Theoretical round. Sketches for solutions. For jury job ONLY

Note. The given sketches are not full; the team leaders have to give more detailed explanations to students. But the correct solutions in the students' papers (enough for 8 pts) may be shorter.

Note. Jury members should evaluate the student's solutions in essence, and not by looking on formal existence the mentioned sentences or formulae. The formal presence of the mentioned positions in the text is not necessary to give the respective points. Points should be done if the following steps de facto using these positions.

α-1. Mercury mirror. Imagine rotating mercury. At each point, the normal to its surface will be directed at an angle $\gamma = \omega^2 r/g$ to the vertical, where ω is the angular velocity of rotation of mercury, r is the distance from the axis of rotation, g is the gravity acceleration. If the direction of the vertical (vector g) is fixed, then this angle γ is related to the radius of curvature R of the liquid mirror as $\gamma = r/R$. From these two equations we get:

$$R = g/\omega^2 = gT^2/4\pi^2.$$

Substituting the values $g = 9.8$ m/s^2 and $T = 86164$ s, we get:

$$R \approx 1.84 \cdot 10^9 \text{ m} = 1.84 \text{ million km}.$$

The focal length of such a mirror would be:

$$F = R/2 = gT^2/8\pi^2 \approx 0.92 \cdot 10^9 \text{ m} = 920 \text{ thousand km}.$$

But ... The result would be true for a flat Earth. The condition "if the direction of the vertical is constant" is written above not accidentally. In the case of a real Earth, when moving away from the Pole, the direction of the vertical changes 300 times faster (the Earth's radius is 300 times smaller than 1.84 million km). That is, the rotation of the Earth only reduces the curvature of the liquid surface at the pole by 1/300. The mirror designed by the Bear will be convex, its curvature will be exactly the same as the curvature of the Arctic Ocean.

Thus, the correct answer is "**No**", the idea of Polar Bear is impracticable.

β-1. Distance to galaxy. The redshift of this galaxy is:

$$z = \Delta\lambda/\lambda = (\lambda_{H\alpha} - \lambda_{H\beta}) / \lambda_{H\beta} = (6563 \text{ nm} - 4861 \text{ nm}) / 4861 \text{ nm} = 0.35,$$

and equally, regardless of the point of view of scientists regarding the value of the Hubble parameter (data about the wavelengths of Hα and Hβ are found in the table of constants). With such values of z, in estimating problems it is still possible to ignore relativistic effects and use the equations of classical physics. So the galaxy is moving away from us with speed

$$V = cz = 300\,000 \text{ km/s} \times 0{,}35 = 105\,000 \text{ km/s}.$$

According to the Hubble rule, the receding speed of galaxy is proportional to distance to it:

$$V = H_0 \times L,$$

where H_0 is the Hubble parameter.

Since the two scientists used different values of the Hubble parameter (H_1 and H_2), they obtained different distances to the researching galaxy:

$$L_1 = V/H_1, \quad L_2 = V/H_2$$

and, accordingly, observing the same apparent magnitude of the supernova, obtained different values of its absolute magnitude:

$$M = m - 5 \lg (L / 10 \text{ pc}),$$

that is, $M_1 = m - 5 \lg (L_1 / 10 \text{ pc})$, $M_2 = m - 5 \lg (L_2 / 10 \text{ pc})$, therefore

$$M_2 - M_1 = -5 \lg L_2 + 5 \lg L_1 = -5 \lg (L_2 / L_1),$$

$$L_2 / L_1 = 10^{-\Delta M/5},$$

$$L_2 / L_1 = 10^{-0.38/5} = 10^{-0.076} = 0.840.$$

Thus, since $H_1L_1 = V = H_2L_2$ and $H_2 - H_1 = \Delta H_0$, we get:

$$H_1L_1 = H_2L_2,$$

$$H_1L_1 = (H_1 + \Delta H_0)L_1(L_2/L_1),$$

$$H_1 = H_1(L_2/L_1) + \Delta H_0(L_2/L_1),$$

$$H_1(1 - L_2/L_1) = \Delta H_0(L_2/L_1),$$

$$H_1 = \Delta H_0(L_2/L_1) / (1 - L_2/L_1).$$

Numerically: $H_1 = 14 \text{ km/s/Mpc} \times 0{,}84 / 0{,}16 \approx 73 \text{ km/s/Mpc},$

$$H_2 = H_1 + \Delta H_0 = 87 \text{ km/s/Mpc}.$$

It remains to use the formula $V = H_0 \times L$ and calculate the distance to the galaxy. However, what is the value of H_0 should we take? It turns out that the points of view of the scientists differ on this issue:

- According to the point of view of the first scientist,

$$L_1 = V/H_1 = 105\,000 \text{ km/s} / 73 \text{ km/s/Mpc} = 1.44 \text{ Gpc}.$$

- From the point of view of the second scientist,

$$L_2 = V/H_2 = 105\,000 \text{ km/s} / 87 \text{ km/s/Mpc} = 1.21 \text{ Gpc}.$$

- And, if we use the average value of the Hubble parameter taken from the table (H_0 = 68 km/s/Mpc), it turns out that

$$L_0 = V/H_0 = 105\,000 \text{ km/s} / 68 \text{ km/s/Mpc} = 1.54 \text{ Gpc}.$$

α-2. Great oppositions of Mars. On the morning of July 27, 2018, Mars.

2.1. During the opposition, the planet is at a point approximately (up to an angular distance from the ecliptic at this moment) opposite to the Sun, that is, located where the Sun was six months ago. The Sun is in the constellation Capricorn on January 27.

2.2. The oppositions are repeated every synodic period. The synodic period of Mars is:

$$T_S = 1/(1/T_E - 1/T_M),$$

$$T_S = 1/(1/365.256 - 1/686.98) \;\; 779.93 \text{ days},$$

Approximately 15 years passed from 2003 to 2018, and there are $15 \cdot T_E / T_S \approx 7.02$ synodic periods of Mars during this period. So, the opposition 2018 was to occur after 7 synodic periods, earlier on August 28.

The easiest way to calculate the date is as follows. For 7 synodic periods of Mars $7 \times 779.93 \approx 5459$ days pass. From August 28, 2003, to August 28, 2018, $365 \times 15 + 4 = 5479$ days passed. Thus, in 2018, the opposition of Mars should be 20 days earlier than August 28, that is, August 8.

Similarly, we calculate the date of opposition 2035. $32 \cdot T_E / T_S \approx 14.99$. 15 synodic periods: $15 \times 779.93 \approx 11699$ days. From August 28, 2003, to August 28, 2035, $365 \times 32 + 8 = 11688$ days pass. Thus, it turns out that in 2035 the opposition of Mars should be 11 days later than August 28, that is, September 8.

2.3. The difference appears from the fact that in our calculations we take the average angular velocity of Mars in orbit, and in periods close to great oppositions, Mars is near the perihelion of its orbit, and its speed is larger. As a result, the difference in the angular velocities of the Earth and Mars is smaller. Therefore, to change the position of Mars relative to the Earth takes more time. That is why, the farther the opposition from the perihelion point of Mars orbit (which is at the ecliptic longitude that the Earth passes around August 28), the greater the error. Calculations in the approximation of circular orbits give about 1.5 times the underestimated interval between the date of the calculated opposition and the date on which the "greatest oppositions" take place (that is, August 28).

2.4. The position of Mars during the opposition of 2035 is closer to the position of the perihelion of the orbit than in 2018, therefore in the opposition of 2035 Mars will be brighter.

β-2. Martian observations. Martians will observe the Sun, the Earth and the Moon in the constellation opposite to that in which we observe the full Moon and the opposition of Mars in the end of July. That is, this is exactly the constellation in which at the end of July the Sun is located for observers at the Earth. In the constellation of Cancer.

2.1. Using the data of the ephemeris, we can interpolate the values of the right ascension and declination of the Moon and Mars at the time of the mid-point of the lunar eclipse. For the Moon, we obtain $\alpha_L = 20^h28^m$, $\delta_L = -18°55'$. For Mars at the same time – $\alpha_M = 20^h32^m$, $\delta_M = -25°33'$. We see that the difference in right ascensions is negligible compared with the difference in declinations, therefore the angular distance between the centers of the Moon and Mars at this time with great accuracy will be $\Delta\delta = 6°38'$.

The center of the moon will be at a distance

$$L_0 = R_{EL} \times \sin\Delta\delta = 4{,}06\cdot10^5 \text{ км} \times 0.116 \approx 4.69\cdot10^4 \text{ km}$$

from the line connecting the centers of Mars and Earth, and the northern edge of the Moon is at a distance

$$L_1 = L_0 - R_E + R_L = 4.69\cdot10^4 \text{ km} - 0.64\cdot10^4 \text{ km} + 0.17\cdot10^5 \text{ km} \approx 4{,}23\cdot10^4 \text{ km}$$

from the line connecting the center of Mars and the northern edge of the Earth. The angle between the line connecting the northern points of the Earth and the Moon, and the line connecting the centers of Mars and Earth, will be

$$\varphi = \arcsin(L_1/R_{EL}) \approx 5°59'.$$

Thus, to observe the transit of the entire disk of the Moon across the disk of the Earth, the Martian orbital station must be at a distance

$$L_2 = R_{EM} \times \sin\varphi = 0.386 \text{ au} \times 1.496\cdot10^8 \text{ km/au} \times 0.104 \approx 6.02\cdot10^6 \text{ km}.$$

from the center of Mars. This means that the orbital station must have a semi-major axis of the orbit equal to at least

$$L_2 = (L_2 + R_M) / 2 \approx L_2 / 2 \approx 3 \cdot 10^6 \text{ km}.$$

Orbiting of bodies around Mars in orbit with such a large semi-axis is impossible, such an orbit goes beyond Hill's sphere.

$$R_H = L_{SM} \times (M_M/3(M_S+M_M))^{1/3} \approx R_{SM} \times (M_M/3M_S)^{1/3} \approx 1/210\ R_{SM} \approx 1.1\cdot10^6 \text{ km}.$$

2.2. This part of the problem is solved similarly to the previous one. In order to observe the passage of the Earth along the solar disk, the Martians must hit a point that is directly opposite to the position of the center of the Sun for the earth observer (with an accuracy of the angular radius of the Sun seen from Mars). From the ephemeris of the Sun, we see that at the time of the Mars opposition, the declination of such a point is approximately $\delta_{-S} = -19°13'$. For Mars, this value is $\delta_M = -25°30'$. The angular distance between the desired point and Mars at this time with high accuracy will be $\Delta\delta = 6°17'$. With a correction for the angular radius of the Sun, the minimum angle required for Martians from Mars will be $\psi = 6°06'$.

Thus, in order to observe the passage of the Earth across the disk of the Sun, the Martian orbital station must be at a distance

$$L_2 = R_{EM} \times \sin\psi = 0.386 \text{ au} \times 1.496 \cdot 10^8 \text{ km/au} \times 0.106 \approx 6.14 \cdot 10^6 \text{ km}.$$

from the center of Mars. This means that the orbital station must have a semi major axis of the orbit equal of at least

$$L_2 = (L_2 + R_M) / 2 \approx L_2 / 2 \approx 3 \cdot 10^6 \text{ km}.$$

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αβ-3. Sunset at Colombo.

3.1. We see in the picture that half of the solar disk disappeared over the horizon. This is exactly half of the disk with great accuracy, since we see its vertical edges strictly on the horizon. In zero approximation, this occurs at 18:00 local time. The accuracy of the fact, that the center of the Sun is on the horizon at the photo, can be estimated at 2′ or ±8 seconds in units of time.

Let us consider what corrections to this time need to be taken into account, and what is the accuracy of each the correction.

1. Longitudinal correction. The local time of Sri Lanka is UT+05:30, and it corresponds to the longitude 82°30′ East, the photographer was at the longitude 79°51′, i.e. 2°39′ to the west. This means that for him the sunset occurs 2°39′ × 4 min/° = 10 min 36 sec later. The accuracy of this correction corresponds to 0.5′ in longitude or 2 seconds in time.

2. Correction of the equation of time. According to the schedule, we find that on September 30 all events related to the Sun occur 10 minutes earlier. The accuracy of the correction is determined by the graph and is approximately half a minute.

3. The center of the Sun crosses the mathematical horizon strictly in the west and at 18:00 of true solar time only at the moments of equinoxes. The evening of September 30 in Colombo is 7 days and 11 hours later than the autumnal equinox. During this time, the Sun has moved along the ecliptic by about 7.4°. The declination of the Sun is approximately equal to $\delta = -23{,}44° \times \sin 7{,}4° \approx -3.0°$. Given the latitude of the place as 6.9°, we obtain that the Sun is $3.0° \times \sin 6.9° \approx 0.36° \approx 22'$ lower than at the same time during the autumnal equinox. The rate of lowering the Sun is $15'/\text{min} \times \cos\delta \times \cos\varphi$ ($\cos\delta$ arises due to the fact that the Sun moves not along a large circle, but along a small circle, and $\cos\varphi$ is due to the inclination of the trajectory). In our case these coefficients are very close to 1, so the Sun goes down 22′ in 1 minute 28 seconds. That is, this correction is minus 1 minute 28 seconds. The correction accuracy is 4 seconds.

4. Refraction. The refraction near the horizon is 35′. The Sun passes down this angle in 2 minutes and 20 seconds. This is a correction with a plus sign. Correction accuracy is 4 seconds.

5. Lowering of the horizon. It is said that the height of the camera above sea level was approximately h = 6.5 m, with the radius of Earth R ≈ 6400 km, this gives lowering of the horizon for $(2h/R)^{1/2} \approx 1.4 \cdot 10^{-3}$ rad ≈ 5′. This correction is plus more 20 seconds. The accuracy of the correction corresponds to 2 seconds in time.

In total, the sum all corrections is:

+10 min 36 sec – 10 min – 1 min 28 sec + 2 min 20 sec + 20 sec = 1 min 48 sec. The largest error is given by the inaccuracy in determining the correction due to the equation of time on graph. That is, the event occurred at 18:01:48 with an accuracy of half a minute. Or at **18:02** if to write with a reasonable accuracy.

3.2. Civil twilight ends when the center of the Sun appears 6° below the mathematical horizon. As we calculated above, at the time of the shooting, the center of the Sun was at 35′ + 5′ = 40′ and was moving downward with speed

$$15'/\text{min} \times \cos\delta \times \cos\varphi = 14{,}87'/\text{min}.$$

The remaining 5°20′ before reaching -6° the Sun will pass in 21.5 minutes. That is, the answer to the second question of the problem is **21.5** minutes.

3.3. The height of Adams Pk, as we see on the map, is equal to 2243 m. Lowering the horizon from this peak will be $(2h_A/R)^{1/2} \approx 2{,}65 \cdot 10^{-2}$ rad ≈ 1°31′. This is 1°26′ more than when photographing from the seafront. This effect raises the apparent position of the solar disk above the visible horizon (by 1°26′ compared to the camera on the seafront). On the other hand, measurements on the map (28 mm from the coast to the peak with a scale of 200 km to 40 mm) show that Adams Pk is 140 km East of Colombo, which corresponds to enlarging longitude by 360° ×140 km / (40076 km × cosφ) = 1°15′. The apparent position of the solar disk is lowered due to this effect. The difference of the effects is 11′ or one third of the angular diameter of the Sun. Thus, from Adams Pk at this time the position of the Sun will be visible, at which 1/6 of its diameter is below the horizon, and 5/6 above the horizon. This picture should be drawn.

αβ-4. Geostationary satellite.

4.1. Geostationary satellites are located above the same equator point. Therefore, each actual satellite is always visible at one point for a stationary observer on Earth.

It should be noted that the geostationary satellite should not be confused with a geosynchronous one, the orbital period of which is exactly equal to the orbital period of the Earth, but which has an orbit with a nonzero eccentricity or slight inclination with respect to the equator. Such a satellite in the sky describes loops and eights for observers on the Earth.

A geostationary satellite has a circular equatorial orbit with diameter R under the conditions:

$$\omega^2 R_{st} = GM/R_{st}^2,$$

$$R_{st}^3 = GM/\omega^2 = GMT^2/4\pi^2,$$

where G is the gravitational constant, M is the mass of the Earth, T is the period of the Earth's revolution around its axis ($23^h56^m04^s$). The calculations give the result

$$R_{st} = 42\ 160 \text{ km}.$$

It is obvious that of all the possible geostationary satellites, the smallest zenith distance has a satellite located on the meridian of the observer. From the drawing we see that the zenith distance is z = φ+ψ, where φ is the latitude, and ψ is the angle at which the "equatorial plane – Colombo" segment is visible from the satellite. We can use the latitude of the place in Colombo, which is indicated as "seafront in the center", 06°54').

$$\mathrm{tg}\psi = r\sin\varphi/(R - r\cos\varphi),$$

The calculations give

$$\psi \approx 1°14′.$$

$$z = 08°08′.$$

4.2. This satellite can be observed all the dark time of day from the moment when 2^m stars become visible (approximately the end of evening civil twilight) to the corresponding moment in the morning, except the possible period when the satellite falls into the shadow of the Earth. The indicated "dark time of day" is 11^h12^m ($12^h - 2\times24^m$) in average, and can continue in Colombo from 11^h00^m in summer to 11^h24^m in winter (seasons are indicated according to the European calendar). It is easy to understand that in winter and in summer the satellite is never eclipsed by the Earth (and therefore visible all night), but in periods close to the equinoxes, such eclipses in the middle of the night are regular. In maximum they last approximately the time $\tau = 24^h \times 0.9 \cdot D_E / 2\pi R_{st}$ (we use here just 24^h, since the movement is synodic, D_E is the diameter of the Earth, and the coefficient 0.9 appears due to the narrowing of the earth shadow cone.

$$\tau_{max} = 62 \text{ min}.$$

Thus, in periods close to the equinoxes, when the "dark time of day" is 11^h12^m, the duration of visibility can be reduced by 62 minutes, that is, to 10^h20^m. Thus, the answer to the second question of the problem is: from 10^h20^m to 11^h24^m.

4.3. Let us compare the light fluxes coming to the observer (O) from the satellite (S) and from the full Moon (L).

If W = light flux from the Sun near Earth,

α_i = albedo of the observed objects,

$S_i = \pi r_i^2 = \pi d_i^2/4$, where r_i and d_i are the radius and the diameter of the observed objects respectively,

R_i = distances from the observed objects to the observer:

Flux from Moon to observer: $I_L = W \cdot \alpha_L \cdot S_L / (2\pi R_{LO}^2)$.

Flux from satellite to observer: $I_S = W \cdot \alpha_S \cdot S_S / (4\pi R_{SO}^2)$.

(We do not take into account effects related to diagrams of scattering and random/mirror scattering. In the student's solutions it's also not really necessary to understand the difference between the coefficients of 2π and the 4π in $2\pi R_{LO}^2$ and $4\pi R_{SO}^2$.)

Since the satellite was made of polished metal, we may assume $\alpha_S = 1$.

Thus,

$$I_L / I_S = \alpha_L \cdot (S_L / S_S) \cdot (4\pi R_{SO}^2 / 2\pi R_{LO}^2) = 2\alpha_L \cdot (d_L/d_S)^2 \cdot (R_{SO}/R_{LO})^2.$$

Necessary data may be taken from the Table of planetary data. Calculations may then give us:

$$I_S/I_L = 10^{-0.4(mS-mL)} = 10^{-0.4 \cdot (2+12.7)} \approx 1.3 \cdot 10^{-6},$$

and $$d_S = 3.475 \cdot 10^6 \text{ m} \times (2 \cdot 0.07 \cdot 1.3 \cdot 10^{-6})^{1/2} \cdot 3.6 \cdot 10^7 \text{ m} / 3.8 \cdot 10^8 \text{ m} \approx 140 \text{ m}.$$

αβ-5. Oort cloud. The fact that every century we observe from 10 to 20 bright comets means that approximately once every 6 years 8 months ($\tau \approx 2.1 \cdot 10^8$ s) two comet bodies collide in the outer Oort cloud, resulting in one of them rushes into the inner regions of the solar system. Collisions occur because all these bodies move chaotically with typical velocities less than the second cosmic velocity (otherwise they would leave this area) and more than half of the first cosmic velocity (otherwise they would move into the inner cloud areas). To estimate the average their velocities, we take the first cosmic velocity for the average distance (R+r)/2 = 0.5 ly:

$$u = (GM/[(R+r)/2])^{1/2} \approx 170 \text{ m/s},$$

where M is the mass of the Sun, R and r are the outer and inner radii of the cloud. (One can also use the formula $u \sim R^{-1/2}$, 0.5 ly is about 30,000 times larger than the distance from the Earth to the Sun, therefore, the speed of the Earth in its orbit should be divided by the square root of 30,000). So we obtain $u \approx 170$ m/s.

Let us find how the frequency of collisions is related to the number of comet bodies in the Oort cloud. If there are N comet bodies in total, then their concentration in space is

$$n = N / [4/3\ \pi\ (R^3 - r^3)] \approx N / 4/3\ \pi R^3 \quad \text{(bodies per m}^3\text{)}.$$

(The fact $r^3 \ll R^3$ was taken into account.) We will use the collision model in which each comet body moves relative to the others with the same average velocity u. During the time τ, the comet body will cover the distance $u \cdot \tau$ and in its movement will capture the volume

$$V = u \cdot \tau \cdot S = u \cdot \tau \cdot \pi d^2/4,$$

where d is the average diameter of the comet body, we will consider it equal to 2.5 km. If another comet body appears in this volume, then a collision occurs, and the probability of such an event is equal to

$$\alpha = n \cdot V_1 = n \cdot u \cdot \tau \cdot \pi d^2/4,$$

Since any of the N comet bodies can collide, the total number of collisions will be equal to:

$$K = \alpha \cdot N = n \cdot u \cdot \tau \cdot \pi d^2/4 \cdot n \cdot 4/3 \cdot \pi R^3 = u \cdot \tau \cdot \pi^2 n^2 d^2 R^3/3,$$

$$K = \alpha \cdot N = N / (4/3\ \pi R^3) \cdot u \cdot \tau \cdot \pi d^2/4 \cdot N = 3 \cdot u \cdot \tau \cdot N^2 d^2/16R^3,$$

Each collision leads to the fact that with some probability β our body will be sent to the inner regions of the Solar System. This probability can be estimated from different models, let us estimate it as an approximate ratio of solid angles:

$$\beta = \pi r^2/4\pi R^2 \approx 0.015,$$

Thus, in order that one body goes to the inner regions of the solar system, it is necessary to fulfill the condition

$$K\cdot\beta = 1,$$

$$3\cdot u\cdot\tau\cdot N^2\, d^2/16R^3\cdot\beta = 1,$$

$$N = 4R^{3/2}/d(3\beta u\tau)^{1/2}.$$

Calculations give: $N = 2.6\cdot10^{16}$,

with an accuracy of one significant digit: $N \approx 3\cdot10^{16}$.

If the concentration of the comet bodies is n (bodies per m^3), then each body has a volume of space 1/n, and, respectively, the average distance will $(1/n)^{1/3}$.

$$n \approx N / 4/3\, \pi R^3 \approx 3^{1/2}/\pi d(\beta u\tau R^3)^{1/2},$$

$$L \approx 1/n^{1/3} \approx R^{1/2}(\pi d)^{1/3}(\beta u\tau/3)^{1/6},$$

$$L \approx 4\cdot10^{10}\ \text{m} \approx 40\ \text{mln.km}.$$

This is roughly the minimum distance between the Earth and Venus.

To estimate the total mass of the comet bodies, we assume that their average density is equal to the density of ice ($\rho \approx 900$ kg/m^3), and the average volume is $v = 1/6\ \pi d^3$. Thus, the average mass of one comet body is:

$$m = \rho\cdot v = 1/6\ \pi d^3\ \rho \approx 7\cdot10^{12}\ \text{kg},$$

and the sum of the mass of all the bodies:

$$M = N\cdot m \approx 2\cdot10^{29}\ \text{kg}.$$

This is one hundred masses of Jupiter or one-tenth of the mass of the Sun.