The circular chromatic index of graphs of high girth*

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Abstract

We show that for each $\varepsilon > 0$ and each integer $\Delta \geq 1$, there exists a number g such that for any graph G of maximum degree Δ and girth at least g, the circular chromatic index of G is at most $\Delta + \varepsilon$.

1 Introduction

A *proper edge-coloring* of a graph G is a coloring of all the edges of G such that every two incident edges receive distinct colors. The smallest number of colors for which there is a proper edge-coloring is called the *chromatic index* and denoted by $\chi'(G)$. Recall that the chromatic index of a simple graph

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with maximum degree Δ is either Δ or $\Delta + 1$ by the classical theorem of Vizing [24]. A graph is said to be of Vizing class I if its chromatic index is Δ; all other graphs are of Vizing class II. In this paper, we study circular edge-colorings of graphs with large girth and we show that their circular chromatic index is close to Δ.

The concept of circular coloring of graphs was introduced by Vince [23]. A *proper circular ℓ-coloring* of a graph G, for a real $\ell \geq 1$, is a coloring of the vertices of G by real numbers from the interval $[0, \ell)$, such that the difference modulo ℓ of the colors γ_1 and γ_2 assigned to two adjacent vertices is at least one, i.e., $1 \leq |\gamma_1 - \gamma_2| \leq \ell - 1$. A circular ℓ-coloring can also be viewed as a coloring by points on a circle of circumference ℓ in such a way that a pair of adjacent vertices receive colors which are at distance at least 1 on the circle. The smallest real ℓ for which there is a proper circular ℓ-coloring is called the *circular chromatic number* of G and denoted by $\chi_c(G)$ (the minimum is always attained and it is a rational number [5, 23]). It can be shown that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. For further results on circular coloring, the reader is referred to the survey [27] on the subject.

A *circular ℓ-edge-coloring* of a graph G is just a circular ℓ-coloring of the line graph $L(G)$. The *circular chromatic index* $\chi'(G)$ of G is defined to be the circular chromatic number of $L(G)$. It is not difficult to show that $\Delta \leq \chi'_c(G) \leq \Delta + 1$ for every simple graph G with maximum degree Δ.

Jaeger and Swart [12] conjectured that there is an integer g such that each snark has girth at most g (actually, they conjectured that $g = 6$ suffices). Recall that the *girth* of a graph is the length of its shortest cycle, and that a *snark* is a cubic bridgeless graph belonging to the Vizing class II [6, 25] (i.e., of chromatic index 4). We remark that the definition of snarks is often taken to include further assumptions (as, e.g., in [14]) to avoid trivial cases.

The conjecture of Jaeger and Swart has become known as the Girth conjecture. More than 15 years after it was stated, the conjecture was disproved by Kochol [14] who constructed cyclically 5-edge-connected snarks with arbitrarily large girths. The authors [13] recently showed that the Girth conjecture is true when relaxed to the circular edge-coloring in the following sense: for each $\varepsilon > 0$, there is a number g such that the circular chromatic index of any cubic bridgeless graph of girth at least g is at most $3 + \varepsilon$.

In the present paper, we extend this result to graphs of higher degree. This provides an affirmative answer to the question presented in [13, Problem 2]: is it true that for any Δ and $\varepsilon > 0$, there exists an integer g such that every Δ-regular graph G of girth at least g has $\chi'_c(G) \leq \Delta + \varepsilon$? Actually, we prove the result for graphs of maximum degree Δ (which are not necessarily Δ-regular).

In general, large girth does not imply that the circular chromatic number of a graph is small. For any integers $k \geq 1$ and $g \geq 3$, there is a graph G with girth at least g, and with $\chi(G) = \chi_c(G) = k$ as shown by Steffen and Zhu [21]. On the other hand, Galluccio, Goddyn and Hell [10] proved the following for graphs avoiding a fixed graph H as a minor: For every graph H and every $\varepsilon > 0$, there is an integer g such that the circular chromatic number of each graph whose girth is at least g and which does not contain H as a minor is at most $2 + \varepsilon$.

A well-known question related to the girth and the circular chromatic number is the Pentagon problem (see, e.g., [18]): does every cubic graph G of sufficiently large girth admit a homomorphism into the 5-cycle? (Recall that a *homomorphism* from G to H is a mapping from $V(G)$ to $V(H)$ carrying edges of G to edges of H.) It is not hard to see that the above homomorphism exists if and only if the circular chromatic number of G is at most $5/2$. Let us stress, however, that in the present paper, we are concerned with the circular chromatic index, i.e., the circular chromatic number of the line graph. Thus, our results have no direct implications for the Pentagon problem.

Similarly as in [13], the proof of our main result uses the concept of independent systems of representatives [2–4, 9, 11, 16, 17]. The technique used in [13] can be easily generalized to the case of Δ-regular $(\Delta - 1)$-edge-connected graphs. However, when the connectivity assumption is removed, the technique fails to work. To be able to use it, one must find a way to get rid of small edge-cuts.

Small edge-cuts may indeed be dangerous: for instance, Afshani et al. [1], motivated by the question of Zhu [27, Question 8.4], showed that the circular chromatic index of any cubic bridgeless graph is at most $11/3$. However, their result does not hold for graphs with bridges (nor, for that matter, for bridgeless graphs of maximum degree 3). In addition, it is known [8,20] that the so-called fractional chromatic index $\chi'_f(G)$ of every Δ-edge-connected Δ-regular graph G of even order is equal to Δ, but the statement is false without the connectivity assumption.

Our results imply an upper bound to $\chi'_f(G)$, since $\chi'_f(G) \leq \chi'_c(G)$ (see [27] for details). Specifically: for each Δ and each $\varepsilon > 0$, there exists an integer g such that the fractional chromatic index of every graph with maximum degree Δ and girth at least g is at most $\Delta + \varepsilon$ (Corollary 16). However, this result on the fractional chromatic index can also be easily proved directly without using our results on the circular chromatic index.

The outline of the paper is as follows. In Section 3, we show, in a series of lemmas, that the edge set of any cyclically Δ-edge-connected graph of maximum degree Δ can be decomposed into $\lceil \Delta/2 \rceil$ factors of maximum degree at most 2, one of which is a matching if Δ is odd. In Section 4, we recall our major tool, a theorem on independent systems of representatives, and some related results. Section 5 contains a proof of a crucial extension property for circular edge-colorings of trees. This is used in Section 6 to prove a special case of our main result for the class of Δ-decomposable graphs, but in a stronger form that allows a subgraph to be precolored. The general case is proved in Section 7.

2 Notation

We use standard terminology and notation of graph theory. For background information on the subject, consult, e.g., [7, 26]. The symbols $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of a graph G, respectively. The *order* of G is the number of its vertices, and the *size* of G is the number of its edges. The result of the *contraction* of a subgraph H of G is denoted by G/H; this may introduce multiple edges, but any loops that arise are discarded. A *matching* in G is a set of pairwise disjoint edges. For an integer k, a *k-factor* of G is a spanning subgraph of G in which each vertex has degree precisely k.

The *distance* of two edges e and e' is the length of a shortest path whose first edge is e and whose last edge is e' decreased by one. In particular, edges at distance one are incident edges. If H is a subgraph of a graph G and H contains at least one edge, then the distance of an edge $e \in E(G)$ from H is the distance of e from the closest edge of H. We write $N(H, k)$ for the subgraph induced by all the edges whose distance from H is at most k. We also use $N(H, k)$ if H is just a set of edges, and we abbreviate $N(\{e\}, k)$ as $N(e, k)$. The *neighborhood* $N(H)$ of a subgraph $H \subseteq G$ is defined to be $N(H, 1)$. Finally, $V_d(G)$ stands for the set of all the vertices of a graph G whose degree in G is d.

3 Decompositions of graphs

In this section, we introduce the notions of decomposable graphs and graph decompositions which play a key role in our arguments.

Let G be a graph of maximum degree Δ. For an integer k, a *sub-k-factor* of G is any spanning subgraph of maximum degree at most k. Note that a 1-factor of a graph is a perfect matching, while a sub-1-factor is just a matching. An *ℓ-decomposition* of a graph G is a decomposition of G into $\lceil \ell/2 \rceil$ edge-disjoint sub-2-factors, one of which is required to be a matching if ℓ is odd. If G has an ℓ-decomposition, then it is said to be *ℓ-decomposable*. For our purposes, Δ-decomposable graphs of maximum degree Δ will be of interest. It is easy to find examples of graphs that are not of this type (consider any cubic graph with no perfect matching). However, graphs of even maximum degree always have a Δ-decomposition:

Lemma 1 *Every graph of maximum degree $2k$, $k \geq 1$, has a $2k$-decomposition.*

Proof: Let G be a graph of maximum degree $2k$. It is easy to see that one may add vertices and edges to G so as to obtain a $2k$-regular multigraph G'. By [19] (see also [7, Corollary 2.1.5]), the edges of G' can be decomposed into k 2-factors. This yields a $2k$-decomposition of G.

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Let G be a connected graph. An *edge-cut* in G is a minimal set T of edges of G such that $G \setminus T$ is disconnected. Note that if T is an edge-cut, then $G \setminus T$ has exactly two components. The graph G is k-edge-connected if it contains no edge-cut of size at most $k-1$. An edge-cut T is *essential* if each component of $G \setminus T$ contains at least one edge. The graph G is *essentially k-edge-connected* if $|E(G)| > k$ and G does not contain any essential edge-cut of size at most $k-1$. An edge-cut T is *cyclic* if each component of $G \setminus T$ contains a cycle. As before, G is *cyclically k-edge-connected* if $|E(G)| > k$ and G contains no cyclic edge-cut of size at most $k-1$.

In the rest of this section, we prove an analogue of Lemma 1 for graphs of odd maximum degree Δ, provided that they are cyclically Δ-edge-connected. The main step is to prove the existence of a matching which covers all vertices of degree Δ.

Our approach is based on the following classical theorem of Tutte [22]:

Theorem 2 *A multigraph G contains a 1-factor if and only if for each $S \subset V(G)$, the following holds:*

$$c_{\text{odd}}(G \setminus S) \leq |S|,$$

where $c_{\text{odd}}(H)$ denotes the number of components of H of odd order.

The proof of the following lemma follows ideas used to prove the theorem of Petersen [19] on the existence of 1-factors in bridgeless cubic graphs:

Lemma 3 *Let G be a connected multigraph. Suppose that all the vertices of G have the same odd degree $\Delta \geq 3$, except possibly for one vertex u of degree at most Δ. If G does not contain any edge-cut of size less than Δ, with a possible exception of the cut formed by all the edges incident with u, then G contains a matching that covers $V_\Delta(G)$.*

Proof: We distinguish two cases regarding the parity of the order of G. Assume first that $|V(G)|$ is even. We use Theorem 2 to show that G contains a 1-factor. Consider a subset $S \subset V(G)$. If $S = \emptyset$, then $c_{\text{odd}}(G \setminus S) = 0$ as G is connected. Otherwise, the number of edges between the vertices of S and $V(G) \setminus S$ is at most $\Delta|S|$, since each vertex of S has degree at most Δ. On the other hand, G contains at most one edge-cut of size less than Δ, and so all but at most one component of $G \setminus S$ are joined to the vertices of S by at least Δ edges. In particular,

$$\Delta(c_{\text{odd}}(G \setminus S) - 1) < \Delta|S|,$$

which implies $c_{\text{odd}}(G \setminus S) \leq |S|$ as required. By Theorem 2, G has a 1-factor.

Assume now that $|V(G)|$ is odd. Let δ be the degree of u. Since Δ is odd, $\delta < \Delta$ by the hand-shaking lemma. Thus, it is enough to show that the graph $G' = G \setminus \{u\}$ has a 1-factor. Note that G' is connected: otherwise, a proper subset of the edges incident with u would form an edge-cut in G, which was assumed not to be the case.

We again use Theorem 2 in order to show the existence of a 1-factor in G'. Let $S' \subset V(G')$ and set $S = S' \cup \{u\}$. If $S' = \emptyset$, then $S = \{u\}$ and the graph $G \setminus S = G' \setminus S'$ is connected. Since its order is even, the condition of Theorem 2 is satisfied. If $S' \neq \emptyset$, then the number of edges between S and $V(G) \setminus S$ is at most $\Delta|S'| + \delta$. Each component C of $G \setminus S$ is joined by at least Δ edges to the vertices of S. Indeed, if not, then the edges incident with C

form an edge-cut of size smaller than Δ. By the assumption of the lemma, the edges forming this edge-cut must be exactly all the edges incident with the vertex u. Then, the graph $G[C \cup \{u\}]$ is a component of the graph G, but since G is connected, we infer that $S' = \emptyset$. Hence, each component of $G \setminus S$ is joined by at least Δ edges to the vertices of S. Since there are at most $\Delta|S'|+\delta < \Delta(|S'|+1)$ such edges, the graph $G \setminus S = G' \setminus S'$ consists of at most $|S'|$ components. In particular, $c_{\mathrm{odd}}(G' \setminus S') \leq |S'|$, and G' contains a 1-factor by Theorem 2.

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As announced before, we now strengthen Lemma 3 to the class of essentially Δ-edge-connected graphs:

Lemma 4 *Every essentially Δ-edge-connected multigraph G of odd maximum degree $\Delta \geq 3$ has a matching which covers all the vertices of $V_\Delta(G)$.*

Proof: As long as G contains a pair of non-adjacent vertices v and v' whose degrees sum up to at most Δ, identify v and v' (preserving multiple edges if they arise). Just like G, the resulting multigraph G' is essentially Δ-edge-connected.

If G' contains a pair of vertices u and v both of whose degrees are smaller than Δ, add the edge uv to G'. (This preserves the essential edge-connectivity since the sum of the degrees of u and v is larger than Δ.) Repeat this process until there is no such pair of vertices, and call the resulting multigraph G''. If G'' is Δ-regular, it is also Δ-edge-connected. Otherwise, G'' contains exactly one vertex z of degree smaller than Δ, and the only edge-cut in G'' of size less than Δ consists of the edges incident with z. In each case, Lemma 3 implies that G'' has a matching which covers $V_\Delta(G'')$. The matching consisting of the corresponding edges in G has the required property.

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Before we handle cyclically Δ-edge-connected graphs, we need to observe the following property of matchings in trees:

Lemma 5 *Let T be a tree of maximum degree Δ and $X \subseteq V(T)$. Set $T_\Delta = V_\Delta(T) \setminus X$. If no vertex of T_Δ has Δ neighbors in X, and at most one vertex of T_Δ has $\Delta-1$ neighbors in X, then there is a matching M in $T \setminus X$ covering all vertices of T_Δ.*

Proof: We can assume that T_Δ is non-empty, for otherwise the claim holds for trivial reasons. Let r be a vertex of T_Δ with $\Delta - 1$ neighbors in X, if there is one. Otherwise, choose r to be an arbitrary vertex of T_Δ. Orient the edges of the tree T away from the vertex r.

Let M_0 be a set of edges obtained by choosing, for each vertex $v \in T_\Delta$, one outgoing edge vw ending in a vertex $w \notin X$. Such a choice can always be made, since if $v \neq r$, then at most $\Delta - 2$ out of the $\Delta - 1$ outgoing edges end in X (and a similar argument applies to r).

Finally, we change M_0 into a matching M by removing certain edges. For every maximal directed path P in M_0, remove every second edge (starting with the second one from the beginning of P). Clearly, M is a matching, and since each P ends in a vertex not contained in T_Δ, M still covers all the vertices of T_Δ.

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We are now able to extend Lemma 4 to cyclically Δ-edge-connected graphs:

Lemma 6 *For any odd integer $\Delta \geq 3$, every cyclically Δ-edge-connected graph G of maximum degree Δ has a matching covering $V_\Delta(G)$.*

Proof: We proceed by induction on the size of G. If G is essentially Δ-edge-connected, then the claim follows from Lemma 4. Thus, let C be an essential edge-cut of size at most $\Delta - 1$; by the cyclic connectivity assumption, at least one of the components A and B of $G - C$ is a tree. We may assume that it is the component A.

Since the multigraph G/A is cyclically Δ-edge-connected and its maximum degree is at most Δ, G/A has a matching M_B covering $V_\Delta(G/A)$ by the induction hypothesis (note that the claim is trivially true if the maximum degree of G/A is smaller than Δ). We will use the same symbol M_B for the matching in G comprised of the corresponding edges.

Let T be the tree obtained from G/B by splitting the vertex corresponding to B into $|C|$ vertices of degree one. We define $X \subset V(T)$ to be the set consisting of the counterparts of the vertices covered by M_B in G. Observe that $|X| \leq \Delta$, and if $|X| = \Delta$, then M_B uses an edge of C, and thus X contains two adjacent vertices. It follows easily that the hypothesis of Lemma 5 is satisfied. Hence, there is a matching M_A in $T \setminus X$ covering

$V_\Delta(T) \setminus X$. Again, we use M_A for the corresponding matching in G. Clearly, $M_A \cup M_B$ is a matching which covers $V_\Delta(G)$.

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We use Lemma 6 to find a Δ-decomposition of any cyclically Δ-edge-connected graph:

Proposition 7 *Any cyclically Δ-edge-connected graph G of maximum degree Δ is Δ-decomposable.*

Proof: If Δ is even, then G is Δ-decomposable by Lemma 1. For $\Delta = 1$, the assertion is trivial. Thus, we may assume that $\Delta \geq 3$ is odd. By Lemma 6, the graph G contains a matching M which covers $V_\Delta(G)$. In particular, $G \setminus M$ has even maximum degree $\Delta - 1$. Again by Lemma 1, $G \setminus M$ has a $(\Delta - 1)$-decomposition. Adding M to this $(\Delta - 1)$-decomposition, we obtain a Δ-decomposition of G.

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4 Independent systems of representatives

An important ingredient of our proof is the concept of an independent system of representatives, introduced by Fellows [9]. Throughout this section, let G be a graph with a given partition $\mathcal{P} = \{V_1, \ldots, V_k\}$ of its vertex set V into k disjoint subsets. A set $X \subseteq V$ is an *independent system of representatives (ISR)* with respect to $\mathcal{P}$ if X is an independent set intersecting each member of $\mathcal{P}$ in a single vertex.

Recently, there has been a surge of interest in ISRs, preceded by a result of Haxell [11] which we describe in more detail. Let us start with some terminology. A set D of vertices of the graph G is *totally dominating* if each vertex of G has a neighbor in D. The size of a smallest totally dominating set in G is the *total domination number* $\tilde{\gamma}(G)$ of G. For $I \subseteq \{1, \ldots, k\}$, set

$$V_I = \bigcup_{i \in I} V_i.$$

Finally, if $X \subseteq V(G)$, then $G[X]$ denotes the induced subgraph on the set X. The above mentioned result of Haxell [11] is the following:

Theorem 8 *Let G be a graph and $\mathcal{P} = \{V_1, \dots, V_k\}$ a partition of $V(G)$. If, for each $I \subseteq \{1, \dots, k\}$, it holds that*

$$\tilde{\gamma}(G[V_I]) \geq 2|I|,$$

then G has an ISR with respect to $\mathcal{P}$.

In the present paper, we make use of the following result which follows from Theorem 8 as a simple corollary:

Theorem 9 *Let G be a graph of maximum degree Δ and let $\mathcal{P}$ be a partition of $V(G)$. If each member of $\mathcal{P}$ contains at least 2Δ vertices, then G has an ISR with respect to $\mathcal{P}$.*

Although Theorem 9 is sufficient for our needs, we briefly sketch the subsequent development in the area which started with an extension of Hall's matching theorem to hypergraphs, proved by topological means by Aharoni and Haxell [4]. The result was further generalized by Meshulam [16], and a more elementary proof of the generalization was given by Aharoni et al. [3].

As observed by Ron Aharoni (see [16]), the method of [4] implies a fundamental result on ISRs that underlies a number of known sufficient conditions including Theorem 8. We refer the reader to [2] for a discussion of these, and for the topological definitions. Here, we just state the result:

Theorem 10 *Let G be a graph and $\mathcal{P} = \{V_1, \dots, V_k\}$ a partition of $V(G)$. If the topological connectivity of the independence complex of $G[V_I]$ is at least $|I| - 2$ for each $I \subseteq \{1, \dots, k\}$, then G has an ISR with respect to $\mathcal{P}$.*

In [17], the relation between the properties of the independence complex (in particular, its homology) and the existence of an ISR is elaborated even further.

5 Local modifications of edge-colorings

In this section, we show that if e is an edge of a tree T of maximum degree Δ, then any Δ-edge-coloring of $T \setminus e$ can be modified to a circular $(\Delta + \varepsilon)$-edge-coloring c' of T (where $\varepsilon > 0$ is a given real number) such that c' has any prescribed value on e, and c' agrees with c at all edges that are sufficiently far apart from e.

Lemma 11 *Let T be a tree of maximum degree Δ and $e \in E(T)$. Suppose that c is a proper Δ-edge-coloring of $T \setminus e$. Furthermore, let real numbers $\varepsilon > 0$ and $\gamma \in [0, \Delta + \varepsilon)$ be given. There exists a circular $(\Delta + \varepsilon)$-edge-coloring c' of T such that $c'(e) = \gamma$, and for any edge f at distance at least $\Delta^2/\varepsilon + 2\Delta$ from e,*

$$c'(f) = c(f).$$

Proof: First, note that it may be assumed that e is incident with a leaf of T. Indeed, if this is not the case, then let T_1 and T_2 be the trees obtained by adding the edge e to each component of $T \setminus e$, apply the statement of the lemma to T_1 and T_2 separately, and combine the corresponding $(\Delta+\varepsilon)$-edge-colorings to obtain c'.

By our assumption, e is only incident with at most $\Delta - 1$ other edges of T. Without loss of generality, we may assume that none of these $\Delta - 1$ edges is colored 0. By setting $c(e) = 0$, c is extended to a proper Δ-edge-coloring of T.

If f is an edge of T, $d(f)$ denotes the distance between f and e. Let C be a circle of circumference $\Delta + \varepsilon$. We identify the points of C with numbers from $[0, \Delta+\varepsilon)$ and assume the numbers increase as one traverses C clockwise. A *palette* is any Δ-tuple

$$P = (P(0), \dots, P(\Delta - 1)) \subset C$$

such that for any distinct x and y, the distance between $P(x)$ and $P(y)$ (on C) is at least 1.

Set $D = \lfloor \gamma\Delta/\varepsilon \rfloor$. In the sequel, we shall define palettes $P_0, \dots, P_{D+\Delta}$. For each edge $f \in E(T)$, the value of c' on f will be

$$c'(f) = \begin{cases} \gamma & \text{if } f = e, \\ P_{d(f)}(c(f)) & \text{if } 1 \le d(f) \le D + \Delta, \\ c(f) & \text{otherwise.} \end{cases} \qquad (1)$$

The palette P_0 consists of Δ consecutive, equally spaced points of C, the first of which is γ. In symbols,

$$P_0(x) = x\lambda + \gamma,$$

where $\lambda = 1 + \varepsilon/\Delta$ and (as in all that follows) all arithmetic is performed modulo $\Delta + \varepsilon$. For $i < D$, the palette P_{i+1} is obtained from P_i by a counter-clockwise rotation by ε/Δ. Using (1), this determines the values of c' on all edges at distance at most D from e.

Let us check at this point that the restriction of c' to the edges at distance at most D from e is a proper circular $(\Delta+\varepsilon)$-coloring. Since the distances of colors in a palette are at least 1, the only conflict that can occur is that for some distinct x and y in $\{0,\dots,\Delta-1\}$ and some i, the distance between $P_i(x)$ and $P_{i+1}(y)$ on C is less than 1. If this happens, then clearly $y = x+1$ (modulo Δ), and thus

$$P_{i+1}(x+1) - P_i(x) < 1.$$

However, one easily calculates that

$$\begin{aligned} P_{i+1}(x+1) - P_i(x) &= \Big((x+1)\lambda + \gamma - (i+1)\frac{\varepsilon}{\Delta}\Big) - \Big(x\lambda + \gamma - i\frac{\varepsilon}{\Delta}\Big) \\ &= \lambda - \frac{\varepsilon}{\Delta} = 1. \end{aligned}$$

Hence, we do have a proper coloring so far.

Observe that $P_D(0) \in [0, \varepsilon/\Delta)$. More generally, $P_D(x)$ is close to $x\lambda$:

$$P_D(x) = x\lambda + \gamma - \left\lfloor \frac{\gamma\Delta}{\varepsilon} \right\rfloor \cdot \frac{\varepsilon}{\Delta} \in \Big[x\lambda, x\lambda + \frac{\varepsilon}{\Delta}\Big). \tag{2}$$

Next, we "round the colors down". The ultimate outcome will be the palette $P_{D+\Delta} = (0,1,\dots,\Delta-1)$. The intermediate palettes $P_{D+1},\dots,P_{D+\Delta-1}$ are defined as follows. For $i = 1,\dots,\Delta$, we let

$$P_{D+i}(x) = \begin{cases} x & \text{if } 0 \le x < i, \\ P_D(x) & \text{otherwise.} \end{cases} \tag{3}$$

Note that P_{D+1} equals P_D except for the value at 0. Furthermore, the distance of $P_{D+1}(0)$ from $P_D(\Delta-1)$ is at least 1. Hence, c' is proper on all edges in $N(e, D+1)$.

In fact, c' is proper on all of $N(e, D+\Delta)$. To see this, it suffices to check that the distance between $P_{D+i}(x)$ and $P_{D+i+1}(x+1)$ (where $0 \le i < \Delta$ and $0 \le x < \Delta$) is at least 1. By (3), this distance is either the distance between x and $x+1$ (which is 1), or the distance between $P_D(x)$ and $P_D(x+1)$ (which is $\lambda > 1$), or the distance between x and $P_D(x+1)$ (which is at least $(x+1)\lambda - x > 1$).

Since $P_{D+\Delta}(x) = x$ for all x, the coloring c' coincides with c on edges at distance at least $D+\Delta$ from e. The estimate on the distance from the statement of the lemma follows from

$$D + \Delta \le \frac{\gamma\Delta}{\varepsilon} + \Delta < \frac{(\Delta+\varepsilon)\Delta}{\varepsilon} + \Delta = \frac{\Delta^2}{\varepsilon} + 2\Delta.$$

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6 Coloring Δ-decomposable graphs

For an integer d, a subgraph H of G is said to be *d-closed* if it contains every path P in G such that P joins two vertices from $V(H)$ and the length of P is at most d.

Theorem 12 *For any integer $\Delta \geq 3$ and real $\varepsilon > 0$, there exist positive integers d and g such that the following holds: let G be an arbitrary Δ-decomposable graph of maximum degree Δ and girth at least g. If H is a d-closed subgraph of G with $1 \leq |E(H)| \leq d\Delta$, then any proper $(\Delta+\varepsilon)$-edge-coloring h of the neighborhood of H extends to a $(\Delta+\varepsilon)$-edge-coloring of the entire graph G.*

Proof: Set $\ell = \lceil \Delta^2/\varepsilon + 2\Delta \rceil$, $d = 2\ell + 2$ and $g = 2(d\Delta + 1)\Delta^{2\ell+1}$. We are given a Δ-decomposition

$$\mathcal{F} = (F_0, \ldots, F_{\lceil \Delta/2 \rceil - 1})$$

of G, where $F_{\lceil \Delta/2 \rceil - 1}$ is a matching if Δ is odd. A key idea of the proof is to use the Δ-decomposition to construct a Δ-edge-coloring c of $G \setminus S$, where S will be a set of edges that are "sparse" and, in addition, distant from the subgraph H. Applying Lemma 11 to a suitable neighborhood of each edge of $S \cup \partial H$, where ∂H is the set of edges with precisely one endvertex in H, one can then modify c to a $(\Delta+\varepsilon)$-edge-coloring of the whole graph G which extends the coloring h.

Let $C_1, \ldots, C_k$ be all the odd cycles contained in $F_0, \ldots, F_{\lceil \Delta/2 \rceil - 1}$. Construct an auxiliary graph $\tilde{G}$ whose vertices are all edges e contained in the cycles C_i such that the distance of e from H is at least $2\ell + 2$. Two such edges e and f are joined by an edge in $\tilde{G}$ if they come from different cycles C_i, and their distance in G is at most 2ℓ. The vertex set of $\tilde{G}$ is naturally partitioned into k classes $\tilde{C}_i$ corresponding to the cycles C_i.

In a graph of maximum degree $\Delta \geq 3$, the number of edges at distance less than t from any given edge does not exceed

$$1 + 2(\Delta - 1) + 2(\Delta - 1)^2 + \cdots + 2(\Delta - 1)^{t-1} < 2\Delta^{t-1} < \Delta^t.$$

Consequently, the maximum degree of $\tilde{G}$ is at most $\Delta^{2\ell+1}$, and there are at most $2d\Delta^{2\ell+2}$ edges in $N(H, 2\ell + 1)$. Thus, the size of each $\tilde{C}_i$ is at least $g - 2d\Delta^{2\ell+2} = 2\Delta^{2\ell+1}$.

We aim to use Theorem 9 for $\tilde{G}$. By the choice of g, the hypotheses of Theorem 9 are satisfied, and hence $\tilde{G}$ has an independent system of representatives $S = \{e_1, \ldots, e_k\}$ with respect to the parts $\tilde{C}_i$.

Define an edge-coloring c of $G \setminus S$ as follows. Since S intersects all odd cycles contained in the sub-2-factors of the decomposition $\mathcal{F}$, each graph $F_i \setminus S$ (where $i = 0, \ldots, \lfloor \Delta/2 \rfloor - 1$) is bipartite and can be colored by colors $2i$ and $2i+1$. If Δ is odd, then $F_{\lceil \Delta/2 \rceil - 1}$ is a matching and all of its edges can be colored with the color $\Delta - 1$. Clearly, c is a proper Δ-edge-coloring of $G \setminus S$.

Let ∂H be the set of edges of $N(H)$ not contained in H. For each $e \in \partial H$, let $T_0(e)$ be the subgraph of G induced by e and all edges in $N(e, \ell)$ outside $H \cup \partial H$, and set $T(e)$ to be the component of $T_0(e)$ containing e. For an edge $e \in S$, let $T(e) = N(e, \ell)$. Since the girth of G is larger than $2(\ell + 1)$, each subgraph $T(e)$, $e \in \partial H \cup S$, is a tree. Moreover, for two distinct edges $e_1, e_2 \in \partial H \cup S$, the trees $T(e_1), T(e_2)$ are edge-disjoint. To prove this, it is enough to show that

$$\text{the distance of } e_1 \text{ and } e_2 \text{ is at least } 2\ell + 1. \tag{4}$$

We distinguish three cases. If $e_1, e_2 \in S$, then (4) follows from the fact that S is an ISR for $\tilde{G}$ and the parts $\tilde{C}_i$. If $e_1 \in S$ and $e_2 \in \partial H$ (or vice versa), (4) is ensured by the construction of $\tilde{G}$: the edges of distance less than $2\ell + 1$ from ∂H are not included in $\tilde{G}$ at all. Finally, if both e_1 and e_2 are in ∂H, then (4) holds since H is d-closed.

We now define a $(\Delta + \varepsilon)$-edge-coloring h' of G that extends the given edge-coloring h of the neighborhood of H. For each edge $e \in \partial H \cup S$, use Lemma 11 to find a $(\Delta + \varepsilon)$-coloring h'_e of $T(e)$ that agrees with c on the edges of the tree $T(e)$ incident with leaves, and such that

$$h'_e(e) = \begin{cases} h(e) & \text{if } e \in \partial H, \\ 0 & \text{if } e \in S. \end{cases}$$

Now define, for all edges f of G,

$$h'(f) = \begin{cases} h'_e(f) & \text{if } f \text{ is contained in some tree } T(e), \\ h(f) & \text{if } f \in E(H), \\ c(f) & \text{otherwise.} \end{cases}$$

It is straightforward to check that h' is a proper $(\Delta + \varepsilon)$-edge-coloring which extends h.

■

7 The main result

Before we are able to prove our main result, Theorem 15, we must establish an auxiliary lemma on cyclic edge-cuts in graphs.

Lemma 13 *Let $k \geq 2$ be a given integer and let G be a connected graph with more than k edges. If G is not cyclically k-edge-connected, then it contains a cyclic edge-cut of size at most $k-1$ which splits G into two components A and B such that G/A is cyclically k-edge-connected or the size of G/A is at most k.*

Proof: Let C be a cyclic edge-cut of G whose size is at most $k-1$, chosen such that the order of a component B of $G \setminus C$ is as small as possible. Let A be the other component of $G \setminus C$. We need to show that the graph G/A has no cyclic edge-cut of size at most $k-1$. Assume the opposite and let C'_A be such an edge-cut. Let C' be the set of edges of G corresponding to those in C'_A. Clearly, C' is a cyclic edge-cut of G, $|C'| \leq k-1$ and one of the components of $G \setminus C'$ has fewer vertices than B — a contradiction.

■

Consider a subgraph H of a graph G and an integer $d \geq 1$. A path P in G is a *d-connecting path* for H if

(i) the endvertices of P are in H,

(ii) no edge nor internal vertex of P is contained in H,

(iii) the length of P is at most d.

The *d-connector* $\mathrm{conn}_d(H)$ of H is the smallest d-closed subgraph of G which contains H. A sequence $(P_1, \ldots, P_k)$ of paths in G is a *construction sequence* for $\mathrm{conn}_d(H)$ if, for each $i = 0, \ldots, k-1$, the path P_{i+1} is d-connecting for

$$H^{(i)} = H \cup P_1 \cup \cdots \cup P_i,$$

and $H^{(k)} = \mathrm{conn}_d(H)$. Clearly, there is at least one construction sequence for $\mathrm{conn}_d(H)$. Although the construction sequence is not uniquely determined, the d-connector itself is.

We now show that if the girth of a graph G is sufficiently large, then the d-connector (as well as its neighborhood) of a subgraph with few edges is a forest of small size.

Lemma 14 *Let $d > \ell \geq 1$. Suppose that a subgraph H of a graph G contains at most ℓ edges and no isolated vertices. If the girth of G is greater than $(d+1)\ell$, then the neighborhood of the d-connector of H is a forest. Furthermore, the size of $\mathrm{conn}_d(H)$ is less than $d\ell$.*

Proof: Let $(P_1, \ldots, P_k)$ be a construction sequence for $\mathrm{conn}_d(H)$. Let $c(H)$ denote the number of components of H. Observe that $c(H) \leq \ell$. We claim that the endvertices of each P_{i+1} are in different components of

$$H^{(i)} = H \cup P_1 \cup \cdots \cup P_i.$$

If not, then let i_0 be the smallest index i violating the claim. Clearly, $H^{(i_0+1)}$ contains a cycle. Furthermore, $i_0 < \ell$. To see this, observe that the numbers of components of the graphs

$$H, H^{(1)}, H^{(2)}, \ldots, H^{(i_0)}$$

form a decreasing sequence; since $c(H) \leq \ell$, i_0 must be less than ℓ.

Now $i_0 < \ell$ implies that the size of $H^{(i_0+1)}$, and hence also the length of the cycle it contains, is at most $\ell + i_0 d \leq \ell(d+1)$. This contradicts the fact that the girth of G is greater than $\ell(d+1)$.

We have shown that each path P_{i+1} joins vertices in different components of $H^{(i)}$. Since H itself contains no cycles (by the girth assumption), we infer that $\mathrm{conn}_d(H)$ is a forest. Its neighborhood must be a forest as well. This follows from the fact that no two vertices of $\mathrm{conn}_d(H)$ have a common neighbor outside $\mathrm{conn}_d(H)$ (as there is no 2-connecting path for $\mathrm{conn}_d(H)$).

It remains to prove the stated bound on the size of $\mathrm{conn}_d(H)$. Since $k < c(H)$ (recall that k stands for the number of paths in the construction sequence), the size of $\mathrm{conn}_d(H)$ is at most $\ell + d(\ell - 1) < d\ell$ as claimed.

■

We are now ready to prove our main theorem:

Theorem 15 *For any integer $\Delta \geq 1$ and real $\varepsilon > 0$, there exists a positive integer g such that if G is a graph of maximum degree Δ and girth at least g, then*

$$\chi'_c(G) \leq \Delta + \varepsilon.$$

Proof: We may clearly assume that $\Delta \geq 3$. Fix $d \geq \Delta$ and $g \geq (d+1)\Delta$ large enough for the statement of Theorem 12 to hold. The proof proceeds by induction on the size of G. The assertion of the theorem is trivial if $|E(G)| \leq \Delta$. Furthermore, we may assume that G is connected.

If G is cyclically Δ-edge-connected, then G has a Δ-decomposition by Proposition 7. Note that this includes the case when G is a tree. By Theorem 12 (where we set H to be any subgraph consisting of a single edge), $\chi'_c(G) \leq \Delta + \varepsilon$. Otherwise, since $|E(G)| > \Delta$, Lemma 13 implies that there is a cyclic edge-cut F of size at most $\Delta - 1$ whose removal splits G into components A and B such that G/A is cyclically Δ-edge-connected or the size of G/A is at most Δ. Let F' denote the d-connector of F in the union of B and F. Lemma 14, used with $\ell = \Delta - 1$, implies that $N(F')$ is a forest. Since B contains a cycle, $N(F') \setminus F$ has fewer edges than B. Thus, if we set G_A to be the union of A with $N(F')$, the size of G_A is smaller than the size of G. At the same time, G_A has girth at least g and its maximum degree is at most Δ. By induction (needed only if the maximum degree is precisely Δ), G_A has a $(\Delta + \varepsilon)$-edge-coloring c_A.

The graph G/A has maximum degree at most Δ. We claim that it has a Δ-decomposition. As shown above, G/A is either cyclically Δ-edge-connected (in which case it has a Δ-decomposition by Proposition 7), or it has at most Δ edges (in which case there is a trivial Δ-decomposition). Therefore, $B \cup F$ is Δ-decomposable as well. Since F' is a d-closed subgraph of $B \cup F$ of size less than $d\Delta$, the precoloring of $N(F')$ induced by c_A extends to a $(\Delta + \varepsilon)$-coloring c_B of $B \cup F$ by Theorem 12. The combination of c_A and c_B is the desired $(\Delta + \varepsilon)$-coloring of G.

■

An immediate corollary of Theorem 15 is the following:

Corollary 16 *For any integer $\Delta \geq 1$ and real $\varepsilon > 0$, there exists a positive integer g such that if G is a graph of maximum degree Δ and girth at least g, then*

$$\chi'_f(G) \leq \Delta + \varepsilon,$$

where χ'_f denotes the fractional chromatic index.

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