $m_3 \geq 4/5$:

Proof of Theorem 1. Fix a cubic bridgeless graph G. Define $w_1 \in \mathbb{R}^E$ to have the value $1/3$ on all edges $e \in E$. It is easy to verify that w_1 is a fractional perfect matching of G: Clearly, the conditions (i) and (ii) hold for every vertex v of G. Since G is bridgeless, every odd cut of G is comprised of at least 3 edges and thus condition (iii) also holds. Note that $w_1(C) = 1$ for each 3-cut C of G. Observe further that since G is cubic, all 3-cuts in G are odd (and more generally, a cut C in G is odd if and only if $|C|$ is odd). Hence, by Lemma 3, there is a perfect matching M_1 intersecting each 3-cut in a single edge. We remark that the existence of M_1 can be alternatively shown by induction on the size of G.

We now use M_1 to define the following vector $w_2 \in \mathbb{R}^E$:

$$w_2(e) = \begin{cases} 1/5 & \text{if } e \in M_1, \\ 2/5 & \text{otherwise.} \end{cases}$$

We verify that w_2 is a fractional perfect matching of G: the conditions (i) and (ii) clearly hold. Let C be an odd cut of G. The size of C is odd and it is at least 3. If C is a 3-cut, then $w(e) = 1/5$ for exactly one of the edges e contained in C and $w(e) = 2/5$ for the remaining two edges. If the size of C is 5 or more, then $w(C) \geq 5 \cdot 1/5 = 1$. Hence, condition (iii) also holds.

For each $e \in E$, set $c_2(e) = 1 - \chi^{M_1}(e)$. By Lemma 3, there exists a perfect matching M_2 such that

$$c_2 \cdot \chi^{M_2} \geq c_2 \cdot w_2 = \frac{2}{5} \cdot \frac{2}{3} |E| = \frac{4}{15} |E| \ .$$

Since $c_2 \cdot \chi^{M_2}$ is just $|M_2 \setminus M_1|$, it follows that

$$|M_1 \cup M_2| = (\frac{1}{3} + \frac{4}{15}) \cdot |E| = \frac{3}{5} |E| \, .$$

We conclude that $m_2 = 3/5$.

It remains to establish the lower bound on m_3. Note that if C is a 5-cut contained in M_1, then $w_2(C) = 1$. Hence, by Lemma 3, we may assume that if C is a 5-cut contained in M_1, then $|C \cap M_2| = 1$. Similarly, M_2 contains exactly one edge of each 3-cut. We now consider the following vector $w_3 \in \mathbb{R}^E$:

$$w_3(e) = \begin{cases} 1/7 & \text{if } e \in M_1 \cap M_2, \\ 2/7 & \text{if } e \in (M_1 \cup M_2) \setminus (M_1 \cap M_2), \\ 3/7 & \text{otherwise.} \end{cases}$$

Again, we verify that w_3 is a fractional perfect matching of G. The conditions (i) and (ii) hold trivially. Let us consider an odd cut C of G. If C is a 3-cut, then it contains exactly one edge e_1 contained in M_1 and exactly one edge e_2 contained if M_2. If $e_1 = e_2$, then $w_3(C) = 1/7 + 2 \cdot 3/7 = 1$. If $e_1 \neq e_2$, then $w_3(C) = 2 \cdot 2/7 + 3/7 = 1$. If C is a 5-cut that is not fully contained in M_1, then $|C \cap M_1| \leq 3$ (recall that C is an odd cut). Hence, $w_3(C) \geq 3 \cdot 1/7 + 2 \cdot 2/7 = 1$. If $C \subseteq M_1$, then $|C \cap M_2| = 1$ by the choice of M_2. We infer that $w_3(C) \geq 1/7 + 4 \cdot 2/7 > 1$. Finally, if the size of C is 7 or more, then $w_3(C) \geq 7 \cdot 1/7 = 1$. We conclude that w_3 is a fractional perfect matching of G.

Set $c_3(e) = 1 - \chi^{M_1 \cup M_2}(e)$. By Lemma 3, there exists a perfect matching M_3 such that

$$c_3 \cdot \chi^{M_3} = |M_3 \setminus (M_1 \cup M_2)| \geq \frac{3}{7} \cdot |E \setminus (M_1 \cup M_2)| = c_3 \cdot w_3 \ .$$

Consequently,

$$\begin{aligned} |M_1 \cup M_2 \cup M_3| &= |M_1 \cup M_2| + |M_3 \setminus (M_1 \cup M_2)| \\ &\geq \frac{3}{5}\,|E| + \frac{3}{7} \cdot \frac{2}{5}\,|E| = \frac{27}{35}\,|E| \, . \end{aligned}$$

We infer that $m_3 \geq 27/35$. □

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