

0. Cvičení: Opakování derivace a integrály

Derivace

Příklady: Určete derivace následujících funkcí

1. $f(x) = e^{5x}(-5 \cos x + 12 \sin x)$

$$f'(x) = 5e^{5x}(-5 \cos x + 12 \sin x) + e^{5x}(5 \sin x + 12 \cos x) = -13e^{5x} \cos x + 65e^{5x} \sin x$$

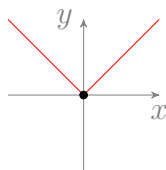
2. $f(x) = \frac{4}{-10+x} = 4(-10+x)^{-1}$

$$f'(x) = -4(-10+x)^{-2} = \frac{-4}{(-10+x)^2}$$

3. $f(x) = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$

$$f'(x) = \frac{1}{2} \frac{1-x}{1+x} \frac{1(1-x) - (1+x)(-1)}{(1-x)^2} = \frac{1}{2} \frac{1-x}{1+x} \frac{2}{(1-x)^2} = \frac{1}{1-x^2}$$

4. $f(x) = |x|$, $x_0 = 0$,



$$f'(0^+) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

$$f'(0^-) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$$

→ derivace v bodě $x_0 = 0$ neexistuje (limita zprava se nerovná limitě zleva)

5. Najděte rovnici tečny a normály funkce $f(x) = \frac{1}{x} \ln \frac{1}{x}$ v bodě $M = [1, ?]$.

Rovnice tečny $t : y - y_0 = f'(x_0)(x - x_0)$,

rovnice normály $n : y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$.

$$f(1) = \frac{1}{1} \ln \frac{1}{1} = 0 \rightarrow M = [1, f(1)] = [1, 0],$$

$$f'(x) = -\frac{1}{x^2} \ln \frac{1}{x} - \frac{1}{x} x \frac{1}{x^2} = -\frac{1}{x^2} \ln \frac{1}{x} - \frac{1}{x^2}, f'(1) = -1.$$

$$t : y - 0 = -1(x - 1)$$

$$t : y = -x + 1$$

$$n : y - 0 = 1(x - 1)$$

$$n : y = x - 1$$

Integrály

Příklady: Vypočtěte následující neurčité integrály (úpravy):

$$1. \int \left(1 - \frac{8}{x^2}\right) \sqrt{x} \sqrt{x} dx = \int (1 - 8x^{-2}) (x \cdot x^{1/2})^{1/2} dx = \int (1 - 8x^{-2}) \underbrace{(x^{3/2})^{1/2}}_{x^{3/4}} dx$$

$$= \int (x^{3/4} - 8x^{-5/4}) dx = \frac{4}{7}x^{7/4} + 8 \cdot 4x^{-1/4} + C = \frac{4}{7}x^{7/4} + 32x^{-1/4} + C$$

$$2. \int \frac{\sqrt{x^4+x^{-4}+2}}{x^3} dx = \int \frac{\sqrt{(x^2+x^{-2})^2}}{x^3} dx = \int \frac{|x^2+x^{-2}|}{x^3} dx = \int \left(\frac{1}{x} + \frac{1}{x^5}\right) dx = -\ln|x| - \frac{1}{4x^4} + C$$

Příklady: Vypočtěte následující neurčité integrály (per partes):

Na otevřeném intervalu platí $\int u(x)v(x)' dx = u(x)v(x) - \int u(x)'v(x) dx$

$$1. \int \ln x dx$$

$$= \left| \begin{array}{cc} D & I \\ \ln x & 1 \end{array} \right| = x \ln x - \int 1 dx = x \ln x - x + C$$

(Diagram: Blue arrow from ln x to x, red arrow from 1 to -x)

$$2. \int x^2 \cos x dx$$

$$= \left| \begin{array}{cc} D & I \\ x^2 & \cos x \\ 2x & -\sin x \\ 2 & -\cos x \end{array} \right| = x^2 \sin x + 2x \cos x - 2 \int \cos x dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

(Diagram: Blue arrows from x^2 to sin x, 2x to -sin x, 2 to -cos x; Red arrows from cos x to -sin x, -sin x to -cos x)

Příklady: Vypočtěte následující neurčité integrály (substituce):

$$1. \int \frac{1}{x \ln^3 x} dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \end{array} \right| = \int \frac{1}{t^3} dt = -\frac{1}{2t^2} + C = -\frac{1}{2\ln^2 x} + C$$

$$2. \int \frac{\sin x}{\sqrt{2+\cos x}} dx = \left| \begin{array}{l} t = 2 + \cos x \\ dt = -\sin x dx \end{array} \right| = -\int \frac{1}{\sqrt{t}} dt = -2\sqrt{t} + C = -2\sqrt{2+\cos x} + C$$

Příklady: Vypočtěte následující neurčité integrály (rozklad na parciální zlomky):

$$1. \int \frac{x^3}{x^2+3x+2} dx$$

$$x^3 \div (x^2 + 3x + 2) = x - 3 + \frac{7x + 6}{x^2 + 3x + 2}$$

$$\frac{7x + 6}{x^2 + 3x + 2} = \frac{7x + 6}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1} = \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$$

$$\rightarrow 7x + 6 = A(x + 1) + B(x + 2) = (A + B)x + A + 2B$$

$$\rightarrow \begin{array}{lcl} x^1 : & 7 & = A + B \\ x^0 : & 6 & = A + 2B \end{array}$$

$$\Rightarrow A = 8, \quad B = -1$$

$$\Rightarrow \frac{7x + 6}{x^2 + 3x + 2} = \frac{8}{x + 2} - \frac{1}{x + 1}$$

$$\begin{aligned} \int \frac{x^3}{x^2 + 3x + 2} dx &= \int \left(x - 3 + \frac{7x + 6}{x^2 + 3x + 2} \right) dx = \int \left(x - 3 + \frac{8}{x + 2} - \frac{1}{x + 1} \right) dx = \\ &= \frac{x^2}{2} - 3x + 8 \ln |x + 2| - \ln |x + 1| + C \end{aligned}$$

1. Cvičení: Funkce více proměnných

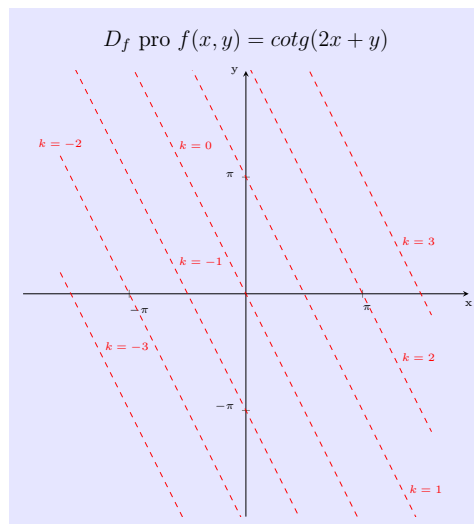
Definiční obory

Příklady: Určete a načrtněte definiční obor následujících funkcí

1. $f(x, y) = \cotg(2x + y)$

$$2x + y \neq k\pi, \quad k \in \mathbb{Z}$$

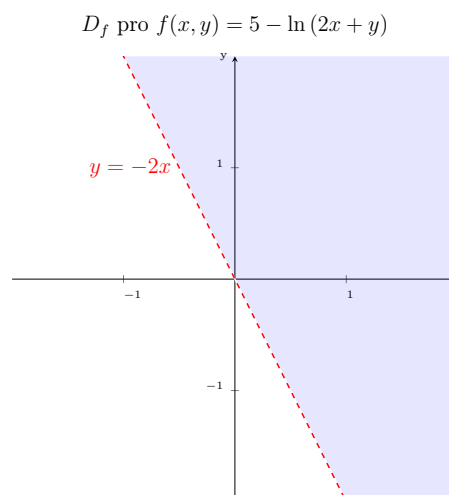
$$D_f = \{[x, y] \in \mathbb{R}^2 : y \neq k\pi - 2x, k \in \mathbb{Z}\}$$



2. $f(x, y) = 5 - \ln(2x + y)$

$$2x + y > 0$$

$$D_f = \{[x, y] \in \mathbb{R}^2 : y > -2x\}$$

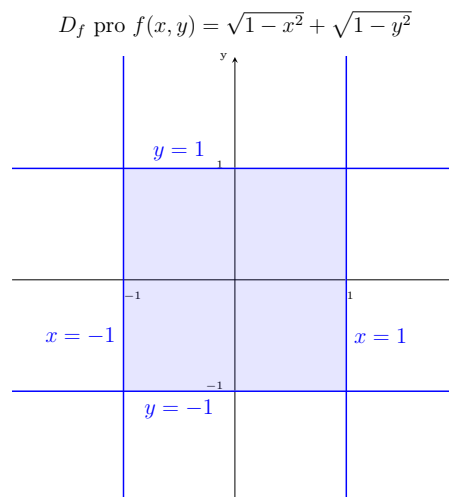


3. $f(x, y) = \sqrt{1 - x^2} + \sqrt{1 - y^2}$

$$1 - x^2 \geq 0 \wedge 1 - y^2 \geq 0$$

$$1 \geq x^2 \wedge 1 \geq y^2$$

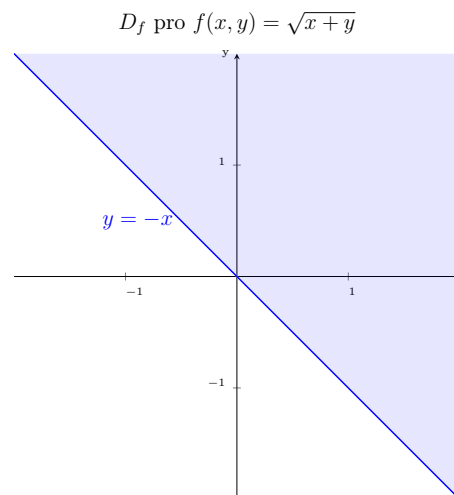
$$D_f = \{[x, y] \in \mathbb{R}^2 : 1 \geq |x| \wedge 1 \geq |y|\}$$



4. $f(x, y) = \sqrt{x + y}$

$$x + y \geq 0$$

$$D_f = \{[x, y] \in \mathbb{R}^2 : y \geq -x\}$$



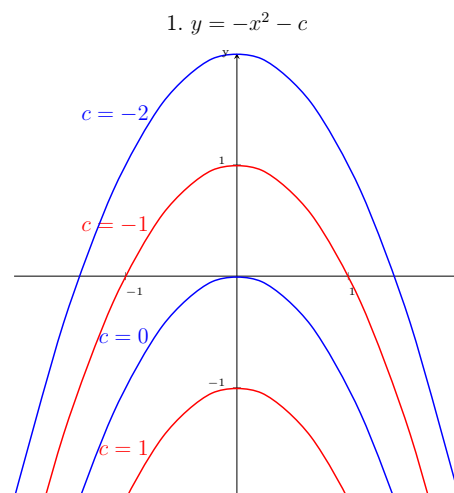
Hladiny funkcí

Příklady: Určete a načrtněte hladiny následujících funkcí

1. $f(x, y) = -x^2 - y$, $D_f = \mathbb{R}^2$

$$f(x, y) = -x^2 - y = c$$

$$y = -x^2 - c, c \in \mathbb{R}$$



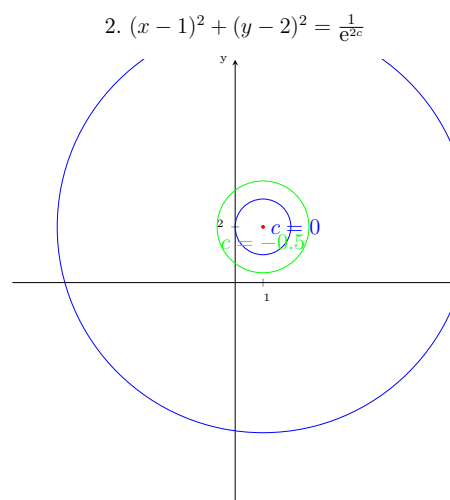
2. $f(x, y) = \ln \left(\frac{1}{\sqrt{(x-1)^2 + (y-2)^2}} \right)$, $D_f = \mathbb{R}^2 - \{(1, 2)\}$

$$\ln \left(\frac{1}{\sqrt{(x-1)^2 + (y-2)^2}} \right) = c$$

$$\frac{1}{\sqrt{(x-1)^2 + (y-2)^2}} = e^c$$

$$\sqrt{(x-1)^2 + (y-2)^2} = \frac{1}{e^c}$$

$$(x-1)^2 + (y-2)^2 = \frac{1}{e^{2c}}, c \in \mathbb{R}$$



Diferenciální počet funkcí více proměnných

Parciální derivace

Příklady: Určete všechny 1. parciální derivace následujících funkcí

1. $f(x, y) = y^x$

$$f'_x(x, y) = y^x \ln y, \quad f'_y(x, y) = xy^{x-1}$$

2. $f(x, y) = ye^{xy}$

$$f'_x(x, y) = y^2 e^{xy}, \quad f'_y(x, y) = xye^{xy} + e^{xy}$$

3. Určete všechny parciální derivace 2. řádu pro funkci $f(x, y) = e^{xe^y}$.

$$f'_x(x, y) = e^{xe^y} \cdot e^y, \quad f'_y(x, y) = e^{xe^y} \cdot xe^y$$

$$f''_{xx}(x, y) = e^{2y} e^{xe^y}$$

$$f''_{xy}(x, y) = e^y e^{xe^y} + e^y e^{xe^y} \cdot xe^y = e^y e^{xe^y} + xe^{2y} e^{xe^y}$$

$$f''_{yx}(x, y) = e^{xe^y} e^y \cdot xe^y + e^{xe^y} e^y = e^y e^{xe^y} + xe^{2y} e^{xe^y}$$

$$f''_{yy}(x, y) = e^{xe^y} \cdot xe^y \cdot xe^y + e^{xe^y} \cdot xe^y = x^2 e^{xe^y} e^{2y} + xe^{xe^y} e^y$$

4. Ukažte, že funkce $f(x, y) = \frac{y}{y^2 - x^2}$ vyhovuje rovnici $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}$.

$$\frac{\partial f}{\partial x} = \frac{2xy}{(y^2 - x^2)^2}$$

$$L = \frac{\partial^2 f}{\partial x^2} = \frac{2y(y^2 - x^2)^2 - 2xy \cdot 2(y^2 - x^2)(-2x)}{(y^2 - x^2)^4} = \frac{2y^3 + 6x^2 y}{(y^2 - x^2)^3}$$

$$\frac{\partial f}{\partial y} = -\frac{x^2 + y^2}{(y^2 - x^2)^2}$$

$$P = \frac{\partial^2 f}{\partial y^2} = -\frac{2y(y^2 - x^2)^2 - (y^2 + x^2) \cdot 2(y^2 - x^2)2y}{(y^2 - x^2)^4} = \frac{2y^3 + 6x^2 y}{(y^2 - x^2)^3}$$

$$L = P$$

Gradient a směrová derivace

1. Určete směr a velikost největšího růstu funkce $f(x, y) = \frac{x}{2y+x}$ v bodě $M = [1, -1]$. Dále derivaci funkce f v bodě $P = [1, 1]$ ve směru vektoru $\mathbf{s} = (1, -2)$.

$$\text{grad } f(x, y) = \left(\frac{2y}{(2y+x)^2}, -\frac{2x}{(2y+x)^2} \right)$$

$$\text{grad } f(M) = (-2, -2)$$

$$|\text{grad } f(M)| = 2\sqrt{2}$$

$$\frac{\partial f(P)}{\partial \mathbf{s}} = \left(\frac{2}{9}, -\frac{2}{9} \right) \cdot \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right) = \frac{2\sqrt{5}}{15}$$

2. Určete derivaci funkce $f(x, y, z) = x^2 + y^2 + z^2$ v bodě $M = [1, -2, 0]$ ve směru vektoru $\mathbf{s} = (1, 1, 1)$.

$$\text{grad } \varphi(x, y, z) = (2x, 2y, 2z)$$

$$\text{grad } \varphi(M) = (2, -4, 0)$$

$$|\mathbf{s}| = \sqrt{3}$$

$$\frac{\partial f(M)}{\partial \mathbf{s}} = \text{grad } \varphi(M) \cdot \frac{\mathbf{s}}{|\mathbf{s}|} = (2, -4, 0) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{3}$$

3. Určit maximální hodnotu derivace ve směru pro funkci $f(x, y, z) = x^2 + y^2 + 2xyz$ v bodě $M = [1, 3, 2]$.

$$\text{grad } \varphi(x, y, z) = (2x + 2yz, 2y + 2xz, 2xy)$$

$$\text{grad } \varphi(M) = (14, 10, 6)$$

$$|\text{grad } \varphi(M)| = \sqrt{332}$$

4. Určete derivaci funkce $f(x, y) = 4 - 2x - \frac{4}{3}y$ v bodě $P = [2, 0]$ podle vektoru $\mathbf{s} = (-2, 0)$.

$$\text{grad } f(x, y) = (-2, -\frac{4}{3})$$

$$\text{grad } f(P) = (-2, -\frac{4}{3})$$

$$\frac{\partial f(P)}{\partial \mathbf{s}} = \text{grad } \varphi(M) \cdot \mathbf{s} = (-2, -\frac{4}{3}) \cdot (-2, 0) = 4$$

$$x(t) = 2 - 2t, \quad y(t) = 0 + 0t, \quad t \in \mathbb{R}$$

$$F(t) = f(x(t), y(t)) = 4 - 2(2 - 2t) = 4t$$

$$F'(t) = 4 \rightarrow F'(0) = 4 \rightarrow \frac{\partial f(M)}{\partial \mathbf{s}} = 4$$

5. Určete gradient a jeho velikost pro funkci $f(x, y) = 2x^2 + y^2$ v bodě $A = [1, 1]$, $B = [-1, 2]$, nakreslete obrázek. Dále určete, ve kterých bodech roviny je gradient kolmý k ose x .

$$\text{grad } f(x, y) = (4x, 2y)$$

$$\text{grad } f(A) = (4, 2), \quad |\text{grad } f(A)| = \sqrt{20}$$

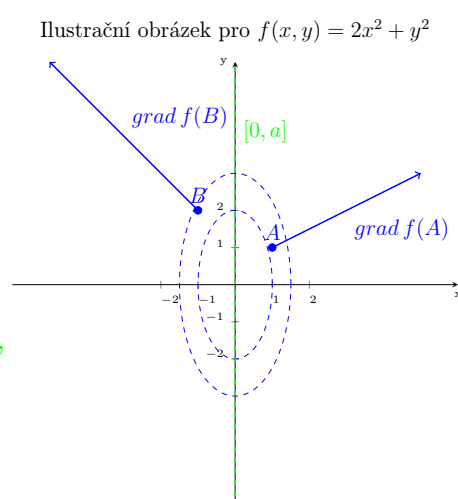
$$\text{grad } f(B) = (-4, 4), \quad |\text{grad } f(B)| = \sqrt{32}$$

$$(4x, 2y) \cdot (1, 0) = 0$$

$$4x = 0 \rightarrow x = 0$$

$$\text{Gradient je kolmý k ose } x \text{ v bodech } [a, 0],$$

$$a \in \mathbb{R}$$



6. Určete gradient a jeho velikost pro funkci $f(x, y) = x^2 + 2y^2$ v bodech $A = [1, 1]$, $B = [-1, 2]$, $C = [2, -1]$ a nakreslete obrázek. Dále určete, ve kterých bodech roviny svírá gradient s osou x úhel $\varphi = \frac{\pi}{4}$.

$$\text{grad } f(x, y) = (2x, 4y)$$

$$\text{grad } f(A) = (2, 4), \quad |\text{grad } f(A)| = \sqrt{20}$$

$$\text{grad } f(B) = (-2, 8), \quad |\text{grad } f(B)| = \sqrt{68}$$

$$\text{grad } f(C) = (4, -4), \quad |\text{grad } f(C)| = \sqrt{32}$$

$$(2x, 4y) \cdot (1, 0) = |(2x, 4y)| \cdot |(1, 0)| \cdot \cos \frac{\pi}{4}$$

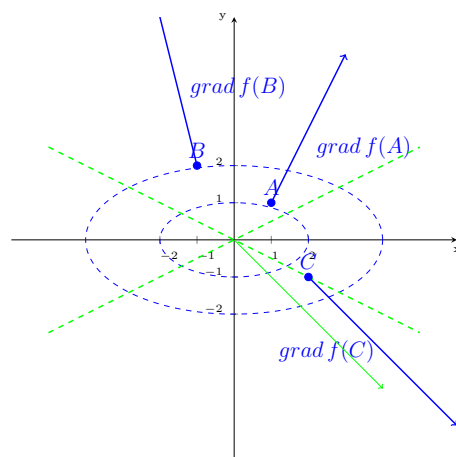
$$2x = \sqrt{4x^2 + 16y^2} \cdot \frac{\sqrt{2}}{2}$$

$$4x^2 = \frac{2}{4}(4x^2 + 16y^2)$$

$$2x^2 = 8y^2$$

$$\frac{1}{2}|x| = |y|$$

Ilustrační obrázek pro $f(x, y) = x^2 + 2y^2$



Totální diferenciál, tečná nadrovina

1. Určete totální diferenciál funkce $f(x, y) = \frac{z}{x^2 + y^2}$.

$$df = -\frac{2xz}{(x^2 + y^2)^2} dx - \frac{2yz}{(x^2 + y^2)^2} dy + \frac{1}{x^2 + y^2} dz$$

2. Určete tečnou nadrovinu funkce $f(x, y) = 9x^2 + y^2 - 25$ v bodech $T = [1, 3, ?]$.

$$T = [1, 3, -7]$$

$$\text{Rovnice tečné nadroviny } \tau : z - z_0 = f'_x(T)(x - x_0) + f'_y(T)(y - y_0)$$

$$f'_x(x, y) = 18x, \quad f'_x(T) = 18$$

$$f'_y(x, y) = 2y, \quad f'_y(T) = 6$$

$$\tau : z + 7 = 18(x - 1) + 6(y - 3)$$

$$0 = 18x + 6y - z - 43$$

Derivace implicitní funkce

Příklady: Určete derivace implicitně zadané funkce $F(x, y(x)) = 0$.

1. $y = x + \ln y$

$$F'_x(x, y) = 1$$

$$F'_y(x, y) = \frac{1}{y} - 1$$

$$y'(x) = \frac{y}{y-1}$$

$$2. \ y - xe^y + x = 0, \ A = [2, 0]$$

$$\begin{aligned} F'_x(x, y) &= -e^y + 1 \\ F'_y(x, y) &= 1 - xe^y \end{aligned}$$

$$\begin{aligned} y'(x) &= \frac{e^y - 1}{1 - xe^y} \\ y'(2) &= \frac{1 - 1}{1 - 2} = 0 \end{aligned}$$

2. Cvičení: Diferenciální počet funkcí více proměnných

Lokální extrémy funkce více proměnných

Příklady: Najděte lokální extrémy následujících funkcí

1. $f(x, y) = 3xy$

$$D_f = \mathbb{R}^2$$

Nutné podmínky:

$$f'_x(x, y) = 3y = 0 \rightarrow y = 0$$

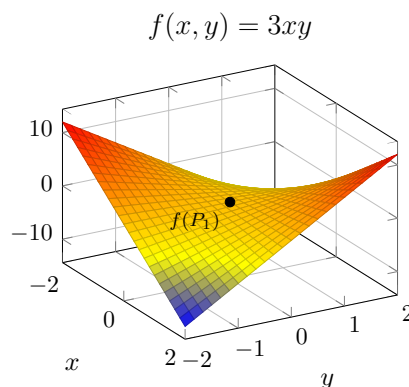
$$f'_y(x, y) = 3x = 0 \rightarrow x = 0$$

$$\rightarrow \text{Stacionární bod: } P_1 = [0, 0]$$

Postačující podmínky:

$$H(x, y) = H(P_1) = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

$$D_1(P_1) = 0, D_2(P_1) = -9 < 0 \rightarrow \text{nelze rozhodnout.}$$



2. $f(x, y) = x^2 + 2xy + 2y^2 - 3x - 5y$

$$D_f = \mathbb{R}^2$$

$$f'_x(x, y) = 2x + 2y - 3 = 0 \quad / \quad -1$$

$$f'_y(x, y) = 2x + 4y - 5 = 0$$

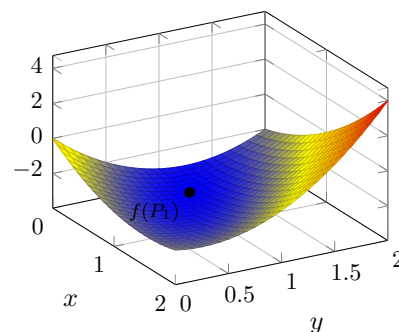
$$\Rightarrow 2y - 2 = 0 \rightarrow y = 1, x = \frac{1}{2}$$

$$\text{Stacionární bod: } P_1 = [\frac{1}{2}, 1]$$

$$H(x, y) = H(P_1) = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$$D_1(P_1) = 2 > 0, D_2(P_1) = 4 > 0 \rightarrow \text{v bodě } P_1 \text{ nastává lokální minimum } f(P_1) = -\frac{13}{4}.$$

$f(x, y) = x^2 + 2xy + 2y^2 - 3x - 5y$



3. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

$$D_f = \mathbb{R}^2$$

$$f'_x(x, y) = 6x^2 + y^2 + 10x = 0$$

$$f'_y(x, y) = 2xy + 2y = 0 \rightarrow y(x + 1) = 0$$

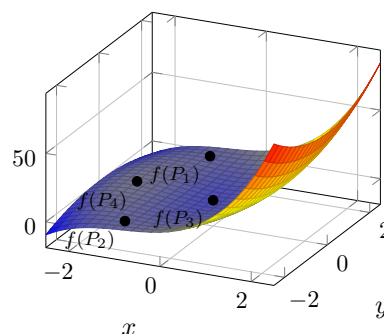
$$\rightarrow x = -1 \vee y = 0$$

$$x = -1: y^2 = 4 \rightarrow y = \pm 2$$

$$y = 0: 3x^2 + 5x = 0 \rightarrow x = 0, x = -\frac{5}{3}$$

$$\text{Stacionární body: } P_1 = [-1, 2], P_2 = [-1, -2], \\ P_3 = [0, 0], P_4 = [-\frac{5}{3}, 0]$$

$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$



$$H(x, y) = \begin{bmatrix} 12x + 10 & 2y \\ 2y & 2x + 2 \end{bmatrix}$$

$$H(P_1) = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix}, \quad H(P_2) = \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix}$$

$D_1(P_1) = -2 < 0, D_2(P_1) = -16 < 0 \rightarrow$ v bodě P_1 nenastává lokální extrém.

$D_1(P_2) = -2 < 0, D_2(P_2) = -16 < 0 \rightarrow$ v bodě P_2 nenastává lokální extrém.

$$H(P_3) = \begin{bmatrix} 10 & 0 \\ 0 & 2 \end{bmatrix}, \quad H(P_4) = \begin{bmatrix} -10 & 0 \\ 0 & -\frac{4}{3} \end{bmatrix}$$

$D_1(P_3) = 10 > 0, D_2(P_3) = 20 > 0 \rightarrow$ v bodě P_3 nastává lokální minimum, $f(P_3) = 0$.

$D_1(P_4) = -10, D_2(P_4) = \frac{40}{3} \rightarrow$ v bodě P_4 nastává lokální maximum, $f(P_4) = \frac{125}{27}$.

4. $f(x, y) = x^3 + 3xy^2 - 15x - 12y$

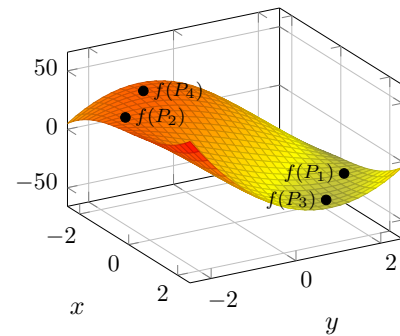
$$f(x, y) = x^3 + 3xy^2 - 15x - 12y$$

$$D_f = \mathbb{R}^2$$

$$f'_x(x, y) = 3x^2 + 3y^2 - 15 = 0 \rightarrow x^2 + y^2 - 5 = 0$$

$$f'_y(x, y) = 6xy - 12 = 0 \rightarrow y = \frac{2}{x}, x \neq 0$$

Vyloučené $x = 0$ rovnici nesplňuje.



$$x^2 + \frac{4}{x^2} - 5 = 0 \rightarrow x^4 - 5x^2 + 4 = 0 \rightarrow (x^2 - 1)(x^2 - 4) = 0 \rightarrow x = \pm 1 \vee x = \pm 2$$

$$x = 1 : y = 2 \quad x = -1 : y = -2 \quad x = 2 : y = 1 \quad x = -2 : y = -1$$

Stacionární body: $P_1 = [1, 2], P_2 = [-1, -2], P_3 = [2, 1], P_4 = [-2, -1]$

$$H(x, y) = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

$$H(P_1) = \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix}, \quad H(P_2) = \begin{bmatrix} -6 & -12 \\ -12 & -6 \end{bmatrix}$$

$D_1(P_1) = 6 > 0, D_2(P_1) = -108 < 0 \rightarrow$ v bodě P_1 nenastává lokální extrém.

$D_1(P_2) = -6 < 0, D_2(P_2) = -108 < 0 \rightarrow$ v bodě P_2 nenastává lokální extrém.

$$H(P_3) = \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix}, \quad H(P_4) = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix}$$

$D_1(P_3) = 12 > 0, D_2(P_3) = 108 > 0 \rightarrow$ v bodě P_3 nastává lokální minimum, $f(P_3) = -28$.

$D_1(P_4) = -12 < 0, D_2(P_4) = 108 > 0 \rightarrow$ v bodě P_4 nastává lokální maximum, $f(P_4) = 28$.

3. Cvičení: Integrální počet funkcí více proměnných

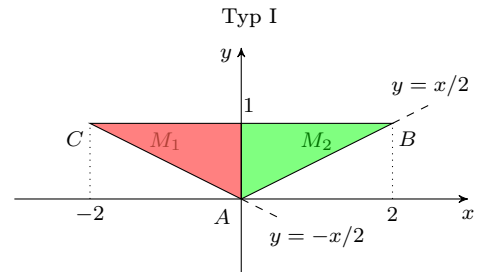
Dvojný integrál

Příklady: Pro integrál $\iint_M f(x, y) dx dy$ určete oba typy mezí a načrtněte množinu M :

1. Množina M je trojúhelník s vrcholy $A = [0, 0]$, $B = [2, 1]$, $C = [-2, 1]$.

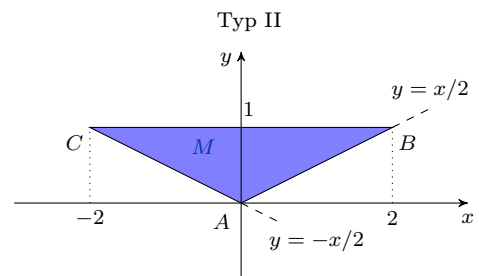
$$M_1: \quad -2 \leq x \leq 0, \\ -\frac{x}{2} \leq y \leq 1$$

$$M_2: \quad 0 \leq x \leq 2, \\ \frac{x}{2} \leq y \leq 1$$



$$\iint_M f(x, y) dx dy = \int_{-2}^0 \int_{-x/2}^1 f(x, y) dy dx + \int_0^2 \int_{x/2}^1 f(x, y) dy dx$$

$$M: \quad 0 \leq y \leq 1, \\ -2y \leq x \leq 2y$$

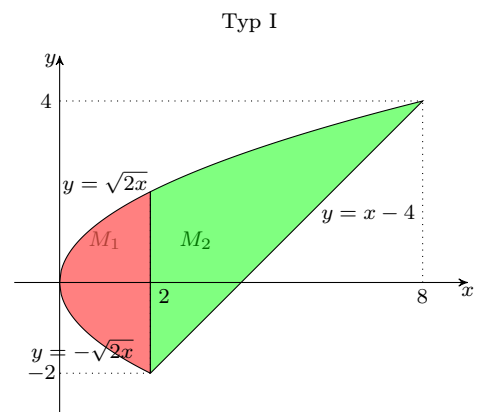


$$\iint_M f(x, y) dx dy = \int_0^1 \int_{-2y}^{2y} f(x, y) dx dy$$

2. Množina M je definována rovnostmi $y = x - 4$, $y^2 = 2x$.

$$M_1: \quad 0 \leq x \leq 2, \\ -\sqrt{2x} \leq y \leq \sqrt{2x}$$

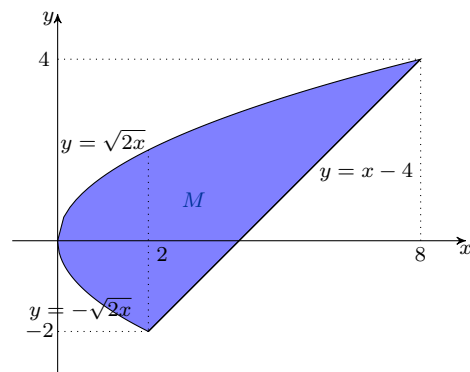
$$M_2: \quad 2 \leq x \leq 8, \\ x - 4 \leq y \leq \sqrt{2x}$$



$$\iint_M f(x, y) dx dy = \int_0^2 \int_{-\sqrt{2x}}^{\sqrt{2x}} f(x, y) dy dx + \int_2^8 \int_{x-4}^{\sqrt{2x}} f(x, y) dy dx$$

Typ II

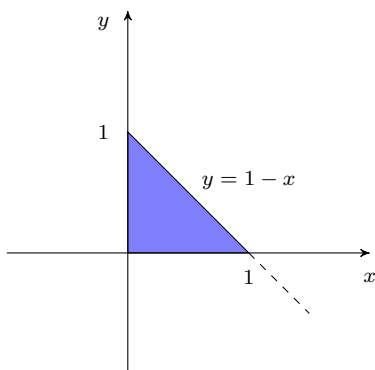
$$M: \quad -2 \leq y \leq 4, \\ \frac{y^2}{2} \leq x \leq y+4$$



$$\iint_M f(x, y) \, dx \, dy = \int_{-2}^4 \int_{y^2/2}^{y+4} f(x, y) \, dx \, dy$$

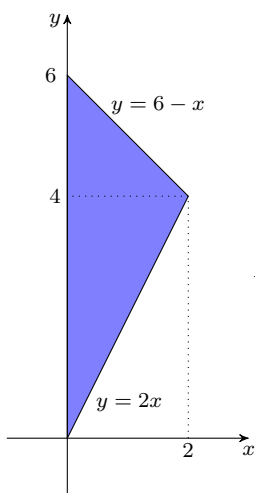
Příklady: Jsou dány následující integrály, načrtněte integrační oblast a zaměňte pořadí integrování:

$$1. \quad I = \int_0^1 \int_0^{1-x} f(x, y) \, dy \, dx$$



$$I = \int_0^1 \left(\int_0^{1-x} f(x, y) \, dy \right) dx = \int_0^1 \left(\int_0^{1-y} f(x, y) \, dx \right) dy$$

$$2. \quad I = \int_0^4 \int_0^{\frac{y}{2}} f(x, y) \, dx \, dy + \int_4^6 \int_0^{6-y} f(x, y) \, dx \, dy$$

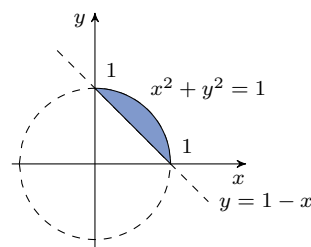


$$I = \int_0^4 \left(\int_0^{\frac{y}{2}} f(x, y) \, dx \right) dy + \int_4^6 \left(\int_0^{6-y} f(x, y) \, dx \right) dy = \int_0^2 \left(\int_{2x}^{6-x} f(x, y) \, dy \right) dx$$

Příklady: Spočítejte následující dvojné integrály a načrtněte integrační oblasti:

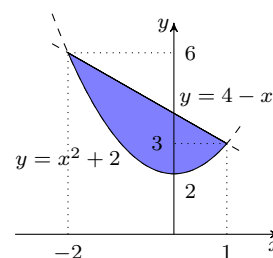
1. $I = \iint_M xy^2 dx dy$, kde M je určena vztahy: $x^2 + y^2 - 1 \leq 0$, $x + y - 1 \geq 0$.

$$\begin{aligned} I &= \int_0^1 \left(\int_{1-y}^{\sqrt{1-y^2}} xy^2 dx \right) dy = \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_{1-y}^{\sqrt{1-y^2}} dy = \\ &= \int_0^1 (y^3 - y^4) dx = \left[\frac{1}{4} y^4 - \frac{1}{5} y^5 \right]_0^1 = \frac{1}{20}. \end{aligned}$$



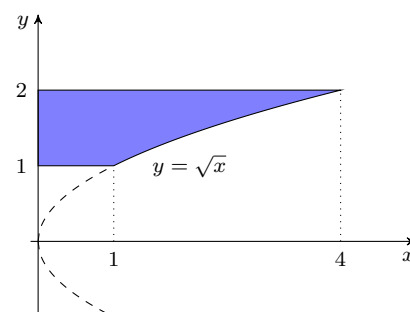
2. $I = \iint_M y dx dy$, kde M je určena vztahy: $x^2 - y + 2 = 0$, $x + y - 4 = 0$.

$$\begin{aligned} I &= \int_{-2}^1 \left(\int_{x^2+2}^{4-x} y dy \right) dx = \int_{-2}^1 \left[\frac{1}{2} y^2 \right]_{x^2+2}^{4-x} dx = \\ &= \frac{1}{2} \int_{-2}^1 (-x^4 - 3x^2 - 8x + 12) dx = \\ &= \frac{1}{2} \left[-\frac{1}{5} x^5 - x^3 - 4x^2 + 12x \right]_{-2}^1 = \frac{81}{5}. \end{aligned}$$



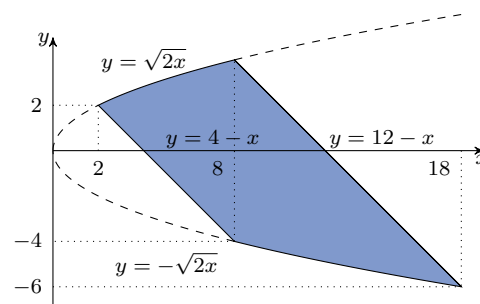
3. $I = \iint_M e^{\frac{x}{y}} dx dy$, kde M je určena vztahy: $x = 0$, $y = 1$, $y = 2$, $y^2 = x$.

$$\begin{aligned} I &= \int_1^2 \left(\int_0^{y^2} e^{\frac{x}{y}} dx \right) dy = \int_1^2 \left[y e^{\frac{x}{y}} \right]_0^{y^2} dy = \int_1^2 (y e^y - y) dy = \\ &= \left[\begin{matrix} D & I \\ y & e^y \\ 1 & e^y \end{matrix} \right] = \left[y e^y - e^y - \frac{1}{2} y^2 \right]_1^2 = e^2 - \frac{3}{2}. \end{aligned}$$



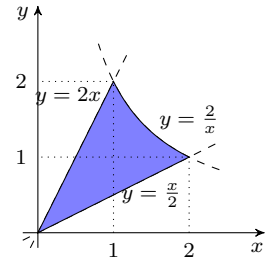
4. $I = \iint_M dx dy$, kde M je určena vztahy: $x + y = 4$, $x + y = 12$, $y^2 = 2x$.

$$\begin{aligned} &= \int_2^8 \left(\int_{4-x}^{\sqrt{2x}} dy \right) dx + \int_8^{18} \left(\int_{-\sqrt{2x}}^{12-x} dy \right) dx = \\ &= \int_2^8 (\sqrt{2x} + x - 4) dx + \int_8^{18} (\sqrt{2x} - x + 12) dx = \\ &= \left[\frac{2\sqrt{2}}{3} x^{3/2} + \frac{1}{2} x^2 - 4x \right]_2^8 + \\ &+ \left[\frac{2\sqrt{2}}{3} x^{3/2} - \frac{1}{2} x^2 + 12x \right]_8^{18} = \frac{74}{3} + \frac{122}{3} = \frac{196}{3}. \end{aligned}$$



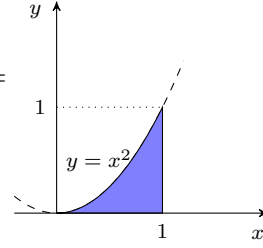
5. $I = \iint_M (x^2 + y) dx dy$, kde M je určena vztahy: $y = \frac{x}{2}$, $y = 2x$, $xy = 2$, $x \geq 0$.

$$\begin{aligned} I &= \int_0^1 \left(\int_{\frac{x}{2}}^{2x} (x^2 + y) dy \right) dx + \int_1^2 \left(\int_{\frac{x}{2}}^{\frac{2}{x}} (x^2 + y) dy \right) dx = \\ &= \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{\frac{x}{2}}^{2x} dx + \int_1^2 \left[x^2 y + \frac{y^2}{2} \right]_{\frac{x}{2}}^{\frac{2}{x}} dx = \\ &= \int_0^1 \left(\frac{3}{2}x^3 + \frac{15}{8}x^2 \right) dx + \int_1^2 \left(2x + \frac{2}{x^2} - \frac{x^3}{2} - \frac{x^2}{8} \right) dx = \\ &= \left[\frac{3}{8}x^4 + \frac{5}{8}x^3 \right]_0^1 + \left[x^2 - \frac{2}{x} - \frac{x^4}{8} - \frac{x^3}{24} \right]_1^2 = \frac{17}{6}. \end{aligned}$$



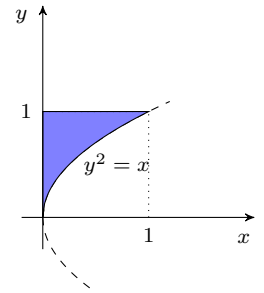
6. $I = \iint_M (x^2 + y^2) dx dy$, kde M je určena vztahy: $x = 0$, $x = 1$, $y = 0$, $y = x^2$.

$$\begin{aligned} I &= \int_0^1 \left(\int_0^{x^2} (x^2 + y^2) dy \right) dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{x^2} dx = \\ &= \int_0^1 \left(x^4 + \frac{x^6}{3} \right) dx = \left[\frac{x^5}{5} + \frac{x^7}{21} \right]_0^1 = \frac{1}{5} + \frac{1}{21}. \end{aligned}$$



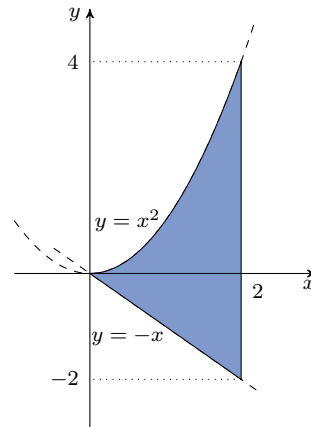
7. $I = \iint_M e^{\frac{x}{y}} dx dy$, kde M je určena vztahy: $x = 0$, $y = 0$, $y = 1$, $y^2 = x$.

$$\begin{aligned} &= \int_0^1 \left(\int_0^{y^2} e^{\frac{x}{y}} dx \right) dy = \int_0^1 \left[y e^{\frac{x}{y}} \right]_0^{y^2} dy = \int_0^1 [y e^y - y] dy = \\ &= \left| \begin{array}{cc} D & I \\ y & e^y \\ 1 & e^y \end{array} \right| = \left[y e^y - e^y - \frac{y^2}{2} \right]_0^1 = \frac{1}{2}. \end{aligned}$$



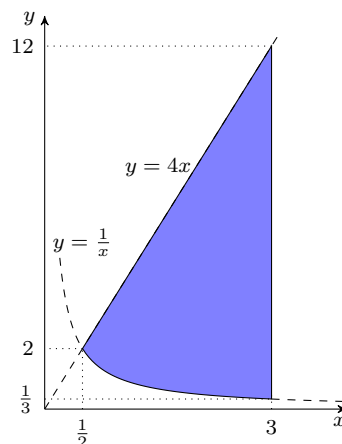
8. $I = \iint_M xy^2 dx dy$, kde M je určena vztahy: $x = 0$, $y = -x$, $x = 2$, $y = x^2$.

$$\begin{aligned} I &= \int_0^2 \left(\int_{-x}^{x^2} xy^2 dy \right) dx = \int_0^2 \left[x \frac{y^3}{3} \right]_{-x}^{x^2} dy = \\ &= \int_0^2 \left(\frac{x^7}{3} + \frac{x^4}{3} \right) dy = \frac{1}{3} \left[\frac{x^8}{8} + \frac{x^5}{5} \right]_0^2 = \frac{2^6}{5}. \end{aligned}$$



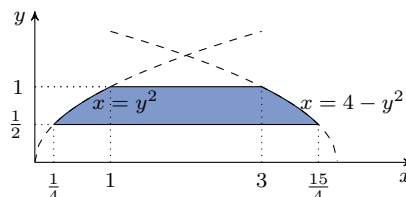
9. $I = \iint_M \frac{x^2}{y^2} dx dy$, kde M je určena vztahy: $xy = 1$, $y = 4x$, $x = 3$.

$$\begin{aligned} I &= \int_{1/2}^3 \left(\int_{\frac{1}{x}}^{4x} \frac{x^2}{y^2} dy \right) dx = \int_{1/2}^3 x^2 \left[-\frac{1}{y} \right]_{\frac{1}{x}}^{4x} dx = \\ &= \int_{1/2}^3 \left(-\frac{x}{4} + x^3 \right) dx = \left[-\frac{x^2}{8} + \frac{x^4}{4} \right]_{1/2}^3 = \frac{1225}{64}. \end{aligned}$$



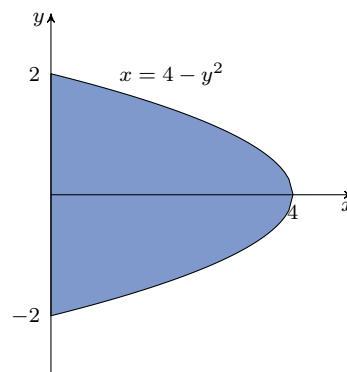
10. $I = \iint_M \frac{y}{x+y^2} dx dy$, kde M je určena vztahy: $y = 1$, $y = \frac{1}{2}$, $x = y^2$, $x = 4 - y^2$.

$$\begin{aligned} I &= \int_{\frac{1}{2}}^1 \left(\int_{y^2}^{4-y^2} \frac{y}{x+y^2} dx \right) dy = \int_{\frac{1}{2}}^1 [y \ln |x+y^2|]_{y^2}^{4-y^2} dy = \\ &= \int_{\frac{1}{2}}^1 (y \ln |4-y^2+y^2| - y \ln |2y^2|) dy = \\ &= \int_{\frac{1}{2}}^1 (y \ln 4 - y \ln |2y^2|) dy = \int_{\frac{1}{2}}^1 (y \ln 2 - 2y \ln y) dy = \\ &= \left[\frac{D}{\ln y} \frac{I}{\frac{1}{y}} \right] = \left[\frac{y^2}{2} \ln 2 - 2 \frac{y^2}{2} \ln y + \frac{y^2}{2} \right]_{\frac{1}{2}}^1 = \frac{1}{8} \ln 2 + \frac{3}{8} \end{aligned}$$



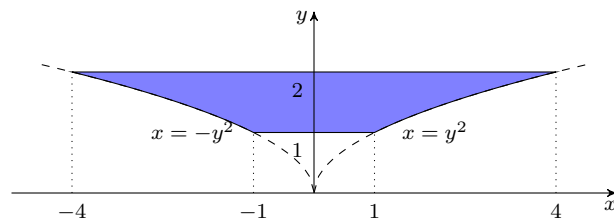
11. $I = \iint_M x^3 y dx dy$, kde M je určena vztahy: $x = 4 - y^2$, $x \geq 0$.

$$\begin{aligned} I &= \int_{-2}^2 \left(\int_0^{4-y^2} x^3 y dx \right) dy = \int_{-2}^2 \left[\frac{x^4}{4} y \right]_0^{4-y^2} dy = \\ &= \int_{-2}^2 \left(64y - 64y^3 + 24y^5 - 4y^7 + \frac{y^9}{4} \right) dy = \\ &= \left[32y^2 - 16y^4 + 4y^6 - \frac{1}{2}y^8 + \frac{y^{10}}{40} \right]_{-2}^2 = 0 \end{aligned}$$



12. $I = \iint_M (x^2 + y^2) dx dy$, kde M je určena vztahy: $y = 1$, $y = 2$, $x = -y^2$, $x = y^2$.

$$\begin{aligned}
 I &= \int_1^2 \left(\int_{-y^2}^{y^2} (x^2 + y^2) dx \right) dy = \\
 &= \int_1^2 \left[\frac{x^3}{3} + xy^2 \right]_{-y^2}^{y^2} dy = \\
 &= \int_1^2 \left(\frac{2}{3}y^6 + 2y^4 \right) dy = \\
 &= \left[\frac{2}{21}x^7 + \frac{2y^5}{5} \right]_1^2 = \frac{2572}{105}
 \end{aligned}$$



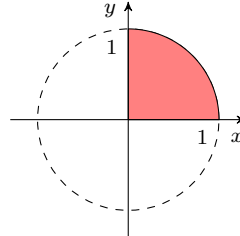
4. Cvičení: Integrální počet funkcí více proměnných

Dvojný integrál - substituce

Příklady: Vypočtete integrál $\iint_M f(x, y) dx dy$ a načrtněte množinu M :

1. $I = \iint_M \arctg \frac{y}{x} dx dy$, kde M je určena nerovnostmi $x \geq 0, y \geq 0, x^2 + y^2 \leq 1$.

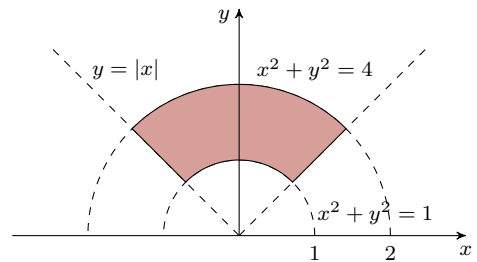
$$\begin{aligned} x &= r \cos \varphi, & 0 \leq r \leq 1, \\ y &= r \sin \varphi, & 0 \leq \varphi \leq \frac{\pi}{2}, \\ |J| &= r \end{aligned}$$



$$I = \int_0^{\frac{\pi}{2}} \left(\int_0^1 \arctg \left(\frac{r \sin \varphi}{r \cos \varphi} \right) r dr \right) d\varphi = \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2} \varphi \right]_0^1 d\varphi = \frac{1}{2} \left[\frac{\varphi^2}{2} \right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{16}$$

2. $I = \iint_M 2(x^2 + y^2) dx dy$, kde M je určena nerovnostmi $|x| \leq y, 1 \leq x^2 + y^2 \leq 4$.

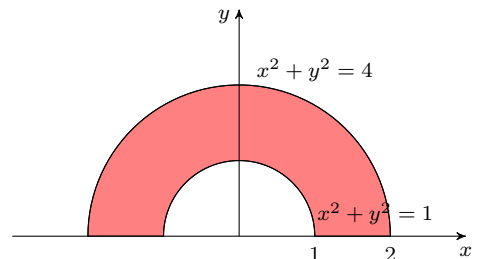
$$\begin{aligned} x &= r \cos \varphi, & 1 \leq r \leq 2, \\ y &= r \sin \varphi, & \frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}, \\ |J| &= r \end{aligned}$$



$$I = 2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\int_1^2 (r^2 \cos^2 \varphi + r^2 \sin^2 \varphi) r dr \right) d\varphi = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[\frac{r^4}{2} \right]_1^2 d\varphi = \frac{15}{2} [\varphi]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \frac{15\pi}{4}$$

3. $I = \iint_M \frac{1}{\sqrt{x^2 + y^2}} dx dy$, kde M je určena nerovnostmi $y \geq 0, 1 \leq x^2 + y^2 \leq 4$.

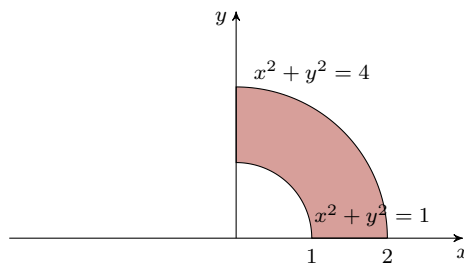
$$\begin{aligned} x &= r \cos \varphi, & 1 \leq r \leq 2, \\ y &= r \sin \varphi, & 0 \leq \varphi \leq \pi, \\ |J| &= r \end{aligned}$$



$$I = \int_0^{\pi} \left(\int_1^2 \frac{1}{\sqrt{(r \cos \varphi)^2 + (r \sin \varphi)^2}} r dr \right) d\varphi = \int_0^{\pi} [r]_1^2 d\varphi = \int_0^{\pi} d\varphi = [\varphi]_0^{\pi} = \pi$$

4. $I = \iint_M x \arctg \frac{y}{x} dx dy$, kde M je určena nerovnostmi $x \geq 0, y \geq 0, 1 \leq x^2 + y^2 \leq 4$.

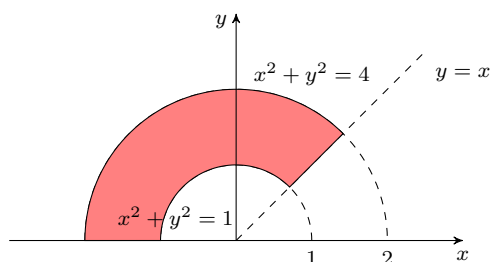
$$\begin{aligned} x &= r \cos \varphi, & 1 \leq r \leq 2, \\ y &= r \sin \varphi, & 0 \leq \varphi \leq \frac{\pi}{2}, \\ |J| &= r \end{aligned}$$



$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \left(\int_1^2 (r^2 \varphi \cos \varphi) r dr \right) d\varphi = \int_0^{\frac{\pi}{2}} \varphi \cos \varphi \left[\frac{r^3}{3} \right]_1^2 d\varphi = \frac{7}{3} \int_0^{\frac{\pi}{2}} \varphi \cos \varphi d\varphi = \\ &= \begin{vmatrix} D & I \\ \varphi & \cos \varphi \\ 1 & \sin \varphi \end{vmatrix} = \frac{7}{3} [\varphi \sin \varphi + \cos \varphi]_0^{\frac{\pi}{2}} = \frac{7}{3} \left(\frac{\pi}{2} - 1 \right) \end{aligned}$$

5. $I = \iint_M \frac{x}{\sqrt{x^2 + y^2}} dx dy$, kde M je určena nerovnostmi $y \geq x, y \geq 0, 1 \leq x^2 + y^2 \leq 4$.

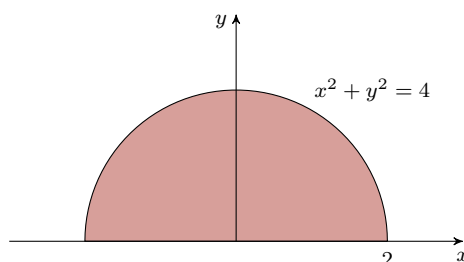
$$\begin{aligned} x &= r \cos \varphi, & 1 \leq r \leq 2, \\ y &= r \sin \varphi, & \frac{\pi}{4} \leq \varphi \leq \pi, \\ |J| &= r \end{aligned}$$



$$I = \int_{\frac{\pi}{4}}^{\pi} \left(\int_1^2 \frac{r \cos \varphi}{\sqrt{(r \cos \varphi)^2 + (r \sin \varphi)^2}} r dr \right) d\varphi = \int_{\frac{\pi}{4}}^{\pi} \frac{3}{2} \cos \varphi d\varphi = \frac{3}{2} [\sin \varphi]_{\frac{\pi}{4}}^{\pi} = -\frac{3}{4} \sqrt{2}$$

6. $I = \iint_M dx dy$, kde M je určena nerovnostmi $y \geq 0, x^2 + y^2 \leq 4$.

$$\begin{aligned} x &= r \cos \varphi, & 0 \leq r \leq 2, \\ y &= r \sin \varphi, & 0 \leq \varphi \leq \pi, \\ |J| &= r \end{aligned}$$

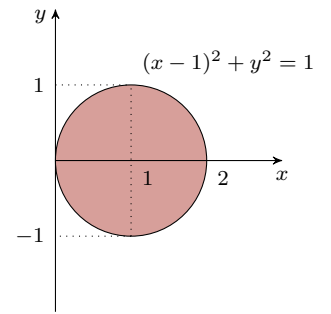


$$I = \int_0^{\pi} \left(\int_0^2 r dr \right) d\varphi = \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^2 d\varphi = \int_0^{\pi} 2 d\varphi = 2\pi$$

7. $I = \iint_M \sqrt{x^2 + y^2} dx dy$, kde M je určena nerovnostmi $x^2 + y^2 - 2x \leq 0$.

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad |J| = r$$

$$\begin{aligned} x^2 + y^2 \leq 2x &\rightarrow r^2 \leq 2r \cos \varphi \rightarrow 0 \leq r \leq 2 \cos \varphi, \\ \Rightarrow \cos \varphi \geq 0 &\rightarrow \frac{3}{2}\pi \leq \varphi \leq \frac{\pi}{2} \end{aligned}$$

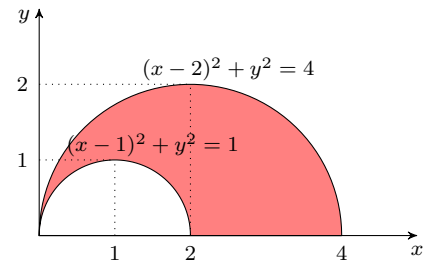


$$\begin{aligned} &= \int_{\frac{3}{2}\pi}^{\frac{\pi}{2}} \left(\int_0^{2 \cos \varphi} r^2 dr \right) d\varphi = \int_{\frac{3}{2}\pi}^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^{2 \cos \varphi} d\varphi = \frac{8}{3} \int_{\frac{3}{2}\pi}^{\frac{\pi}{2}} \underbrace{\cos^3 \varphi}_{(1-\sin^2 \varphi) \cos \varphi} d\varphi = \\ &= \left| \begin{array}{l} t = \sin \varphi \\ dt = \cos \varphi d\varphi \\ \int (1-t^2) dt = t - \frac{t^3}{3} + C \end{array} \right| = \frac{8}{3} \left[\sin \varphi - \frac{\sin^3 \varphi}{3} \right]_{\frac{3}{2}\pi}^{\frac{\pi}{2}} = \frac{8}{3} \left(2 - \frac{2}{3} \right) = \frac{32}{9} \end{aligned}$$

8. $I = \iint_M xy dx dy$, kde M je určena nerovnostmi $x^2 + y^2 - 2x \geq 0$, $x^2 + y^2 - 4x \leq 0$, $y \geq 0$.

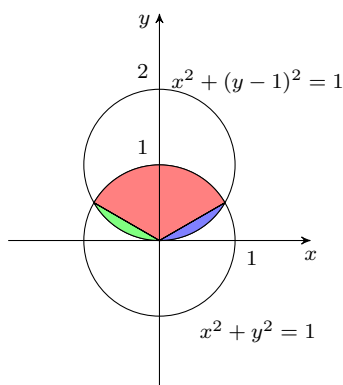
$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad |J| = r$$

$$\begin{aligned} x^2 + y^2 \geq 2x &\rightarrow r^2 \geq 2r \cos \varphi \rightarrow r \geq 2 \cos \varphi \geq 0, \\ x^2 + y^2 \leq 4x &\rightarrow r^2 \leq 4r \cos \varphi \rightarrow 0 \leq r \leq 4 \cos \varphi, \\ \Rightarrow 2 \cos \varphi \leq r \leq 4 \cos \varphi \\ \Rightarrow \cos \varphi \geq 0 \\ y \geq 0 &\rightarrow r \sin \varphi \geq 0 \rightarrow \sin \varphi \geq 0 \\ \Rightarrow \cos \varphi \geq 0 \quad \wedge \quad \sin \varphi \geq 0 &\Rightarrow \varphi \in \langle 0, \frac{\pi}{2} \rangle \end{aligned}$$



$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \left(\int_{2 \cos \varphi}^{4 \cos \varphi} r^3 \sin \varphi \cos \varphi dr \right) d\varphi = \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi \left[\frac{r^4}{4} \right]_{2 \cos \varphi}^{4 \cos \varphi} d\varphi = \int_0^{\frac{\pi}{2}} 60 \sin \varphi \cos^5 \varphi d\varphi = \\ &= \left| \begin{array}{l} t = \cos \varphi \\ dt = -\sin \varphi d\varphi \\ -\int t^5 dt = -\frac{t^6}{6} + C \end{array} \right| = -60 \left[\frac{1}{6} \cos^6 \varphi \right]_0^{\frac{\pi}{2}} d\varphi = 10 \end{aligned}$$

9. $I = \iint_M \sqrt{x^2 + y^2} dx dy$, kde M je určena nerovnostmi $x^2 + y^2 \leq 1$, $x^2 + y^2 - 2y \leq 0$.



$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad |J| = r$$

$$x^2 + y^2 \leq 1 \rightarrow 0 \leq r \leq 1,$$

$$x^2 + y^2 \leq 2y \rightarrow r^2 \leq 2r \sin \varphi \rightarrow 0 \leq r \leq 2 \sin \varphi,$$

$$\Rightarrow \begin{array}{llll} 0 \leq 2 \sin \varphi \leq 1 & \rightarrow & 0 \leq \sin \varphi \leq \frac{1}{2} & \rightarrow \varphi \in \langle 0, \frac{\pi}{6} \rangle \cup \langle \frac{\pi}{6}, \frac{5\pi}{6} \rangle \\ 1 \leq 2 \sin \varphi & \rightarrow & \frac{1}{2} \leq \sin \varphi & \rightarrow \varphi \in \langle \frac{\pi}{6}, \frac{5\pi}{6} \rangle \end{array}$$

$$\Rightarrow r \in \langle 0, 1 \rangle \quad \varphi \in \langle \frac{\pi}{6}, \frac{5\pi}{6} \rangle$$

$$\Rightarrow r \in \langle 2 \sin \varphi, 1 \rangle \quad \varphi \in \langle 0, \frac{\pi}{6} \rangle$$

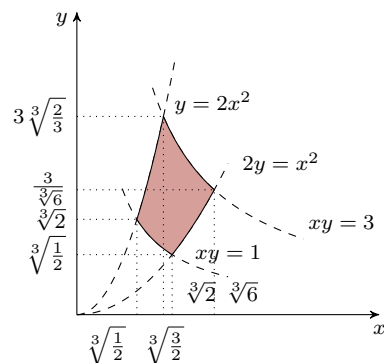
$$\Rightarrow r \in \langle 2 \sin \varphi, 1 \rangle \quad \varphi \in \langle \frac{5\pi}{6}, \pi \rangle$$

$$\begin{aligned} I &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\int_0^1 r^2 dr \right) d\varphi + \int_0^{\frac{\pi}{6}} \left(\int_{2 \sin \varphi}^1 r^2 dr \right) d\varphi + \int_{\frac{5\pi}{6}}^{\pi} \left(\int_{2 \sin \varphi}^1 r^2 dr \right) d\varphi = \\ &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[\frac{r^3}{3} \right]_0^1 d\varphi + \int_0^{\frac{\pi}{6}} \left[\frac{r^3}{3} \right]_{2 \sin \varphi}^1 d\varphi + \int_{\frac{5\pi}{6}}^{\pi} \left[\frac{r^3}{3} \right]_{2 \sin \varphi}^1 d\varphi = \\ &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{3} d\varphi + \int_0^{\frac{\pi}{6}} \left(\frac{1}{3} - \frac{8}{3} \sin^3 \varphi \right) d\varphi + \int_{\frac{5\pi}{6}}^{\pi} \left(\frac{1}{3} - \frac{8}{3} \sin^3 \varphi \right) d\varphi = \\ &= \frac{1}{3} \left(\frac{5\pi}{6} - \frac{\pi}{6} + \frac{\pi}{6} - 0 + \pi - \frac{5\pi}{6} \right) - \frac{8}{3} \int_0^{\frac{\pi}{6}} \sin \varphi (1 - \cos^2 \varphi) d\varphi - \frac{8}{3} \int_{\frac{5\pi}{6}}^{\pi} \sin \varphi (1 - \cos^2 \varphi) d\varphi = \\ &= \left| \begin{array}{l} t = \cos \varphi \\ dt = -\sin \varphi d\varphi \\ -\int (1 - t^2) dt = -t + \frac{t^3}{3} + C \end{array} \right| = \frac{8}{3} \int_0^{\frac{\pi}{6}} \left(\cos \varphi - \frac{1}{3} \cos^3 \varphi \right) d\varphi + \frac{8}{3} \int_{\frac{5\pi}{6}}^{\pi} \left(\cos \varphi - \frac{1}{3} \cos^3 \varphi \right) d\varphi = \\ &= \frac{6\sqrt{3}}{3} - \frac{32}{9} \end{aligned}$$

10. $I = \iint_M x^3 dx dy$, kde M je určena rovnostmi $xy = 1$, $xy = 3$, $y = \frac{x^2}{2}$, $y = 2x^2$. Použijte transformaci: $xy = u$ a $\frac{y}{x^2} = v$.

$$1 \leq u \leq 3,$$

$$\frac{1}{2} \leq v \leq 2$$



$$y = \frac{u}{x}, \quad y = vx^2$$

$$\rightarrow \frac{u}{x} = vx^2 \quad \rightarrow \quad x^3 = \frac{u}{v}$$

$$\rightarrow x = \sqrt[3]{\frac{u}{v}} \quad \rightarrow \quad y = \sqrt[3]{u^2 v}$$

Jacobián:

$$x = x(u, v) = \sqrt[3]{\frac{u}{v}}, \quad y = y(u, v) = \sqrt[3]{u^2 v}.$$

$$|J| = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array} \right| = \left| \begin{array}{cc} \frac{1}{3}u^{-2/3}v^{-1/3} & -\frac{1}{3}u^{1/3}v^{-4/3} \\ \frac{2}{3}u^{-1/3}v^{1/3} & \frac{1}{3}u^{2/3}v^{-2/3} \end{array} \right| = \frac{1}{9}v^{-1} + \frac{2}{9}v^{-1} = \frac{1}{3v}$$

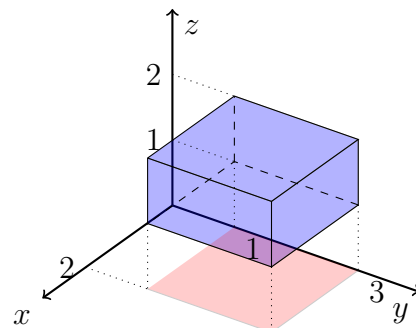
$$\int_{\frac{1}{2}}^2 \left(\int_1^3 \frac{u}{v} \frac{1}{3v} du \right) dv = \frac{1}{3} \int_{\frac{1}{2}}^2 \left[\frac{u^2}{2} \frac{1}{v^2} \right]_1^3 dv = \frac{4}{3} \int_{\frac{1}{2}}^2 \frac{1}{v^2} dv = \frac{4}{3} \left[-\frac{1}{v} \right]_{\frac{1}{2}}^2 = 2$$

5. Cvičení: Trojný integrál

Příklady: Vypočtěte integrál $\iiint_{\Omega} f(x, y, z) dx dy dz$ a načrtněte množinu Ω :

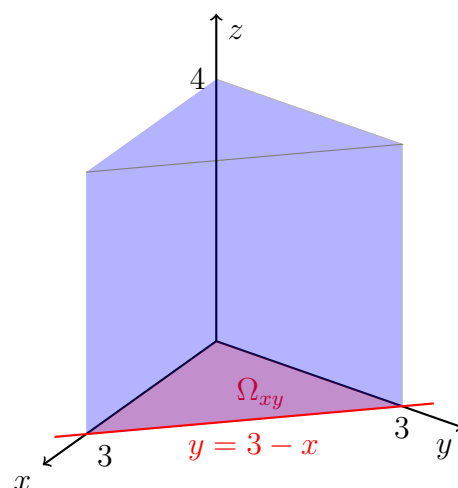
1. $I = \iiint_{\Omega} xy^2 z dx dy dz$, kde $\Omega = \{[x, y, z] \in \mathbb{R}^3, 0 \leq x \leq 2, 1 \leq y \leq 3, 1 \leq z \leq 2\}$.

$$\begin{aligned} I &= \iiint_{\Omega_{xy}} \int_1^2 xy^2 z dz dx dy = \iint_{\Omega_{xy}} \left[xy^2 \frac{z^2}{2} \right]_1^2 dx dy = \\ &= \frac{3}{2} \int_1^3 \int_0^2 xy^2 dx dy = \frac{3}{2} \int_1^3 \left[y^2 \frac{x^2}{2} \right]_0^2 dy = 3 \int_1^3 y^2 dy = \\ &= 3 \left[\frac{y^3}{3} \right]_1^3 = 26 \end{aligned}$$



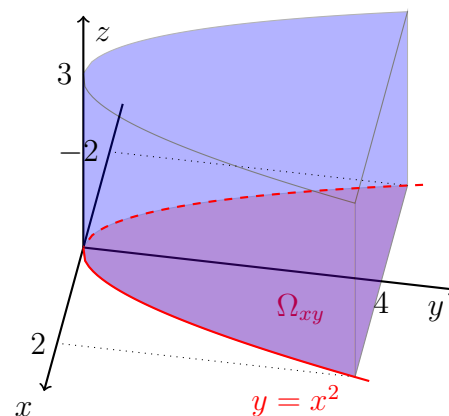
2. $I = \iiint_{\Omega} \frac{x+y}{z+4} dx dy dz$, kde Ω je určena vztahy $x = 0, y = 0, x + y = 3, 0 \leq z \leq 4$.

$$\begin{aligned} I &= \iint_{\Omega_{xy}} \left(\int_0^4 \frac{x+y}{z+4} dz \right) dx dy = \\ &= \iint_{\Omega_{xy}} [(x+y) \ln |z+4|]_0^4 dx dy = \\ &= \int_0^3 \int_0^{3-x} (x+y) \ln 2 dy dx = \ln 2 \int_0^3 \left[xy + \frac{y^2}{2} \right]_0^{3-x} dx = \\ &= \ln 2 \int_0^3 \left(x(3-x) + \frac{(3-x)^2}{2} \right) dx = \\ &= \ln 2 \left[\frac{3}{2}x^2 - \frac{x^3}{3} - \frac{(3-x)^3}{6} \right]_0^3 = \\ &= \ln 2 \left(\frac{27}{2} - \frac{27}{3} + \frac{27}{6} \right) = 9 \ln 2 \end{aligned}$$



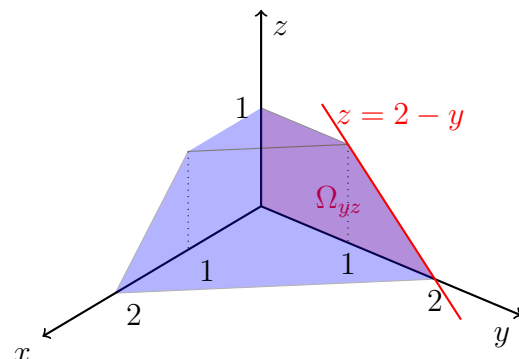
3. $I = \iiint_{\Omega} z dx dy dz$, kde Ω je určena vztahy $y = 4, z = 0, z = 3, x^2 - y = 0$.

$$\begin{aligned} I &= \iint_{\Omega_{xy}} \int_0^3 \frac{z^2}{2} dz dx dy = \frac{9}{2} \int_{-2}^2 \int_{x^2}^4 dy dx = \frac{9}{2} \int_{-2}^2 (4 - x^2) dx = \\ &= \frac{9}{2} \left[4x - \frac{x^3}{3} \right]_{-2}^2 = 48 \end{aligned}$$



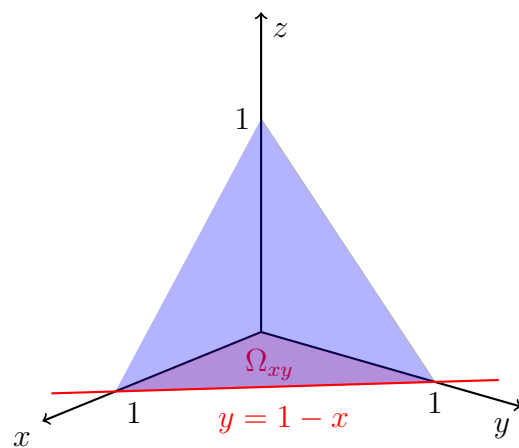
4. $I = \iiint_{\Omega} dx dy dz$, kde Ω je určena vztahy $x = 0$, $y = 0$, $z = 0$, $z = 1$, $x + y + z = 2$.

$$\begin{aligned}
 I &= \iint_{\Omega_{yz}} \int_0^{2-y-z} dx dy dz = \int_0^1 \int_0^{2-z} (2-y-z) dy dz = \\
 &= \int_0^1 \left[2y - \frac{y^2}{2} - zy \right]_0^{2-z} dz = \\
 &= \int_0^1 \left(2(2-z) - \frac{(2-z)^2}{2} - z(2-z) \right) dz = \\
 &= \int_0^1 \left((2-z)^2 - \frac{(2-z)^2}{2} \right) dz = \\
 &= \frac{1}{2} \int_0^1 (2-z)^2 dz = \left[-\frac{(2-z)^3}{6} \right]_0^1 = -\frac{1}{6} + \frac{8}{6} = \frac{7}{6}
 \end{aligned}$$



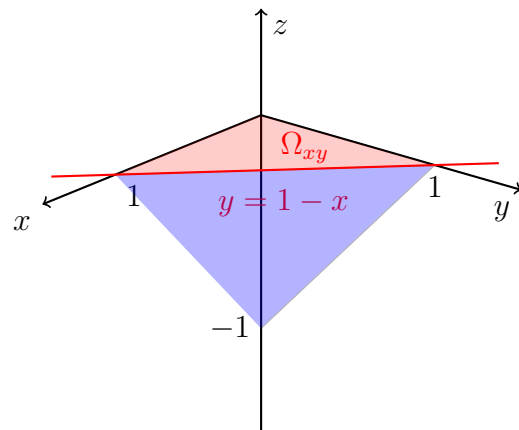
5. $I = \iiint_{\Omega} dx dy dz$, kde Ω je určena vztahy $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

$$\begin{aligned}
 I &= \iint_{\Omega_{xy}} \int_0^{1-x-y} dz dx dy = \int_0^1 \int_0^{1-x} (1-x-y) dy dx = \\
 &= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx = \\
 &= \int_0^1 \left((1-x) - x(1-x) - \frac{(1-x)^2}{2} \right) dx = \\
 &= \int_0^1 \left((1-x)^2 - \frac{(1-x)^2}{2} \right) dx = \frac{1}{2} \int_0^1 (1-x)^2 dx = \\
 &= -\frac{1}{2} \left[\frac{(1-x)^3}{3} \right]_0^1 = \frac{1}{6}
 \end{aligned}$$



6. $I = \iiint_{\Omega} \frac{1}{(x+y-z+1)^3} dx dy dz$, kde Ω je určena vztahy $x = 0, y = 0, z = 0, x + y - z = 1$.

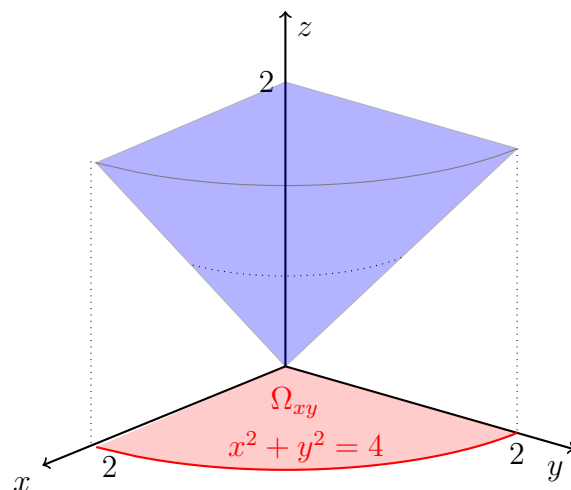
$$\begin{aligned} I &= \iiint_{\Omega_{xy}} \int_{x+y-1}^0 \frac{1}{(x+y-z+1)^3} dz dx dy = \\ &= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{(x+y-z+1)^2} \right]_{x+y-1}^0 dy dx = \\ &= \frac{1}{2} \int_0^1 \int_0^{1-x} \left(\frac{1}{(x+y+1)^2} - \frac{1}{4} \right) dy dx = \\ &= -\frac{1}{2} \int_0^1 \left[\frac{1}{x+y+1} + \frac{1}{4}y \right]_0^{1-x} dx = \end{aligned}$$



$$= -\frac{1}{2} \int_0^1 \left(\frac{1}{2} + \frac{1}{4}(1-x) - \frac{1}{x+1} \right) dx = -\frac{1}{2} \left[\frac{3}{4}x - \frac{x^2}{8} - \ln|x+1| \right]_0^1 = -\frac{1}{4} \left(\frac{3}{2} - \frac{1}{4} \right) = \frac{1}{2} \ln 2 - \frac{5}{16}$$

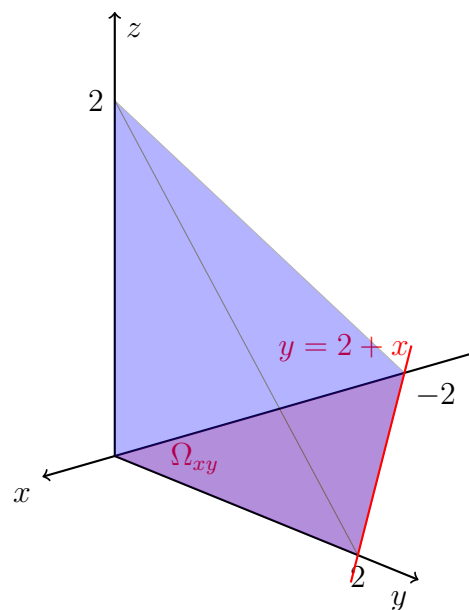
7. $I = \iiint_{\Omega} y dx dy dz$, kde Ω je určena vztahy $x \geq 0, y \geq 0, \sqrt{x^2 + y^2} \leq z \leq 2$.

$$\begin{aligned} &= \iiint_{\Omega_{xy}} \int_{\sqrt{x^2+y^2}}^2 y dz dx dy = \iint_{\Omega_{xy}} [yz]^2_{\sqrt{x^2+y^2}} dx dy = \\ &= \int_0^2 \int_0^{\sqrt{4-x^2}} (2y - y\sqrt{x^2+y^2}) dy dx = \\ &= \left| \begin{array}{l} t = x^2 + y^2 \\ dt = 2y dy \\ \int (-y\sqrt{x^2+y^2}) dy = -\frac{1}{2} \int \sqrt{t} dt = -\frac{1}{3} \sqrt{t^3} \end{array} \right| = \\ &= \int_0^2 \left[y^2 - \frac{1}{3} \sqrt{(x^2+y^2)^3} \right]_0^{\sqrt{4-x^2}} dx = \\ &= \int_0^2 \left(4 - x^2 - \frac{1}{3} \underbrace{\sqrt{4^3}}_8 + \frac{1}{3} \underbrace{\sqrt{(x^2)^3}}_{x^3} \right) dx = \\ &= \left[4x - \frac{x^3}{3} - \frac{8}{3}x + \frac{x^4}{12} \right]_0^2 = \frac{4}{3} \end{aligned}$$



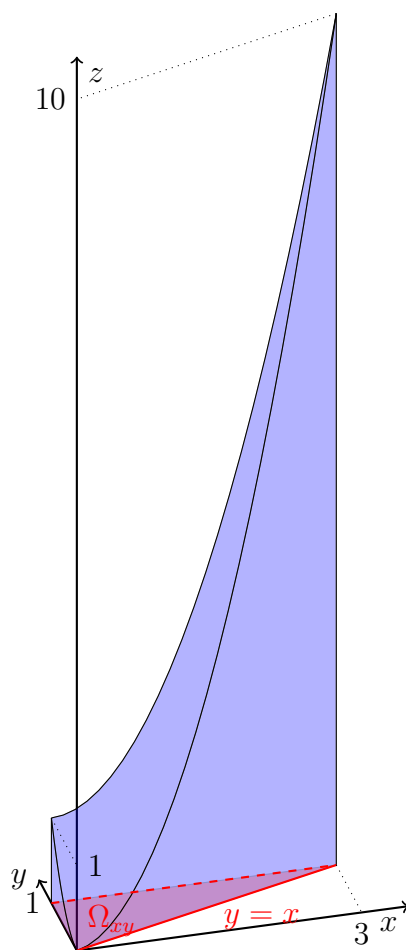
8. $I = \iiint_{\Omega} 1 \, dx \, dy \, dz$, kde Ω je určena vztahy $x = 0$, $y = 0$, $z = 0$, $-x + y + z = 2$.

$$\begin{aligned}
 I &= \iiint_{\Omega_{xy}} \int_0^{2+x-y} dz \, dx \, dy = \int_{-2}^0 \int_0^{2+x} (2+x-y) \, dy \, dx = \\
 &= \int_{-2}^0 \left[2y + xy - \frac{y^2}{2} \right]_0^{2+x} dx = \frac{1}{2} \int_{-2}^0 (2+x)^2 dx = \\
 &= \frac{1}{2} \left[\frac{(2+x)^3}{3} \right]_{-2}^0 = \frac{4}{3}
 \end{aligned}$$



9. Vypočítejte objem tělesa $\Omega = \{[x, y, z] \in \mathbb{R}^3, 0 \leq x \leq 3y, 0 \leq y \leq 1, 0 \leq z \leq x^2 + y^2\}$.

$$\begin{aligned} \iiint_{\Omega} 1 \, dx dy dz &= \iint_{\Omega_{xy}} \left(\int_0^{x^2+y^2} dz \right) dx dy = \\ &= \int_0^1 \int_0^{3y} (x^2 + y^2) dy dx = \int_0^1 \left[\frac{x^3}{3} + xy^2 \right]_0^{3y} dy = \\ &= \int_0^1 12y^3 dy = [3y^4]_0^1 = 3 \end{aligned}$$

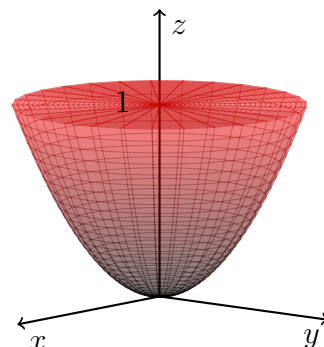


Trojný integrál - substituce do válcových souřadnic

1. Vypočtete integrál $I = \iiint_{\Omega} (x^2 + y^2) dx dy dz$ Ω je určena nerovnostmi $x^2 + y^2 \leq z \leq 1$.

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \\ z &= z \\ |J| &= r \end{aligned}$$

$$\begin{aligned} r &\in \langle 0, 1 \rangle \\ \varphi &\in \langle 0, 2\pi \rangle \\ z &\in \langle r^2, 1 \rangle \end{aligned}$$



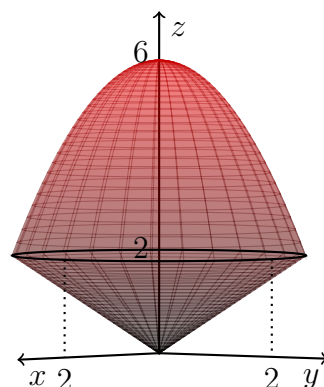
$$x^2 + y^2 \leq z \leq 1 \quad \rightarrow \quad r^2 \leq z \leq 1$$

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 r^2 \cdot r dz dr d\varphi = \int_0^{2\pi} \int_0^1 r^3 [z]_{r^2}^1 dr d\varphi = \int_0^{2\pi} \int_0^1 (r^3 - r^5) dr d\varphi = \\ &= \int_0^{2\pi} \left[\frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 d\varphi = \frac{1}{12} \int_0^{2\pi} d\varphi = \frac{\pi}{6} \end{aligned}$$

2. Vypočtete integrál $I = \iiint_{\Omega} \sqrt{x^2 + y^2} dx dy dz$ Ω je určena nerovnostmi $\sqrt{x^2 + y^2} \leq z \leq 6 - x^2 - y^2$.

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \\ z &= z \\ |J| &= r \end{aligned}$$

$$\begin{aligned} r &\in \langle 0, 2 \rangle \\ \varphi &\in \langle 0, 2\pi \rangle \\ z &\in \langle r, 6 - r^2 \rangle \end{aligned}$$



$$\sqrt{x^2 + y^2} \leq z \leq 6 - x^2 - y^2 \quad \rightarrow \quad r \leq z \leq 6 - r^2$$

Poloměr společné kružnice:

$$\begin{aligned} z^2 &= x^2 + y^2 \\ z &= 6 - x^2 - y^2 \quad \rightarrow \quad x^2 + y^2 = 6 - z \end{aligned}$$

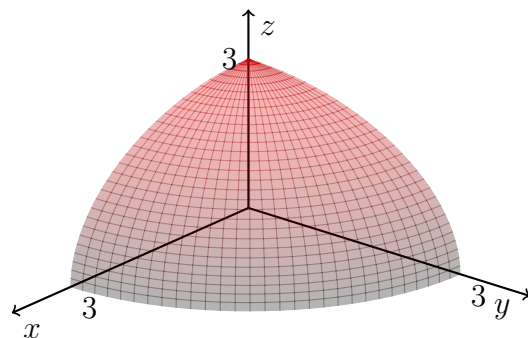
$$\rightarrow 6 - z = z^2 \quad \rightarrow \quad z^2 + z - 6 = 0 \quad \rightarrow \quad \mathbf{z_1 = 2, z_2 = -3}$$

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_r^{6-r^2} r \cdot r dz dr d\varphi = \int_0^{2\pi} \int_0^2 r^2 [z]_r^{6-r^2} dr d\varphi = \int_0^{2\pi} \int_0^2 (6r^2 - r^4 - r^3) dr d\varphi = \\ &= \int_0^{2\pi} \left[2r^3 - \frac{r^5}{5} - \frac{r^4}{4} \right]_0^2 d\varphi = \frac{28}{5} \int_0^{2\pi} d\varphi = \frac{56}{5} \pi \end{aligned}$$

Trojný integrál - substituce do sférických souřadnic

1. Vypočtete integrál $I = \iiint_{\Omega} z \, dx \, dy \, dz$ Ω je určena nerovnostmi $0 \leq z \leq \sqrt{9 - x^2 - y^2}$, $x \geq 0, y \geq 0$.

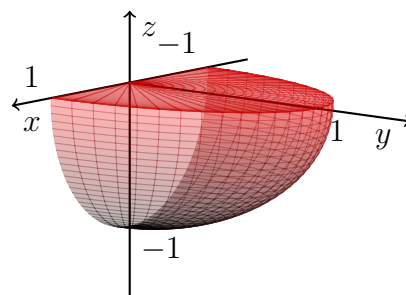
$$\begin{aligned} x &= r \cos \varphi \cos \vartheta \\ y &= r \sin \varphi \cos \vartheta \\ z &= r \sin \vartheta \\ |J| &= r^2 \cos \vartheta \end{aligned} \quad \begin{aligned} r &\in \langle 0, 3 \rangle \\ \varphi &\in \langle 0, \frac{\pi}{2} \rangle \\ \vartheta &\in \langle 0, \frac{\pi}{2} \rangle \end{aligned}$$



$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 r \sin \vartheta \cdot r^2 \cos \vartheta \, dr \, d\varphi \, d\vartheta = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \vartheta \cos \vartheta \left[\frac{r^4}{4} \right]_0^3 d\varphi \, d\vartheta = \\ &= \frac{81}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \vartheta \cos \vartheta \, d\varphi \, d\vartheta = \frac{81}{8} \pi \int_0^{\frac{\pi}{2}} \sin \vartheta \cos \vartheta \, d\vartheta = \left| \begin{array}{l} t = \sin \vartheta \\ dt = \cos \vartheta \, d\vartheta \\ \int t \, dt = \frac{t^2}{2} + C \end{array} \right| = \\ &= \frac{81}{16} \pi [\sin^2 \vartheta]_0^{\frac{\pi}{2}} = \frac{81}{16} \pi \end{aligned}$$

2. Vypočtete integrál $I = \iiint_{\Omega} \frac{1}{\sqrt{x^2 + y^2 + z^2 - 4}} \, dx \, dy \, dz$ Ω je určena nerovnostmi $x^2 + y^2 + z^2 \leq 1, y \geq 0, z \leq 0$.

$$\begin{aligned} x &= r \cos \varphi \cos \vartheta \\ y &= r \sin \varphi \cos \vartheta \\ z &= r \sin \vartheta \\ |J| &= r^2 \cos \vartheta \end{aligned} \quad \begin{aligned} r &\in \langle 0, 1 \rangle \\ \varphi &\in \langle 0, \pi \rangle \\ \vartheta &\in \langle -\frac{\pi}{2}, 0 \rangle \end{aligned}$$



$$\begin{aligned} I &= \int_{-\frac{\pi}{2}}^0 \int_0^{\pi} \int_0^1 \frac{1}{r-4} r^2 \cos \vartheta \, dr \, d\varphi \, d\vartheta = \int_{-\frac{\pi}{2}}^0 \int_0^{\pi} \int_0^1 \left(r + 4 + \frac{16}{r-4} \right) \cos \vartheta \, dr \, d\varphi \, d\vartheta = \\ &= \int_{-\frac{\pi}{2}}^0 \int_0^{\pi} \left[\frac{r^2}{2} + 4r + 16 \ln |r-4| \right]_0^1 \cos \vartheta \, d\varphi \, d\vartheta = \left(\frac{9}{2} + 16 \ln \frac{3}{4} \right) \int_{-\frac{\pi}{2}}^0 \int_0^{\pi} \cos \vartheta \, d\varphi \, d\vartheta = \\ &= \pi \left(\frac{9}{2} + 16 \ln \frac{3}{4} \right) \int_{-\frac{\pi}{2}}^0 \cos \vartheta \, d\vartheta = \pi \left(\frac{9}{2} + 16 \ln \frac{3}{4} \right) [\sin \vartheta]_{-\frac{\pi}{2}}^0 = \pi \left(\frac{9}{2} + 16 \ln \frac{3}{4} \right) \end{aligned}$$

6. Cvičení: Vektorová analýza

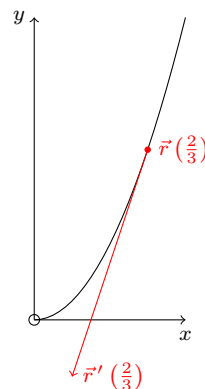
Příklady:

- Je dána křivka $\mathcal{C}: \vec{r}(t) = (\frac{1}{t}, \frac{1}{t^2})$, $t \in (0, \infty)$. Načrtněte křivku \mathcal{C} . Určete tečný vektor v bodě $t = \frac{2}{3}$ a načrtněte. Dále určete bod, ve kterém je tečna rovnoběžná s přímkou $y = x$.

Eliminace parametru t : $t = \frac{1}{x} \rightarrow y = x^2$, $x > 0 \rightarrow$ parabola.

$$\vec{r}'(t) = \left(-\frac{1}{t^2}, -\frac{2}{t^3}\right), t \in (0, \infty)$$

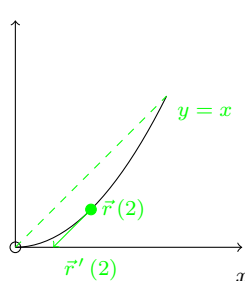
$$\vec{r}'\left(\frac{2}{3}\right) = \left(-\frac{9}{4}, -\frac{27}{4}\right)$$



Přímka $y = x$ má směrový vektor: $\vec{s} = (1, 1)$, tj. všechny rovnoběžné vektory jsou libovolné násobky: $\vec{s}_k = (k, k)$, $k \in \mathbb{R}$.

$$-\frac{1}{t^2} = k \quad \wedge \quad -\frac{2}{t^3} = k \quad \Rightarrow \quad -\frac{1}{t^2} = -\frac{2}{t^3}$$

$$\Rightarrow t = 2 \rightarrow \vec{r}(2) = \left(\frac{1}{2}, \frac{1}{4}\right), \vec{r}'(2) = \left(-\frac{1}{4}, -\frac{1}{4}\right)$$



Příklady: Určete a načrtněte vektorové čáry vektorového pole $\vec{a}(x, y)$.

- $\vec{a}(x, y) = (x, y)$.

$$\frac{dx(t)}{dt} = x \quad \frac{dy(t)}{dt} = y$$

$$\int \frac{dx}{x} = \int dt, x \neq 0 \quad \int \frac{dy}{y} = \int dt, y \neq 0$$

$$\ln|x| = t + C_1, C_1 \in \mathbb{R} \quad \ln|y| = t + C_2, C_2 \in \mathbb{R}$$

$$|x| = e^{t+C_1} = e^{C_1}e^t \quad |y| = e^{t+C_2} = e^{C_2}e^t$$

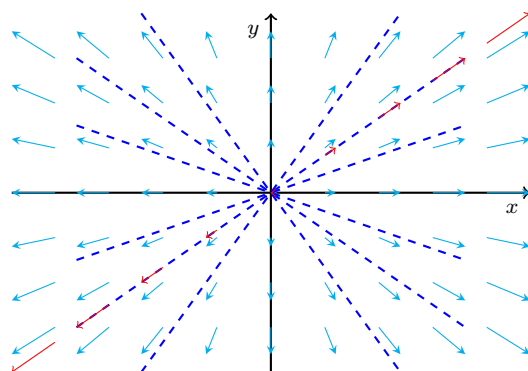
$$x(t) = K_1e^t, K_1 \in \mathbb{R} \quad y(t) = K_2e^t, K_2 \in \mathbb{R}$$

$$\vec{r}(t) = (K_1e^t, K_2e^t), \quad t \in \mathbb{R}, K_1, K_2 \in \mathbb{R}$$

Obecná rovnice (eliminujeme parametr t):

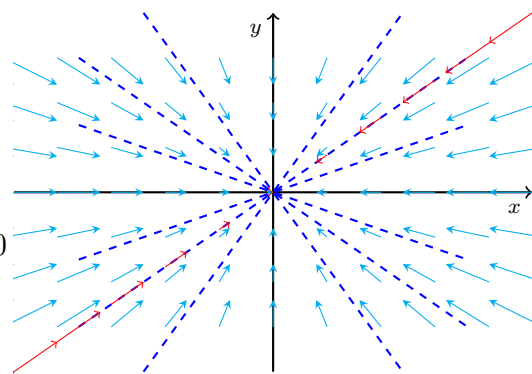
$$t = \ln \frac{x}{K_1} \rightarrow y = K_2e^{\ln \frac{x}{K_1}} \rightarrow y = \frac{K_2}{K_1}x$$

$$y = kx, k \in \mathbb{R}$$



2. $\vec{a}(x, y) = (-x, -y)$.

$$\begin{aligned}\frac{dy(t)}{-y} &= \frac{dx(t)}{-x}, \quad x, y \neq 0 \\ \int \frac{dy(t)}{y} &= \int \frac{dx(t)}{x} \\ \ln |y| &= \ln |x| + C_1 = \ln |x| + \ln C_2, \quad C_1 \in \mathbb{R}, C_2 > 0 \\ \ln |y| &= \ln(C_2|x|) \\ y &= Kx, \quad K \in \mathbb{R}\end{aligned}$$



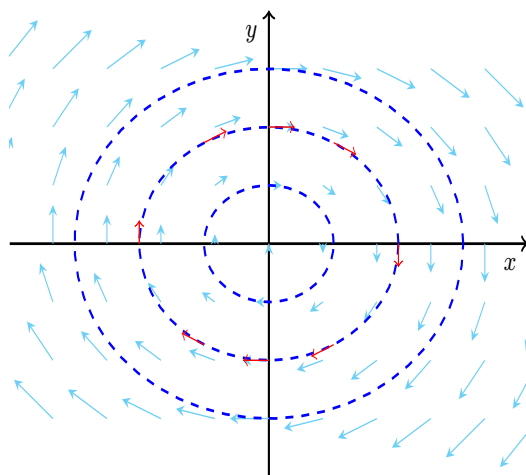
Parametrizace:

$$\vec{r}(t) = (t, kt), \quad t \in \mathbb{R}, k \in \mathbb{R}$$

3. $\vec{a}(x, y) = (y, -x)$.

$$\begin{aligned}\frac{dx(t)}{dt} &= y \\ \frac{d^2x}{dt^2} &= \frac{dy}{dt} \rightarrow \frac{dy(t)}{dt} = -x \\ \frac{d^2x}{dt^2} &= -x\end{aligned}$$

$$x'' + x = 0$$



Charakteristická rovnice: $\lambda^2 + 1 = 0 \rightarrow \lambda_{1,2} = \pm i$

Fundamentální systém: FS = $\{\cos t, \sin t\}$

$$x(t) = C_1 \cos t + C_2 \sin t$$

$$y(t) = \frac{dx(t)}{dt} = -C_1 \sin t + C_2 \cos t$$

$$\vec{r}(t) = (C_1 \cos t + C_2 \sin t, -C_1 \sin t + C_2 \cos t), \quad t \in \mathbb{R}, C_1, C_2 \in \mathbb{R}$$

Obecná rovnice:

$$\begin{aligned}x^2 + y^2 &= C_1^2 \cos^2 t + 2C_1C_2 \cos t \sin t + C_2^2 \sin^2 t + C_1^2 \sin^2 t - 2C_1C_2 \cos t \sin t + C_2^2 \cos^2 t = \\ &= C_1^2(\cos^2 t + \sin^2 t) + C_2^2(\cos^2 t + \sin^2 t) = C_1^2 + C_2^2\end{aligned}$$

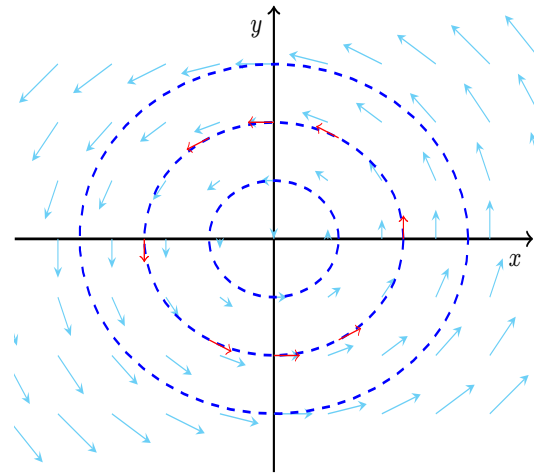
$$x^2 + y^2 = c^2, \quad c \in \mathbb{R}$$

4. $\vec{a}(x, y) = (-y, x)$.

$$\begin{aligned}\frac{dy}{x} &= \frac{dx}{-y}, \quad x, y \neq 0 \\ \int y \, dy &= - \int x \, dx \\ \frac{y^2}{2} &= -\frac{x^2}{2} + C_1, \quad C_1 \in \mathbb{R} \\ y^2 + x^2 &= 2C_1 = C^2, \quad C \in \mathbb{R}\end{aligned}$$

Parametrizace:

$$\vec{r}(t) = (c \sin t, c \cos t), \quad t \in \langle 0, 2\pi \rangle, \quad c \in \mathbb{R}$$



5. $\vec{a}(x, y) = (y, 1)$.

$$\frac{dx(t)}{dt} = y$$

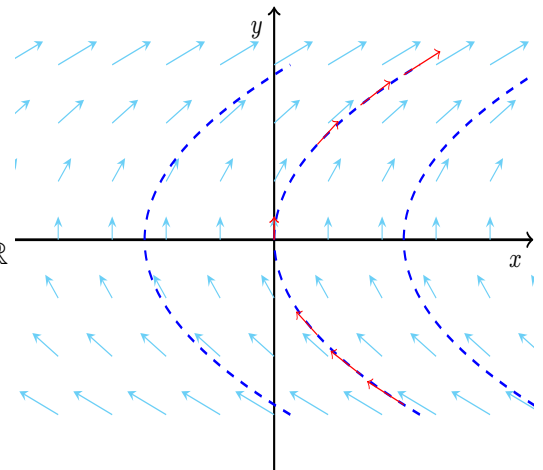
$$\frac{dy(t)}{dt} = 1$$

$$\int dy = \int dt$$

$$y(t) = t + C_1, \quad C_1 \in \mathbb{R}$$

$$\int dx = \int (t + C_1) \, dt$$

$$x(t) = \frac{(t + C_1)^2}{2} + C_2, \quad C_2 \in \mathbb{R}$$

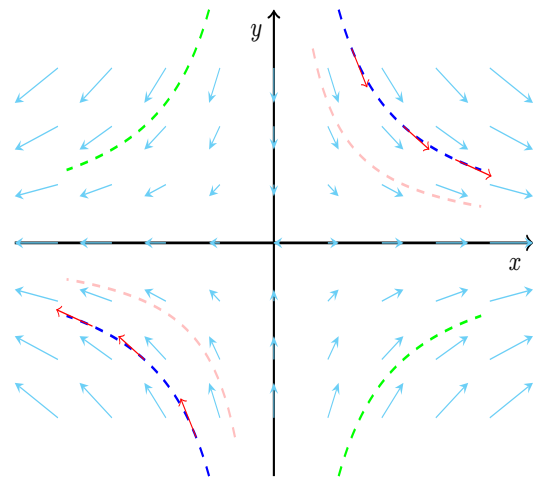


$$\vec{r}(t) = \left(\frac{(t + C_1)^2}{2} + C_2, t + C_1 \right), \quad t \in \mathbb{R}, \quad C_1, C_2 \in \mathbb{R}$$

Obečná rovnice: $y - C_1 = t \rightarrow x = \frac{y^2}{2} + C_2, \quad C_2 \in \mathbb{R}$

6. $\vec{a}(x, y) = (x, -y)$.

$$\begin{aligned}\frac{dx(t)}{dt} &= x & \frac{dy(t)}{dt} &= -y \\ \int \frac{dx}{x} &= \int dt, x \neq 0 & \int \frac{dy}{y} &= - \int dt, y \neq 0 \\ \ln|x| &= -t + C_1, C_1 \in \mathbb{R} & \ln|y| &= t + C_2, C_2 \in \mathbb{R} \\ x(t) &= K_1 e^{-t}, K_1 \in \mathbb{R} & y(t) &= K_2 e^t, K_2 \in \mathbb{R} \\ \vec{r}(t) &= (K_1 e^{-t}, K_2 e^t), t \in \mathbb{R}, K_1, K_2 \in \mathbb{R}\end{aligned}$$

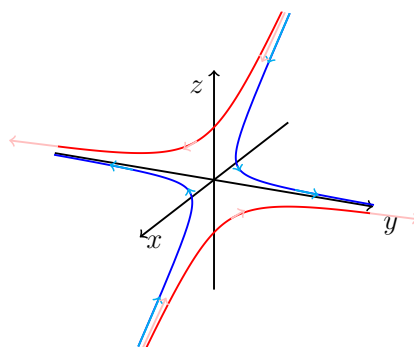
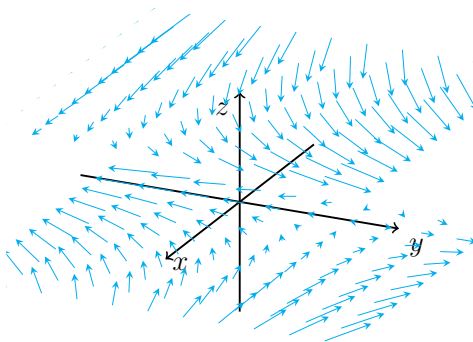


Obečná rovnice: $e^t = \frac{x}{K_1} \rightarrow y = \frac{K_1 K_2}{x}$

$$y = \frac{K}{x}, K \in \mathbb{R}, x \neq 0$$

7. $\vec{a}(x, y, z) = (x, -y, -z)$.

$$\begin{aligned}\frac{dx(t)}{dt} &= x & \frac{dy(t)}{dt} &= -y & \frac{dz(t)}{dt} &= -z \\ \int \frac{dx(t)}{x} &= \int dt, x \neq 0 & \int \frac{dy(t)}{y} &= - \int dt, y \neq 0 & \int \frac{dz(t)}{z} &= - \int dt, z \neq 0 \\ \ln|x| &= t + C_1, C_1 \in \mathbb{R} & \ln|y| &= -t + C_2, C_2 \in \mathbb{R} & \ln|z| &= -t + C_3, C_3 \in \mathbb{R} \\ x(t) &= K_1 e^t, K_1 \in \mathbb{R} & y(t) &= K_2 e^{-t}, K_2 \in \mathbb{R} & y(t) &= K_3 e^{-t}, K_3 \in \mathbb{R} \\ \vec{r}(t) &= (K_1 e^t, K_2 e^{-t}, K_3 e^{-t}), t \in \mathbb{R}, K_1, K_2, K_3 \in \mathbb{R}\end{aligned}$$



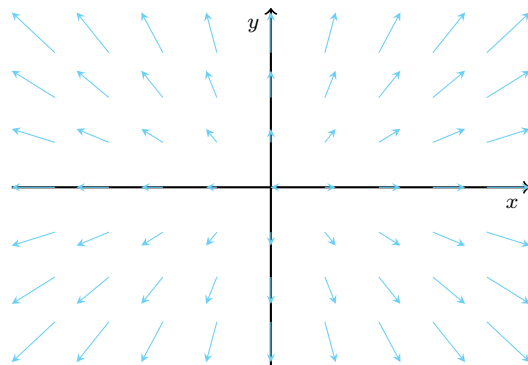
Příklady: Určete, zda je vektorové pole \vec{a} vírové \times nevírové, zřídlové \times nezřídlové.

1. $\vec{a} = (2x, 3y)$

$$\operatorname{div} \vec{a} = \nabla \vec{a} = 2 + 3 = 5$$

$$\operatorname{rot} \vec{a} = \nabla \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 3y & 0 \end{vmatrix} = (0, 0, 0)$$

\Rightarrow zřídlové, nevírové pole

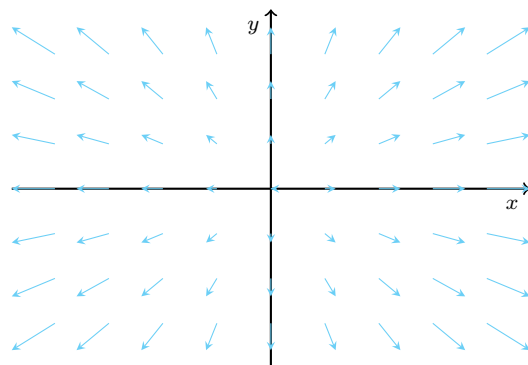


2. $\vec{a} = (2x, 2y)$

$$\operatorname{div} \vec{a} = \nabla \vec{a} = 2 + 2 = 4$$

$$\operatorname{rot} \vec{a} = \nabla \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 0 \end{vmatrix} = (0, 0, 0)$$

\Rightarrow zřídlové, nevírové pole

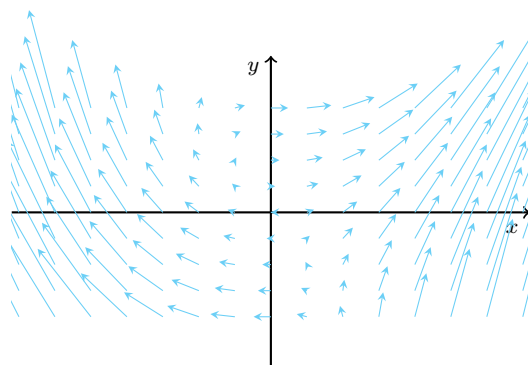


3. $\vec{a} = (2x + y, 3x^2)$

$$\operatorname{div} \vec{a} = \nabla \vec{a} = 2$$

$$\operatorname{rot} \vec{a} = \nabla \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y & 3x^2 & 0 \end{vmatrix} = (0, 0, 6x - 1)$$

\Rightarrow zřídlové, vírové pole



4. $\vec{a} = (x^3, xy, e^{xyz})$.

$$\operatorname{div} \vec{a} = 3x^2 + x + xye^{xyz}$$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & xy & e^{xyz} \end{vmatrix} = (xze^{xyz}, -yze^{xyz}, y)$$

\Rightarrow zřídlové, vírové pole

Příklady: Pro vektorové pole \vec{a} určete, zda je pole potenciálové a najděte potenciál φ .

1. $\vec{a} = (2xe^{x^2+2y}, 2e^{x^2+2y})$

Pole je definováno na \mathbb{R}^2 - jednoduše souvislá oblast.

$$\text{rot } \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1(x, y) & a_2(x, y) & 0 \end{vmatrix} = \left(0, 0, \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

$$\frac{\partial a_1}{\partial y} = 4xe^{x^2+2y} \quad \wedge \quad \frac{\partial a_2}{\partial x} = 4xe^{x^2+2y}$$

$$\frac{\partial a_1}{\partial y} = \frac{\partial a_2}{\partial x} \quad \Rightarrow \quad \text{rot } \vec{a} = \vec{0}$$

\Rightarrow Pole je potenciálové.

$$\vec{a} = \text{grad} \varphi \rightarrow \frac{\partial \varphi}{\partial x} = 2xe^{x^2+2y} \wedge \frac{\partial \varphi}{\partial y} = 2e^{x^2+2y}$$

$$\frac{\partial \varphi}{\partial x} = 2xe^{x^2+2y} \rightarrow \varphi = \int 2xe^{x^2+2y} dx = \left| \begin{array}{l} t = x^2 + 2y \\ dt = 2x dx \\ \int e^t dt = e^t + K \end{array} \right| = e^{x^2+2y} + C$$

$$\frac{\partial \varphi}{\partial y} = 2e^{x^2+2y} \rightarrow \varphi = \int 2e^{x^2+2y} dy = e^{x^2+2y} + C$$

$$\Rightarrow \varphi(x, y) = e^{x^2+2y} + C, C \in \mathbb{R}$$

2. $\vec{a} = (e^{-x} \sin y, -e^{-x} \cos y)$

Pole je definováno na \mathbb{R}^2 - jednoduše souvislá oblast.

$$\frac{\partial a_1}{\partial y} = e^{-x} \cos y \quad \wedge \quad \frac{\partial a_2}{\partial x} = e^{-x} \cos y$$

$$\frac{\partial a_1}{\partial y} = \frac{\partial a_2}{\partial x} \quad \Rightarrow \quad \text{rot } \vec{a} = \vec{0}$$

\Rightarrow Pole je potenciálové.

$$\vec{a} = \text{grad} \varphi \rightarrow \frac{\partial \varphi}{\partial x} = e^{-x} \sin y \wedge \frac{\partial \varphi}{\partial y} = -e^{-x} \cos y$$

$$\frac{\partial \varphi}{\partial x} = e^{-x} \sin y \rightarrow \varphi = \int e^{-x} \sin y dx = -e^{-x} \sin y + C$$

$$\frac{\partial \varphi}{\partial y} = -e^{-x} \cos y \rightarrow \varphi = \int -e^{-x} \cos y dy = -e^{-x} \sin y + C$$

$$\Rightarrow \varphi(x, y) = -e^{-x} \sin y + C, C \in \mathbb{R}$$

3. $\vec{a} = (x, y, z)$

Pole je definováno na \mathbb{R}^3 - jednoduše souvislá oblast.

$$\text{rot } \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0, 0, 0) = \vec{0}$$

\Rightarrow Pole je potenciálové.

$$\vec{a} = \text{grad} \varphi \rightarrow \frac{\partial \varphi}{\partial x} = x \wedge \frac{\partial \varphi}{\partial y} = y \wedge \frac{\partial \varphi}{\partial z} = z$$

$$\begin{array}{lll} \frac{\partial \varphi}{\partial x} = x & \frac{\partial \varphi}{\partial y} = y & \frac{\partial \varphi}{\partial z} = z \\ \varphi = \frac{x^2}{2} + C & \varphi = \frac{y^2}{2} + C & \varphi = \frac{z^2}{2} + C \end{array}$$

$$\Rightarrow \varphi(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + C, C \in \mathbb{R}$$

4. $\vec{a} = (x + yz, y + xz, z + xy)$

Pole je definováno na \mathbb{R}^3 - jednoduše souvislá oblast.

$$\text{rot } \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + yz & y + xz & z + xy \end{vmatrix} = (x - x, y - y, z - z) = \vec{0}$$

\Rightarrow Pole je potenciálové.

$$\vec{a} = \text{grad} \varphi \rightarrow \frac{\partial \varphi}{\partial x} = x + yz \wedge \frac{\partial \varphi}{\partial y} = y + xz \wedge \frac{\partial \varphi}{\partial z} = z + xy$$

$$\begin{array}{lll} \frac{\partial \varphi}{\partial x} = x + yz & \frac{\partial \varphi}{\partial y} = y + xz & \frac{\partial \varphi}{\partial z} = z + xy \\ \varphi = \frac{x^2}{2} + xyz + C & \varphi = \frac{y^2}{2} + xyz + C & \varphi = \frac{z^2}{2} + xyz + C \end{array}$$

$$\Rightarrow \varphi(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + xyz + C, C \in \mathbb{R}$$

5. $\vec{a} = (x, y^2, -z)$

Pole je definováno na \mathbb{R}^3 - jednoduše souvislá oblast.

$$\text{rot } \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y^2 & -z \end{vmatrix} = (0, 0, 0) = \vec{0}$$

\Rightarrow Pole je potenciálové.

$$\vec{a} = \text{grad} \varphi \rightarrow \frac{\partial \varphi}{\partial x} = x \wedge \frac{\partial \varphi}{\partial y} = y^2 \wedge \frac{\partial \varphi}{\partial z} = -z$$

$$\begin{array}{lll} \frac{\partial \varphi}{\partial x} = x & \frac{\partial \varphi}{\partial y} = y^2 & \frac{\partial \varphi}{\partial z} = -z \\ \varphi = \frac{x^2}{2} + C & \varphi = \frac{y^3}{3} + C & \varphi = -\frac{z^2}{2} + C \end{array}$$

$$\Rightarrow \varphi(x, y, z) = \frac{x^2}{2} + \frac{y^3}{3} - \frac{z^2}{2} + C, \quad C \in \mathbb{R}$$

Příklady: Ukažte, že funkce $f(x, y)$ je harmonická, tj. splňuje rovnici $\Delta f = 0$.

1. $f(x, y) = x^4 - 6x^2y^2 + y^4$

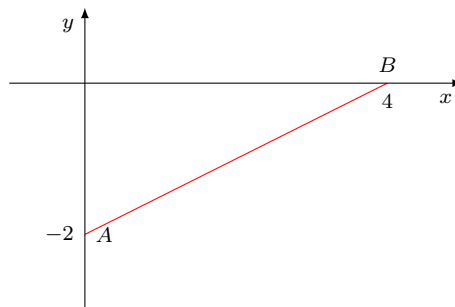
$$\begin{aligned} f'_x &= 4x^3 - 12xy^2, & f''_{xx} &= 12x^2 - 12y^2 \\ f'_y &= -12x^2y + 4y^3, & f''_{yy} &= -12x^2 + 12y^2 \\ \Delta f &= f''_{xx} + f''_{yy} = 12x^2 - 12y^2 - 12x^2 + 12y^2 = 0 \end{aligned}$$

7. Cvičení: Křivkové integrály 1. druhu

Příklady: Vypočtěte dané křivkové integrály 1. druhu.

1. ✓ $\int_{\mathcal{K}} \frac{ds}{x-y}$, kde křivka \mathcal{K} je úsečka AB , $A = [0, -2]$, $B = [4, 0]$.

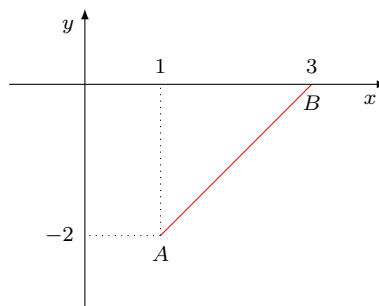
$$\begin{aligned}\vec{s} &= B - A = (4, 2) \\ \vec{r}(t) &= (4t, -2 + 2t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (4, 2); \quad \|\vec{r}'(t)\| = \sqrt{20}\end{aligned}$$



$$\int_{\mathcal{K}} \frac{ds}{x-y} = \int_0^1 \frac{1}{4t+2-2t} \sqrt{20} dt = \sqrt{5} \int_0^1 \frac{1}{t+1} dt = \sqrt{5} [\ln|t+1|]_0^1 = \sqrt{5} \ln 2$$

2. ✓ $\int_{\mathcal{K}} \frac{ds}{x-2y}$, kde křivka \mathcal{K} je úsečka AB , $A = [1, -2]$, $B = [3, 0]$.

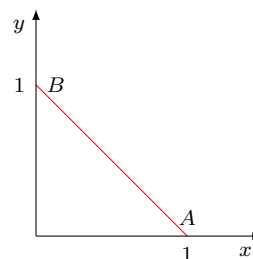
$$\begin{aligned}\vec{s} &= B - A = (2, 2) \\ \vec{r}(t) &= (1+2t, -2+2t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (2, 2); \quad \|\vec{r}'(t)\| = 2\sqrt{2}\end{aligned}$$



$$\int_{\mathcal{K}} \frac{ds}{x-2y} = \int_0^1 \frac{1}{1+2t+4-4t} 2\sqrt{2} dt = 2\sqrt{2} \int_0^1 \frac{1}{5-2t} dt = -\sqrt{2} [\ln|5-2t|]_0^1 = \sqrt{2} \ln \frac{5}{3}$$

3. ✓ $\int_{\mathcal{K}} (x^2 + y^2) ds$, kde křivka \mathcal{K} je úsečka AB , $A = [1, 0]$, $B = [0, 1]$.

$$\begin{aligned}\vec{s} &= B - A = (-1, 1) \\ \vec{r}(t) &= (1-t, t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (-1, 1); \quad \|\vec{r}'(t)\| = \sqrt{2}\end{aligned}$$

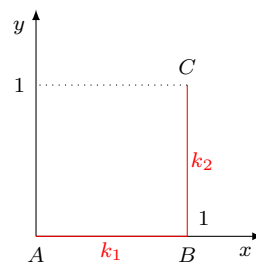


$$\begin{aligned}\int_{\mathcal{K}} (x^2 + y^2) ds &= \int_0^1 ((1-t)^2 + t^2) \sqrt{2} dt = \sqrt{2} \int_0^1 (1-2t+t^2) dt = \sqrt{2} \left[t - t^2 + \frac{2}{3}t^3 \right]_0^1 = \\ &= \frac{2\sqrt{2}}{3}\end{aligned}$$

4. ✓ $\int_{\mathcal{K}} (x^2 + y^2) ds$, kde křivka \mathcal{K} je lomená čára spojující body $A = [0, 0]$, $B = [1, 0]$, $C[1, 1]$.

$$\begin{aligned} k_1 : \vec{s} &= B - A = (1, 0) \\ \vec{r}(t) &= (t, 0); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (1, 0); \quad \|\vec{r}'(t)\| = 1 \end{aligned}$$

$$\begin{aligned} k_2 : \vec{s} &= C - B = (0, 1) \\ \vec{r}(t) &= (1, t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (0, 1); \quad \|\vec{r}'(t)\| = 1 \end{aligned}$$



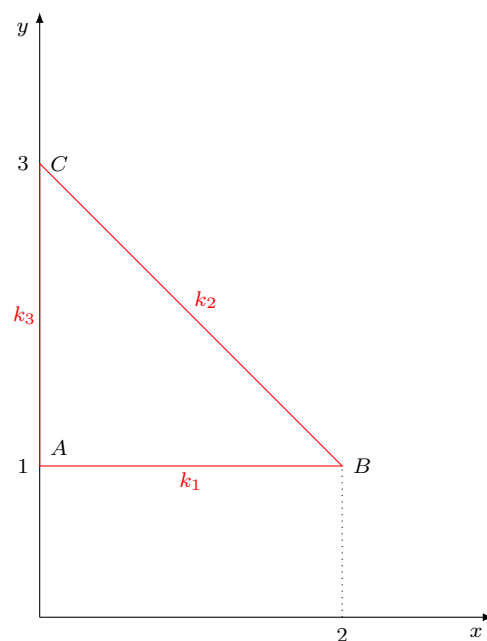
$$\begin{aligned} \int_{\mathcal{K}} (x^2 + y^2) ds &= \int_{k_1} (x^2 + y^2) ds + \int_{k_2} (x^2 + y^2) ds = \int_0^1 t^2 dt + \int_0^1 (1 + t^2) dt = \\ &= \left[t + \frac{2}{3}t^3 \right]_0^1 = \frac{5}{3} \end{aligned}$$

5. ✓ $\int_{\mathcal{K}} (x + y) ds$, kde křivka \mathcal{K} je obvod trojúhelníku ABC , $A = [0, 1]$, $B = [2, 1]$, $C = [0, 3]$.

$$\begin{aligned} k_1 : \vec{s} &= B - A = (2, 0) \\ \vec{r}(t) &= (2t, 1); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (2, 0); \quad \|\vec{r}'(t)\| = 2 \end{aligned}$$

$$\begin{aligned} k_2 : \vec{s} &= C - B = (-2, 2) \\ \vec{r}(t) &= (2 - 2t, 1 + 2t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (-2, 2); \quad \|\vec{r}'(t)\| = \sqrt{8} \end{aligned}$$

$$\begin{aligned} k_3 : \vec{s} &= C - A = (0, 2) \\ \vec{r}(t) &= (0, 1 + 2t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (0, 2); \quad \|\vec{r}'(t)\| = 2 \end{aligned}$$



$$\begin{aligned} \int_{\mathcal{K}} (x + y) ds &= \int_{k_1} (x + y) ds + \int_{k_2} (x + y) ds + \int_{k_3} (x + y) ds = \\ &= \int_0^1 (2t + 1)2 dt + \int_0^1 (2 - 2t + 1 + 2t)\sqrt{8} dt + \int_0^1 (1 + 2t)2 dt = \\ &= 2 [t^2 + t]_0^1 + \sqrt{8} [3t]_0^1 + 2 [t + t^2]_0^1 = 8 + 3\sqrt{8} \end{aligned}$$

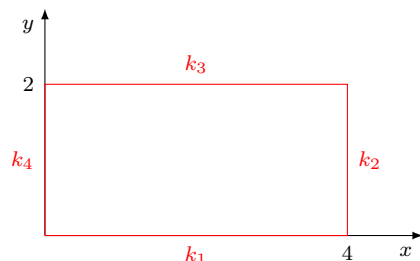
6. $\int_{\mathcal{K}} xy \, ds$, kde křivka \mathcal{K} je obvod obdelníku určeného přímkami $x = 0$, $x = 4$, $y = 0$, $y = 2$.

$$\begin{aligned} k_1 : \vec{s} &= [4, 0] - [0, 0] = (4, 0) \\ \vec{r}(t) &= (4t, 0); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (4, 0); \quad \|\vec{r}'(t)\| = 4 \end{aligned}$$

$$\begin{aligned} k_2 : \vec{s} &= [4, 2] - [4, 0] = (0, 2) \\ \vec{r}(t) &= (4, 2t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (0, 2); \quad \|\vec{r}'(t)\| = 2 \end{aligned}$$

$$\begin{aligned} k_3 : \vec{s} &= [0, 2] - [4, 2] = (-4, 0) \\ \vec{r}(t) &= (4 - 4t, 2); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (-4, 0); \quad \|\vec{r}'(t)\| = 4 \end{aligned}$$

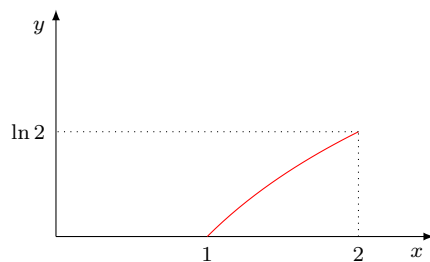
$$\begin{aligned} k_4 : \vec{s} &= [0, 0] - [0, 2] = (0, -2) \\ \vec{r}(t) &= (0, 2 - 2t); \quad t \in \langle 0, 1 \rangle \\ \rightarrow \vec{r}'(t) &= (0, -2); \quad \|\vec{r}'(t)\| = 2 \end{aligned}$$



$$\begin{aligned} \int_{\mathcal{K}} xy \, ds &= \int_{k_1} xy \, ds + \int_{k_2} xy \, ds + \int_{k_3} xy \, ds + \int_{k_4} xy \, ds = \\ &= \int_0^1 4t \cdot 0 \cdot 4 \, dt + \int_0^1 4 \cdot 2t \cdot 2 \, dt + \int_0^1 (4 - 4t)2 \cdot 4 \, dt + \int_0^1 0 \cdot (2 - 2t) \cdot 2 \, dt = \\ &= [8t^2]_0^1 + [32t - 16t^2]_0^1 = 24 \end{aligned}$$

7. $\int_{\mathcal{K}} x^2 \, ds$, kde křivka $\mathcal{K} = \{[x, y] \in \mathbb{R}^2 : x \in \langle 1, 2 \rangle \wedge y = \ln x\}$.

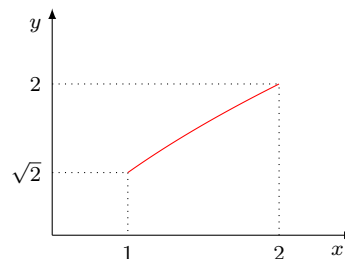
$$\begin{aligned} \vec{r}(t) &= (t, \ln t); \quad t \in \langle 1, 2 \rangle \\ \rightarrow \vec{r}'(t) &= \left(1, \frac{1}{t}\right) \\ \|\vec{r}'(t)\| &= \sqrt{1 + \frac{1}{t^2}} = \sqrt{\frac{t^2 + 1}{t^2}} = \frac{\sqrt{t^2 + 1}}{t} \end{aligned}$$



$$\begin{aligned} \int_{\mathcal{K}} x^2 \, ds &= \int_1^2 t^2 \frac{\sqrt{t^2 + 1}}{t} \, dt = \int_1^2 t \sqrt{t^2 + 1} \, dt = \left| \begin{array}{l} z = t^2 + 1 \\ dz = 2t \, dt \\ \frac{1}{2} \int \sqrt{z} \, dz = \frac{1}{3} z^{\frac{3}{2}} + C \end{array} \right| \\ &= \frac{1}{3} \left[(t^2 + 1)^{\frac{3}{2}} \right]_1^2 = \frac{1}{3} (5\sqrt{5} - 2\sqrt{2}). \end{aligned}$$

8. ✓ $\int_{\mathcal{K}} \frac{x^2}{y} ds$, kde křivka \mathcal{K} je část paraboly $y^2 = 2x$, $y \in \langle \sqrt{2}, 2 \rangle$.

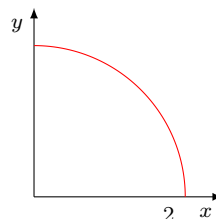
$$\begin{aligned}\vec{r}(t) &= \left(\frac{t^2}{2}, t \right); \quad t \in \langle \sqrt{2}, 2 \rangle \\ \rightarrow \vec{r}'(t) &= (t, 1) \\ \|\vec{r}'(t)\| &= \sqrt{t^2 + 1}\end{aligned}$$



$$\begin{aligned}\int_{\mathcal{K}} \frac{x^2}{y} ds &= \int_{\sqrt{2}}^2 \frac{1}{4} \frac{t^4}{t} \sqrt{1+t^2} dt = \frac{1}{4} \int_{\sqrt{2}}^2 t^3 \sqrt{t^2+1} dt = \left| \begin{array}{l} z = t^2 + 1 \\ dz = 2t dt \\ \frac{1}{8} \int (z-1) \sqrt{z} dz = \frac{1}{8} \int (z^{\frac{3}{2}} - z^{\frac{1}{2}}) dz \\ = \frac{1}{8} \frac{2}{5} z^{\frac{5}{2}} - \frac{1}{8} \frac{2}{3} z^{\frac{3}{2}} + C \end{array} \right| \\ &= \left[\frac{1}{20} (t^2+1)^{\frac{5}{2}} - \frac{1}{12} (t^2+1)^{\frac{3}{2}} \right]_{\sqrt{2}}^2 = \frac{5}{6} \sqrt{5} - \frac{1}{5} \sqrt{3}.\end{aligned}$$

9. ✓ $\int_{\mathcal{K}} y ds$, kde křivka \mathcal{K} je čtvrtina kružnice $x^2 + y^2 = 4$ v I. kvadrantu.

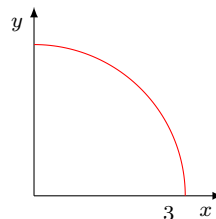
$$\begin{aligned}\vec{r}(t) &= (2 \cos t, 2 \sin t); \quad t \in \langle 0, \frac{\pi}{2} \rangle \\ \rightarrow \vec{r}'(t) &= (-2 \sin t, 2 \cos t) \\ \|\vec{r}'(t)\| &= \sqrt{4 \sin^2 t + 4 \cos^2 t} = 2\end{aligned}$$



$$\int_{\mathcal{K}} y ds = \int_0^{\frac{\pi}{2}} 2 \sin t \cdot 2 dt = 4 [-\cos t]_0^{\frac{\pi}{2}} = 4.$$

10. ✓ $\int_{\mathcal{K}} x ds$, kde křivka \mathcal{K} je čtvrtina kružnice $x^2 + y^2 = 9$ v I. kvadrantu.

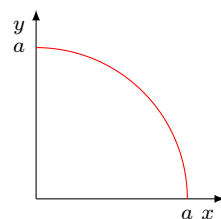
$$\begin{aligned}\vec{r}(t) &= (3 \cos t, 3 \sin t); \quad t \in \langle 0, \frac{\pi}{2} \rangle \\ \rightarrow \vec{r}'(t) &= (-3 \sin t, 3 \cos t) \\ \|\vec{r}'(t)\| &= \sqrt{9 \sin^2 t + 9 \cos^2 t} = 3\end{aligned}$$



$$\int_{\mathcal{K}} x ds = \int_0^{\frac{\pi}{2}} 3 \cos t \cdot 3 dt = 9 [\sin t]_0^{\frac{\pi}{2}} = 9.$$

11. ✓ $\int_{\mathcal{K}} x^2 y ds$, kde křivka \mathcal{K} je oblouk kružnice $x^2 + y^2 = a^2$ s počátečním bodem $[a, 0]$ a koncovým bodem $[0, a]$, kde $a > 0$.

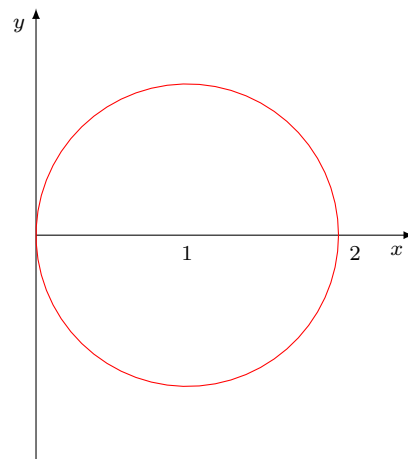
$$\begin{aligned}\vec{r}(t) &= (a \cos t, a \sin t); \quad t \in \langle 0, \frac{\pi}{2} \rangle \\ \rightarrow \vec{r}'(t) &= (-a \sin t, a \cos t) \\ ||\vec{r}'(t)|| &= \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a\end{aligned}$$



$$\begin{aligned}\int_{\mathcal{K}} x^2 y ds &= \int_0^{\frac{\pi}{2}} a^2 \cos^2 t \cdot a \sin t \cdot a dt = \int_0^{\frac{\pi}{2}} a^4 \cos^2 t \cdot \sin t dt = \left| \begin{array}{l} z = \cos t \\ dz = -\sin t dt \\ -a^4 \int z^2 dz = -a^4 \frac{z^3}{3} + C \end{array} \right| = \\ &= -a^4 \left[\frac{\cos^3 t}{3} \right]_0^{\frac{\pi}{2}} = \frac{a^4}{3}.\end{aligned}$$

12. ✓ $\int_{\mathcal{K}} (x^2 + y^2) ds$, kde křivka \mathcal{K} je popsána rovnicí $x^2 + y^2 = 2x$.

$$\begin{aligned}(x-1)^2 + y^2 &= 1 \\ \vec{r}(t) &= (1 + \cos t, \sin t); \quad t \in \langle 0, 2\pi \rangle \\ \rightarrow \vec{r}'(t) &= (-\sin t, \cos t) \\ ||\vec{r}'(t)|| &= \sqrt{\sin^2 t + \cos^2 t} = 1\end{aligned}$$

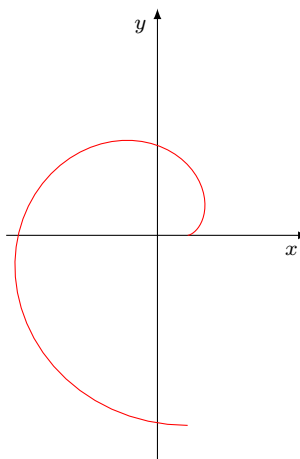


$$\int_{\mathcal{K}} (x^2 + y^2) ds = \int_0^{2\pi} ((1 + \cos t)^2 + \sin^2 t) dt = \int_0^{2\pi} (2 + 2 \cos t) dt = [(2t + 2 \sin t)]_0^{2\pi} = 4\pi.$$

13. ✓ $\int_{\mathcal{K}} (x^2 + y^2) ds$, kde křivka \mathcal{K} je popsána parametrickými rovnicemi $x(t) = a(\cos t + t \sin t)$, $y(t) = a(\sin t - t \cos t)$, $t \in \langle 0, 2\pi \rangle$, $a > 0$.

$$\begin{aligned}\vec{r}(t) &= (a(\cos t + t \sin t), a(\sin t - t \cos t)); \quad t \in \langle 0, 2\pi \rangle \\ \rightarrow \vec{r}'(t) &= (a(-\sin t + \sin t + t \cos t), a(\cos t - \cos t + t \sin t)) = \\ &= (at \cos t, at \sin t)\end{aligned}$$

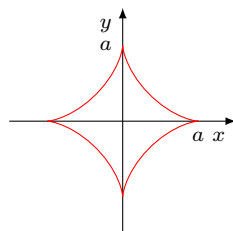
$$||\vec{r}'(t)|| = \sqrt{a^2 t^2 \cos^2 t + a^2 t^2 \sin^2 t} = at$$



$$\begin{aligned}\int_{\mathcal{K}} (x^2 + y^2) ds &= \int_0^{2\pi} (a^2(\cos t + t \sin t)^2 + a^2(\sin t - t \cos t)^2) a t dt = \\ &= a^3 \int_0^{2\pi} (t + t^3) dt = a^3 \left[\frac{t^2}{2} + \frac{t^4}{4} \right]_0^{2\pi} = a^3(2\pi^2 + 4\pi^4).\end{aligned}$$

14. ✓ Vypočtete délku asteroidy $\mathcal{K} : \sqrt[3]{x^2} + \sqrt[3]{y^2} = \sqrt[3]{a^2}$, $a > 0$.

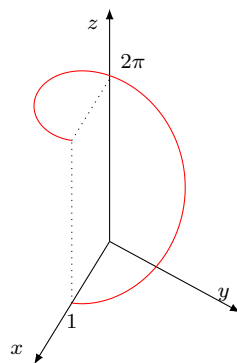
$$\begin{aligned}\vec{r}(t) &= (a \cos^3 t, a \sin^3 t); \quad t \in \langle 0, 2\pi \rangle \\ \rightarrow \vec{r}'(t) &= (-3a \cos^2 t \sin t, 3a \sin^2 t \cos t) \\ \|\vec{r}'(t)\| &= \sqrt{9a^2(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)} \\ &= 3a |\cos t \sin t|\end{aligned}$$



$$\begin{aligned}\int_{\mathcal{K}} 1 ds &= 4 \int_0^{\frac{\pi}{2}} 3a \cos t \sin t dt = \left| \begin{array}{l} z = \sin t \\ dz = \cos t dt \\ \int z dz = \frac{z^2}{2} + C \end{array} \right| = \\ &= 6a [\sin^2 t]_0^{\frac{\pi}{2}} = 6a.\end{aligned}$$

15. ✓ $\int_{\mathcal{K}} \frac{z^2}{x^2 + y^2} ds$, kde křivka \mathcal{K} je první závit šroubovice $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = t$.

$$\begin{aligned}\vec{r}(t) &= (\cos t, \sin t, t); \quad t \in \langle 0, 2\pi \rangle \\ \rightarrow \vec{r}'(t) &= (-\sin t, \cos t, 1) \\ \|\vec{r}'(t)\| &= \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}\end{aligned}$$



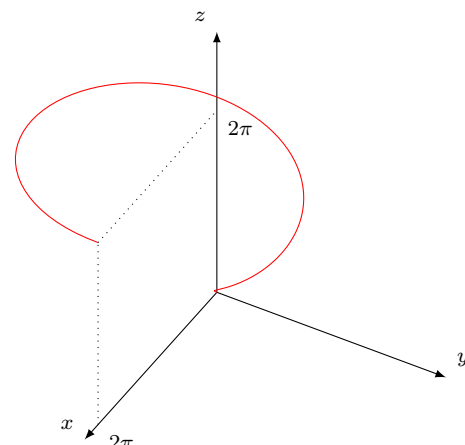
$$\int_{\mathcal{K}} \frac{z^2}{x^2 + y^2} ds = \int_0^{2\pi} \frac{t^2}{\cos^2 t + \sin^2 t} \cdot \sqrt{2} dt = \sqrt{2} \left[\frac{t^3}{3} \right]_0^{2\pi} = \frac{8\sqrt{2}\pi^3}{3}.$$

16. ✓ $\int_{\mathcal{K}} (2\sqrt{x^2 + y^2} - z) ds$, kde křivka \mathcal{K} je 1. závit kuželové šroubovice $x(t) = t \cos t$,
 $y(t) = t \sin t$, $z(t) = t$.

$$\vec{r}(t) = (t \cos t, t \sin t, t); \quad t \in \langle 0, 2\pi \rangle$$

$$\vec{r}'(t) = (\cos t - t \sin t, \sin t + t \cos t, 1)$$

$$\begin{aligned} \|\vec{r}'(t)\| &= \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 + 1} = \\ &= \sqrt{2 + t^2} \end{aligned}$$



$$\begin{aligned} \int_{\mathcal{K}} (2\sqrt{x^2 + y^2} - z) ds &= \int_0^{2\pi} (2\sqrt{t^2 \cos^2 t + t^2 \sin^2 t} - t) \cdot \sqrt{2 + t^2} dt = \\ &= \int_0^{2\pi} t\sqrt{2 + t^2} dt = \left| \begin{array}{l} 2 + t^2 = z \\ 2t dt = dz \\ \frac{1}{2} \int \sqrt{z} dz = \frac{1}{3} z^{\frac{3}{2}} + C \end{array} \right| = \frac{1}{2} \left[(2 + t^2)^{\frac{3}{2}} \right]_0^{2\pi} = \frac{1}{2} ((2 + 4\pi^2)^{\frac{3}{2}} - 2\sqrt{2}). \end{aligned}$$

17. ✓ $\int_{\mathcal{K}} (x + y) ds$, kde \mathcal{K} je průniková křivka ploch $x^2 + y^2 + z^2 = a^2$, $a > 0$, $x = y$ v prvním oktantu.

Parametrizace kružnice $\mathcal{K}(\sigma, S, r)$ o poloměru $r = a$, se středem $S = [0, 0, 0]$, ležící v rovině σ o rovnici $\vec{n}(X - S)$: $\vec{r}(t) = (s_1 + r \cos t \cdot u_1 + r \sin t \cdot v_1, s_2 + r \cos t \cdot u_2 + r \sin t \cdot v_2, s_3 + r \cos t \cdot u_3 + r \sin t \cdot v_3)$, kde \vec{u}, \vec{v} jsou báze vektory lokální kartézské souřadné soustavy roviny σ . Lze brát $\vec{u} = \frac{A-S}{|A-S|}$, kde A je bod na kružnici $\mathcal{K}(\sigma, S, r)$ a $\vec{v} = \vec{u} \times \frac{\vec{n}}{|\vec{n}|}$.

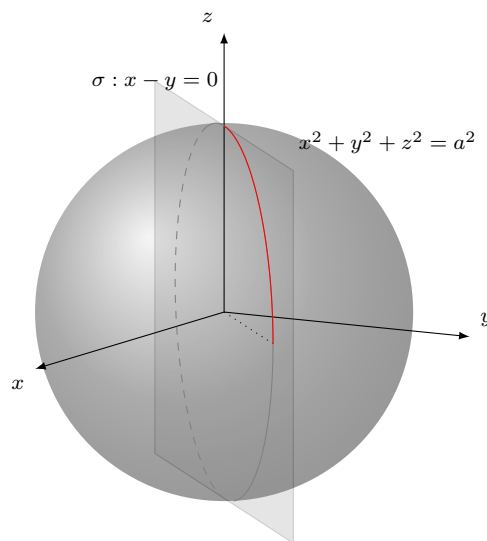
$$\vec{u} = \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, 0 \right), \vec{v} = (0, 0, a)$$

$$\vec{r}(t) = \left(\frac{a^2}{\sqrt{2}} \cos t, \frac{a^2}{\sqrt{2}} \cos t, a^2 \sin t \right),$$

$$t \in \langle 0, \frac{\pi}{2} \rangle$$

$$\vec{r}'(t) = \left(-\frac{a^2}{\sqrt{2}} \sin t, -\frac{a^2}{\sqrt{2}} \sin t, a^2 \cos t \right)$$

$$\begin{aligned} \|\vec{r}'(t)\| &= \sqrt{\frac{a^4}{2} \sin^2 t + \frac{a^4}{2} \sin^2 t + a^4 \cos^2 t} \\ &= a^2 \end{aligned}$$



$$\int_{\mathcal{K}} (x + y) ds = \int_0^{\frac{\pi}{2}} \left(\frac{a^2}{\sqrt{2}} \cos t + \frac{a^2}{\sqrt{2}} \cos t \right) \cdot a^2 dt = \frac{2a^4}{\sqrt{2}} [\sin t]_0^{\frac{\pi}{2}} = \sqrt{4}a^2$$

8. Cvičení: Křivkové integrály 2. druhu

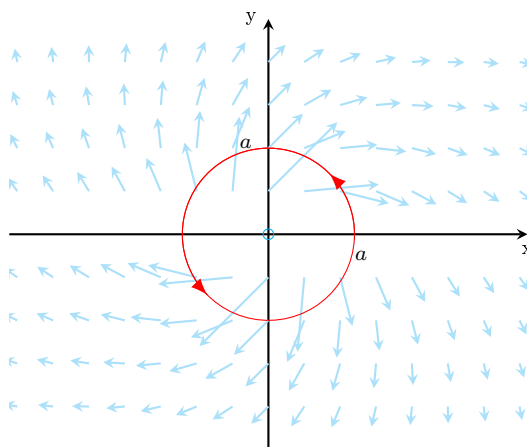
Příklady: Vypočtěte dané křivkové integrály 2. druhu (práce, po uzavřené křivce - cirkulace).

1. $\int_{\mathcal{K}} \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$, kde křivka \mathcal{K} je kladně orientovaná kružnice $x^2 + y^2 = a^2$, $a > 0$.

$$\vec{r}(t) = (a \cos t, a \sin t); \quad t \in \langle 0, 2\pi \rangle$$

$$\vec{r}'(t) = (-a \sin t, a \cos t)$$

souhlasná orientace



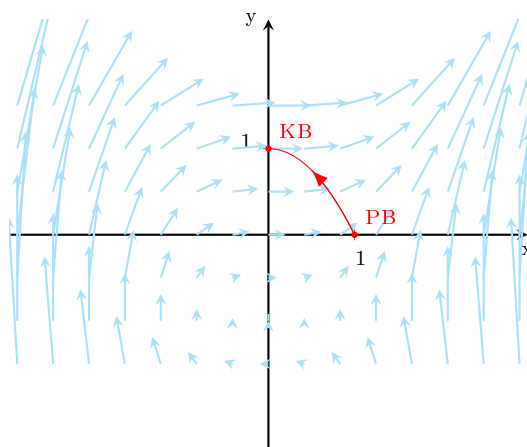
$$\begin{aligned} \int_{\mathcal{K}} \frac{(x+y)dx - (x-y)dy}{x^2+y^2} &= \int_0^{2\pi} \frac{(a \cos t + a \sin t)(-a \sin t) - (a \cos t - a \sin t)(a \cos t)}{a^2} dt = \\ &= \int_0^{2\pi} (-\cos t \sin t - \sin^2 t - \cos^2 t + \sin t \cos t) dt = - \int_0^{2\pi} dt = -2\pi \end{aligned}$$

2. $\int_{\mathcal{K}} \vec{f} d\vec{r}$, kde $\vec{f} = (y+1, x^2)$ a křivka \mathcal{K} je část paraboly $y = 1 - x^2$ s počátečním bodem $[1, 0]$ a koncovým bodem $[0, 1]$.

$$\vec{r}(t) = (t, 1 - t^2); \quad t \in \langle 0, 1 \rangle$$

$$\vec{r}'(t) = (1, -2t)$$

nesouhlasná orientace



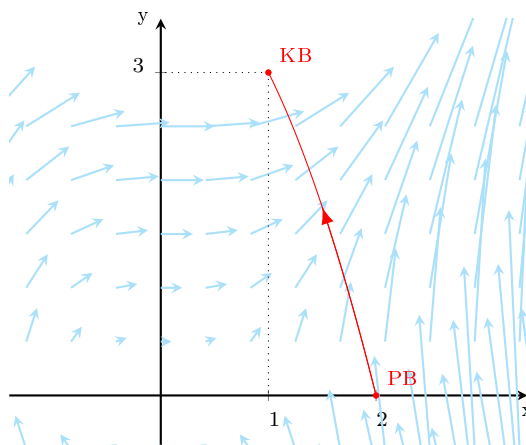
$$\int_{\mathcal{K}} (y+1)dx + x^2 dy = - \int_0^1 ((1-t^2+1) \cdot 1 + t^2 \cdot (-2t)) dt = \left[\frac{t^4}{2} + \frac{t^3}{3} - 2t \right]_0^1 = -\frac{7}{6}$$

3. $\int_{\mathcal{K}} \vec{f} d\vec{r}$, kde $\vec{f} = (y, x^2)$ a křivka \mathcal{K} je část paraboly $y = 4 - x^2$ s počátečním bodem $[2, 0]$ a koncovým bodem $[1, 3]$.

$$\vec{r}(t) = (t, 4 - t^2); \quad t \in \langle 1, 2 \rangle$$

$$\vec{r}'(t) = (1, -2t)$$

nesouhlasná orientace



$$\begin{aligned} \int_{\mathcal{K}} y dx + x^2 dy &= - \int_1^2 ((4 - t^2) \cdot 1 + t^2 \cdot (-2t)) dt = \left[\frac{t^4}{2} + \frac{t^3}{3} - 4t \right]_1^2 = \\ &= 8 + \frac{8}{3} - 8 - \frac{1}{2} - \frac{1}{3} + 4 = \frac{35}{6} \end{aligned}$$

4. $\int_{\mathcal{K}} \vec{f} d\vec{r}$, kde $\vec{f} = (x - y, x)$ a křivka \mathcal{K} je kladně orientovaná hranice čtverce $ABCD$, kde $A = [1, 1]$, $B = [-1, 1]$, $C = [-1, -1]$, $D = [1, -1]$.

$$\overrightarrow{AB} : \vec{r}(t) = (t, 1); \quad t \in \langle -1, 1 \rangle$$

$$\vec{r}'(t) = (1, 0)$$

nesouhlasná orientace

$$\overrightarrow{BC} : \vec{r}(t) = (-1, t); \quad t \in \langle -1, 1 \rangle$$

$$\vec{r}'(t) = (0, 1)$$

nesouhlasná orientace

$$\overrightarrow{CD} : \vec{r}(t) = (t, -1); \quad t \in \langle -1, 1 \rangle$$

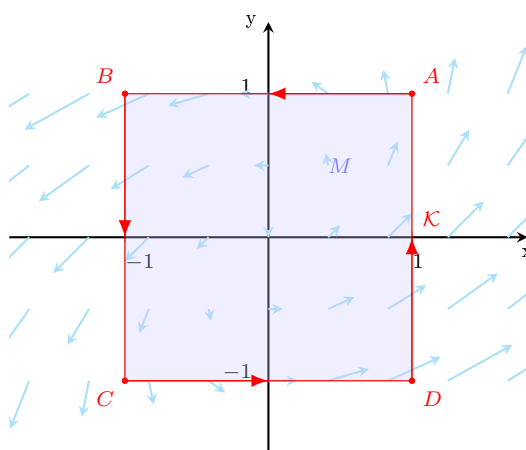
$$\vec{r}'(t) = (1, 0)$$

souhlasná orientace

$$\overrightarrow{DA} : \vec{r}(t) = (1, t); \quad t \in \langle -1, 1 \rangle$$

$$\vec{r}'(t) = (0, 1)$$

souhlasná orientace



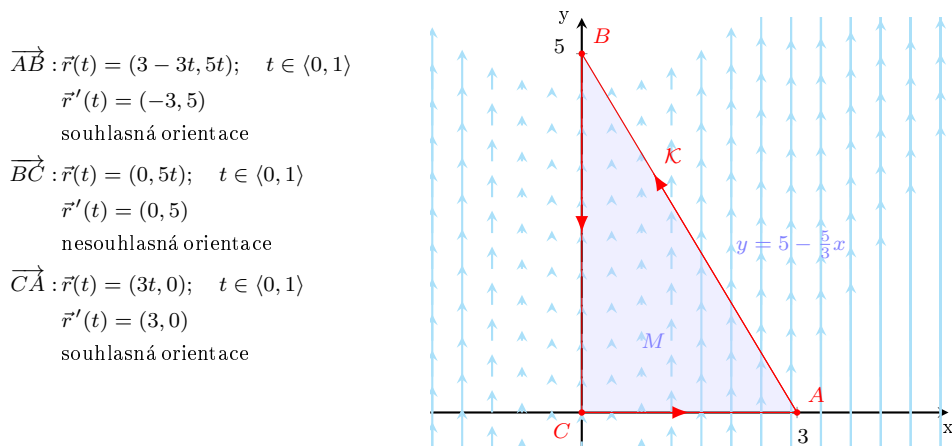
a)

$$\begin{aligned} \int_{\mathcal{K}} (x - y) dx + x dy &= - \int_{-1}^1 ((t - 1) \cdot 1 + t \cdot 0) dt - \int_{-1}^1 ((-1 - t) \cdot 0 + (-1) \cdot 1) dt + \\ &+ \int_{-1}^1 ((t + 1) \cdot 1 + t \cdot 0) dt + \int_{-1}^1 ((1 - t) \cdot 0 + (1) \cdot 1) dt = \left[-\frac{t^2}{2} + t + t + \frac{t^2}{2} + t + t \right]_{-1}^1 = \\ &= 4 + 4 = 8 \end{aligned}$$

b) Greenova věta

$$\oint_{\mathcal{K}} (x - y) dx + x dy = \iint_M (1 + 1) dx dy = 2 \int_{-1}^1 \int_{-1}^1 dx dy = 2S_{ABCD} = 8$$

5. $\int_{\mathcal{K}} \vec{f} d\vec{r}$, kde $\vec{f} = (0, x^2)$ a křivka \mathcal{K} je kladně orientovaná hranice trojúhelníku ohraničeného osami x, y a křivkou $\frac{x}{3} + \frac{y}{5} = 1$.



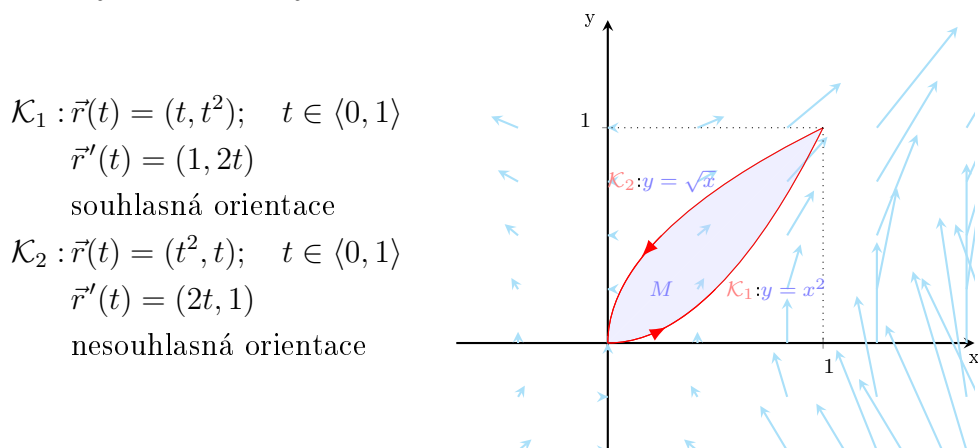
a)

$$\int_{\mathcal{K}} x^2 dy = \int_0^1 5(3 - 3t)^2 dt = 5 \left[-\frac{1}{3} \frac{(3 - 3t)^3}{3} \right]_0^1 = 15$$

b) Greenova věta

$$\begin{aligned} \oint_{\mathcal{K}} x^2 dy &= \iint_M 2x dx dy = 2 \int_0^3 \int_0^{5-\frac{5}{3}x} x dy dx = 2 \int_0^3 \left(5x - \frac{5}{3}x^2 \right) dx = \\ &= 2 \left[\frac{5}{2}x^2 - \frac{5}{9}x^3 \right]_0^3 = 45 - 30 = 15 \end{aligned}$$

6. $\int_{\mathcal{K}} \vec{f} d\vec{r}$, kde $\vec{f} = (xy, x^2)$ a křivka \mathcal{K} je kladně orientovaná hranice obrazce ohraničeného křivkami $y^2 = x$ a $x^2 = y$.



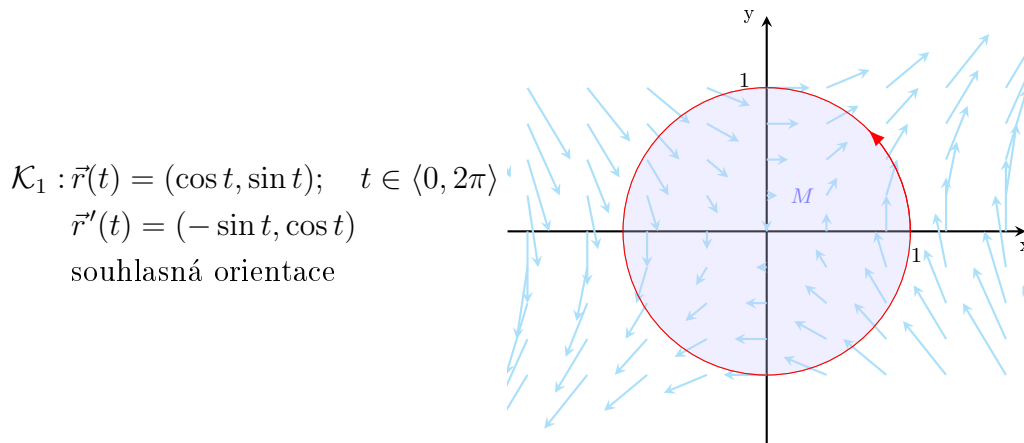
a)

$$\int_{\mathcal{K}} xy \, dx + x^2 \, dy = \int_0^1 (t^3 + 2t^3) \, dt - \int_0^1 (2t^4 + t^4) \, dt = \left[\frac{3t^4}{4} - \frac{3t^5}{5} \right]_0^1 = \frac{3}{20}$$

b) Greenova věta

$$\begin{aligned} \oint_{\mathcal{K}} xy \, dx + x^2 \, dy &= \iint_M x \, dx \, dy = \int_0^1 \int_{x^2}^{\sqrt{x}} x \, dy \, dx = \int_0^1 \left(x^{\frac{3}{2}} - x^3 \right) \, dx = \\ &= \left[\frac{2}{5} x^{\frac{5}{2}} - \frac{x^4}{4} \right]_0^1 = \frac{3}{20} \end{aligned}$$

7. $\int_{\mathcal{K}} \vec{f} \, d\vec{r}$, kde $\vec{f} = (y, x)$ a křivka \mathcal{K} je kladně orientovaná křivka $x^2 + y^2 = 1$.



a)

$$\int_{\mathcal{K}} y \, dx + x \, dy = \int_0^{2\pi} (-\sin^2 t + \cos^2 t) \, dt = \int_0^{2\pi} \cos 2t \, dt = \left[\frac{1}{2} \sin 2t \right]_0^{2\pi} = 0$$

b) Greenova věta

$$\oint_{\mathcal{K}} y \, dx + x \, dy = \iint_M (1 - 1) \, dx \, dy = 0$$

c) Potenciálové pole

$$\text{rot } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = (0, 0, 1 - 1)$$

Pole je potenciálové \rightarrow práce po uzavřené křivce je 0.

Příklady: Vypočtete dané křivkové integrály 2. druhu

1. Ukažte, že křivkový integrál druhého druhu vektorového pole $\vec{f} = (y^2z^3 + z, 2xyz^3 - z, 3xy^2z^2 + x - y)$ nezávisí na integrační cestě v oblasti \mathbb{R}^3 a vypočtete práci, kterou pole vykoná z bodu $A = [1, 1, 1]$ do bodu $B = [-2, 1, -1]$.

\mathbb{R}^3 jednoduše souvislá oblast

$$\begin{aligned} \operatorname{rot} \vec{f} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 + z & 2xyz^3 - z & 3xy^2z^2 + x - y \end{vmatrix} = \\ &= (6xyz^2 - 1 - (6xyz^2 - 1), 3y^2z^2 + 1 - (3y^2z^2 + 1), 2yz^3 - 2yz^3) = \vec{0} \end{aligned}$$

\Rightarrow pole je potenciálové \rightarrow KI 2. druhu nezávisí na integrační cestě.

Potenciál:

$$\left. \begin{aligned} \frac{\partial \varphi}{\partial x} &= y^2z^3 + z \rightarrow \varphi = xy^2z^3 + xz + C \\ \frac{\partial \varphi}{\partial y} &= 2xyz^3 - z \rightarrow \varphi = xy^2z^3 - yz + C \\ \frac{\partial \varphi}{\partial z} &= 3xy^2z^2 + x - y \rightarrow \varphi = xy^2z^3 + xz - yz + C \end{aligned} \right\} \varphi(x, y, z) = xy^2z^3 + xz - yz + C$$

$$\int_{\kappa} (y^2z^3 + z) dx + (2xyz^3 - z) dy + (3xy^2z^2 + x - y) dz = \varphi(B) - \varphi(A) = 4$$

2. Ukažte, že křivkový integrál druhého druhu vektorového pole $\vec{f} = (x^2 - 2yz, y^2 - 2xz, z^2 - 2xy)$ nezávisí na integrační cestě v oblasti \mathbb{R}^3 a vypočtete práci, kterou pole vykoná z bodu $A = [1, 1, 1]$ do bodu $B = [-1, 2, -2]$.

\mathbb{R}^3 jednoduše souvislá oblast

$$\begin{aligned} \operatorname{rot} \vec{f} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - 2yz & y^2 - 2xz & z^2 - 2xy \end{vmatrix} = \\ &= (-2x + 2x, -2y + 2y, -2z + 2z) = \vec{0} \end{aligned}$$

\Rightarrow pole je potenciálové \rightarrow KI 2. druhu nezávisí na integrační cestě.

Potenciál:

$$\left. \begin{aligned} \frac{\partial \varphi}{\partial x} &= x^2 - 2yz \rightarrow \varphi = \frac{x^3}{3} - 2xyz + C \\ \frac{\partial \varphi}{\partial y} &= y^2 - 2xz \rightarrow \varphi = \frac{y^3}{3} - 2xyz + C \\ \frac{\partial \varphi}{\partial z} &= z^2 - 2xy \rightarrow \varphi = \frac{z^3}{3} - 2xyz + C \end{aligned} \right\} \varphi(x, y, z) = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - 2xyz + C$$

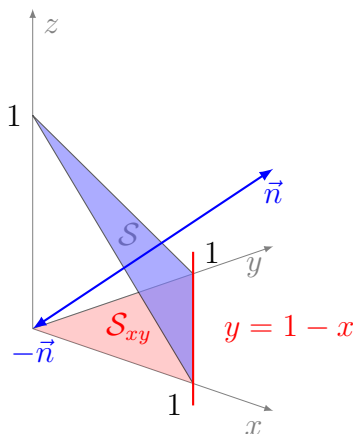
$$\int_{\kappa} (x^2 - 2yz) dx + (y^2 - 2xz) dy + (z^2 - 2xy) dz = \varphi(B) - \varphi(A) = -\frac{22}{3}$$

9. Cvičení: Plošné integrály 1. druhu

Příklady: Vypočtěte dané plošné integrály 1. druhu.

1. Vypočtěte $\iint_{\mathcal{S}} xyz \, dS$, kde \mathcal{S} je část roviny $x + y + z = 1$ v prvním oktantu ($x > 0$, $y > 0$, $z > 0$).

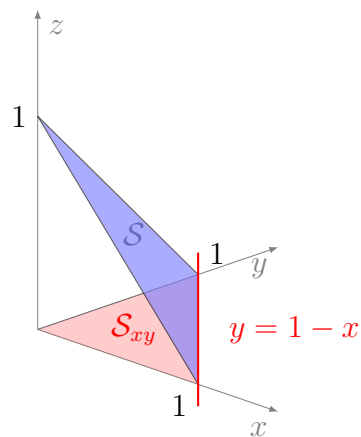
$$\begin{aligned} g(x, y) &= z = 1 - x - y \\ g'_x(x, y) &= -1, \quad g'_y(x, y) = -1 \\ \|\vec{n}\| &= \sqrt{(g'_x)^2 + (g'_y)^2 + 1} = \sqrt{3} \end{aligned}$$



$$\begin{aligned} \iint_{\mathcal{S}} xyz \, dS &= \sqrt{3} \iint_{\mathcal{S}_{xy}} xy(1-x-y) \, dx dy = \sqrt{3} \int_0^1 \int_0^{1-x} (xy - x^2y - xy^2) \, dy \, dx = \\ &= \sqrt{3} \int_0^1 \left[x \frac{y^2}{2} - x^2 \frac{y^2}{2} - x \frac{y^3}{3} \right]_0^{1-x} dx = \sqrt{3} \int_0^1 \left(x(1-x) \frac{(1-x)^2}{2} - x \frac{(1-x)^3}{3} \right) dx = \\ &= \sqrt{3} \int_0^1 \left(x \frac{(1-x)^3}{6} \right) dx = \left| \begin{array}{ccc} D & & I \\ x & \searrow & \frac{(1-x)^3}{6} \\ 1 & \searrow & -\frac{(1-x)^4}{24} \\ 0 & \rightarrow \int & \frac{(1-x)^5}{120} \end{array} \right| = \sqrt{3} \left[-x \frac{(1-x)^4}{24} + \frac{(1-x)^5}{120} \right]_0^1 = \\ &= \frac{\sqrt{3}}{120}. \end{aligned}$$

2. Vypočtete $\iint_S \left(\frac{x}{2} + y + z\right) dS$, kde S je část roviny $x + y + z = 1$ v prvním oktantu ($x > 0$, $y > 0$, $z > 0$).

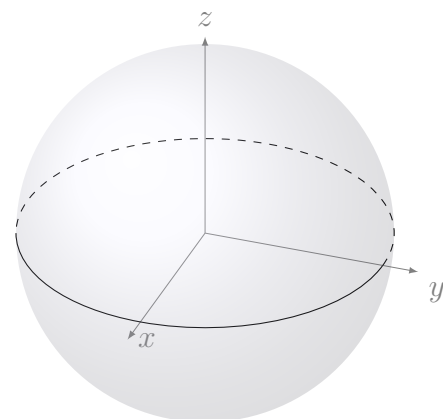
$$\begin{aligned} g(x, y) &= z = 1 - x - y \\ g'_x(x, y) &= -1, \quad g'_y(x, y) = -1 \\ \|\vec{n}\| &= \sqrt{(g'_x)^2 + (g'_y)^2 + 1} = \sqrt{3} \end{aligned}$$



$$\begin{aligned} \iint_S \left(\frac{x}{2} + y + z\right) dS &= \sqrt{3} \iint_{S_{xy}} \left(\frac{x}{2} + y + 1 - x - y\right) dx dy = \sqrt{3} \int_0^1 \int_0^{1-x} \left(1 - \frac{x}{2}\right) dy dx = \\ &= \sqrt{3} \int_0^1 \left[y - \frac{x}{2}y\right]_0^{1-x} dx = \sqrt{3} \int_0^1 \left(1 - \frac{3}{2}x + \frac{x^2}{2}\right) dx = \sqrt{3} \left[x - \frac{3}{4}x^2 + \frac{x^3}{6}\right]_0^1 = \frac{5\sqrt{3}}{12}. \end{aligned}$$

3. Vypočtete $\iint_S 1 dS$, kde S je $x^2 + y^2 + z^2 = R^2$.

$$\begin{aligned}\vec{r}(u, v) &= (R \cos u \cos v, R \sin u \cos v, R \sin v), \\ u &\in \langle 0, 2\pi \rangle, v \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle, \\ \vec{t}_u &= (-R \sin u \cos v, R \cos u \cos v, 0) \\ \vec{t}_v &= (-R \cos u \sin v, -R \sin u \sin v, R \cos v)\end{aligned}$$

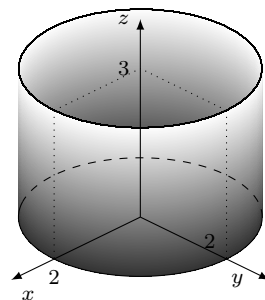


$$\begin{aligned}\vec{n} = \vec{t}_u \times \vec{t}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R \sin u \cos v & R \cos u \cos v & 0 \\ -R \cos u \sin v & -R \sin u \sin v & R \cos v \end{vmatrix} = \\ &= (R^2 \cos u \cos^2 v, R^2 \sin u \cos^2 v, R^2 \sin^2 u \sin v \cos v + R^2 \cos^2 u \sin v \cos v) = \\ &= (R^2 \cos u \cos^2 v, R^2 \sin u \cos^2 v, R^2 \sin v \cos v) \\ |\vec{n}| &= \sqrt{R^4 \cos^2 u \cos^4 v + R^4 \sin^2 u \cos^4 v + R^4 \sin^2 v \cos^2 v} = \\ &= \sqrt{R^4 \cos^4 v + R^4 \sin^2 v \cos^2 v} = \sqrt{R^4 \cos^2 v} = R^2 |\cos v| = R^2 \cos v\end{aligned}$$

$$\begin{aligned}\iint_S 1 dS &= \iint_{\Omega} R^2 |\cos v| du dv = 2R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \cos v du dv = 2\pi R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos v dv = \\ &= 2\pi R^2 [\sin v]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi R^2\end{aligned}$$

4. Vypočtete $\int_{\mathcal{S}} \frac{1}{x^2+y^2+z^2} dS$, kde \mathcal{S} je válcová plocha $x^2 + y^2 = 4$, $0 \leq z \leq 3$.

$$\begin{aligned}\vec{r}(u, v) &= (2 \cos u, 2 \sin u, v) \\ u &\in \langle 0, 2\pi \rangle, \quad v \in \langle 0, 3 \rangle, \\ \vec{r}'_u &= (-2 \sin u, 2 \cos u, 0) \\ \vec{r}'_v &= (0, 0, 1)\end{aligned}$$



$$\begin{aligned}\vec{n} = \vec{r}'_u \times \vec{r}'_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 \sin u & 2 \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2 \cos u, 2 \sin u, 0) = \\ |\vec{n}| &= \sqrt{4 \cos^2 u + 4 \sin^2 u} = 2\end{aligned}$$

$$\begin{aligned}\iint_{\mathcal{S}} \frac{1}{x^2 + y^2 + z^2} dS &= \iint_{\Omega} \frac{1}{4 \cos^2 u + 4 \sin^2 u + v^2} 2 du dv = \int_0^3 \int_0^{2\pi} \frac{2}{4 + v^2} du dv = \\ &= \frac{2}{4} 2\pi \int_0^3 \frac{1}{1 + \left(\frac{v}{2}\right)^2} dv = \pi \left[2 \arctg \frac{v}{2} \right]_0^3 = 2\pi \arctg \frac{3}{2}\end{aligned}$$

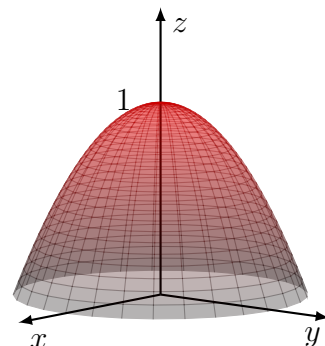
5. Vypočtete $\iint_{\mathcal{S}} (x^2 + y^2) dS$, kde $\mathcal{S} = \{[x, y, z] \in \mathbb{R}^3 : z = 1 - x^2 - y^2 \wedge z \geq 0\}$.

$$\vec{r}(u, v) = (u \cos v, u \sin v, 1 - u^2)$$

$$u \in \langle 0, 1 \rangle, v \in \langle 0, 2\pi \rangle,$$

$$\vec{t}_u = (\cos v, \sin v, -2u)$$

$$\vec{t}_v = (-u \sin v, u \cos v, 0)$$



$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & -2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (2u^2 \cos v, 2u^2 \sin v, u) =$$

$$|\vec{n}| = \sqrt{4u^4 + u^2} = u\sqrt{4u^2 + 1}$$

$$\iint_{\mathcal{S}} (x^2 + y^2) dS = \iint_{\Omega} u^2 u \sqrt{4u^2 + 1} du dv = \int_0^{2\pi} \int_0^1 u^2 u \sqrt{4u^2 + 1} du dv =$$

$$= \left| \begin{array}{l} t = 4u^2 + 1 \\ \frac{t-1}{4} = u^2 \\ dt = 8u du \\ \frac{1}{8} \int \frac{t-1}{4} \sqrt{t} dt = \frac{1}{32} \frac{2}{5} t^{\frac{5}{2}} - \frac{2}{32} t^{\frac{1}{2}} + C \end{array} \right| = \int_0^{2\pi} \left[\frac{1}{80} (4u^2 + 1)^{\frac{5}{2}} - \frac{1}{16} (4u^2 + 1)^{\frac{1}{2}} \right]_0^1 dv =$$

$$= \left(\frac{1}{80} 5^{\frac{5}{2}} - \frac{1}{16} 5^{\frac{1}{2}} - \frac{1}{80} + \frac{1}{16} \right) \int_0^{2\pi} dv = \pi \left(\frac{5}{8} \sqrt{5} - \frac{1}{8} \sqrt{5} - 1/10 \right) = \pi \left(\frac{1}{2} \sqrt{5} - 1/10 \right)$$

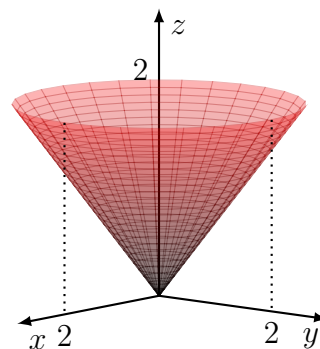
6. Vypočtěte $\iint_{\mathcal{S}} dS$, kde $\mathcal{S} = \left\{ [x, y, z] \in \mathbb{R}^3 : z = \sqrt{x^2 + y^2} \wedge 2 \geq z \geq 0 \right\}$.

$$\vec{r}(u, v) = (u \cos v, u \sin v, u)$$

$$u \in \langle 0, 2 \rangle, v \in \langle 0, 2\pi \rangle,$$

$$\vec{t}_u = (\cos v, \sin v, 1)$$

$$\vec{t}_v = (-u \sin v, u \cos v, 0)$$



$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (-u \cos v, -u \sin v, u)$$

$$|\vec{n}| = \sqrt{u^2 + u^2} = \sqrt{2}u$$

$$\iint_{\mathcal{S}} dS = \iint_{\Omega} \sqrt{2}u \, du \, dv = \sqrt{2} \int_0^{2\pi} \int_0^2 u \, du \, dv = \sqrt{2} \int_0^{2\pi} \left[\frac{u^2}{2} \right]_0^2 dv = 2\sqrt{2} \int_0^{2\pi} dv = 4\pi\sqrt{2}.$$

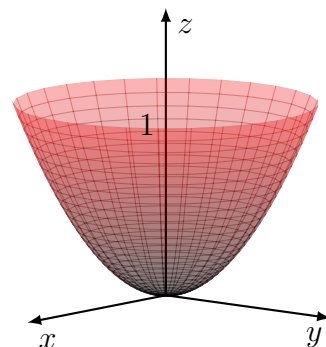
7. Vypočtete $\iint_S \sqrt{1+x^2+y^2} dS$, kde $S = \{[x, y, z] \in \mathbb{R}^3 : 2z = x^2 + y^2 \wedge z \leq 1\}$.

$$\vec{r}(u, v) = (u \cos v, u \sin v, \frac{u^2}{2})$$

$$u \in \langle 0, \sqrt{2} \rangle, v \in \langle 0, 2\pi \rangle,$$

$$\vec{t}_u = (\cos v, \sin v, u)$$

$$\vec{t}_v = (-u \sin v, u \cos v, 0)$$



$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (-u^2 \cos v, -u^2 \sin v, u) =$$

$$|\vec{n}| = \sqrt{u^4 + u^2} = u\sqrt{u^2 + 1}$$

$$\iint_S \sqrt{1+x^2+y^2} dS = \iint_{\Omega} \sqrt{1+u^2 \cos^2 v + u^2 \sin^2 v} u \sqrt{u^2 + 1} du dv =$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} u(1+u^2) du dv = \int_0^{2\pi} \left[\frac{u^2}{2} + \frac{u^3}{3} \right]_0^{\sqrt{2}} dv = \left(1 + \frac{2^{\frac{3}{2}}}{3} \right) \int_0^{2\pi} dv = 2\pi + \frac{4\sqrt{2}}{3}\pi$$

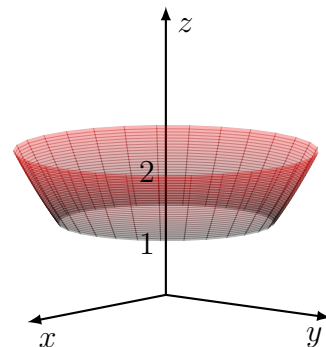
8. Vypočtete $\iint_S \operatorname{arctg} \frac{y}{x} dS$, kde $\mathcal{S} = \{[x, y, z] \in \mathbb{R}^3 : z = x^2 + y^2 \wedge 1 \leq z \leq 2\}$.

$$\vec{r}(u, v) = (u \cos v, u \sin v, u^2)$$

$$u \in \langle 1, \sqrt{2} \rangle, v \in \langle 0, 2\pi \rangle,$$

$$\vec{t}_u = (\cos v, \sin v, 2u)$$

$$\vec{t}_v = (-u \sin v, u \cos v, 0)$$



$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (-2u^2 \cos v, -2u^2 \sin v, u) =$$

$$|\vec{n}| = \sqrt{4u^4 + u^2} = u\sqrt{4u^2 + 1}$$

$$\begin{aligned} \iint_S \operatorname{arctg} \frac{y}{x} dS &= \iint_{\Omega} \operatorname{arctg} \left(\frac{u \sin v}{u \cos v} \right) u \sqrt{4u^2 + 1} du dv = \int_0^{2\pi} \int_1^{\sqrt{2}} v u \sqrt{4u^2 + 1} du dv = \\ &= \left| \begin{array}{l} t = 1 + 4u^2 \\ dt = 8u du \\ \frac{1}{8} \int \sqrt{t} dt = \frac{1}{12} t^{3/2} + C \end{array} \right| = \frac{1}{12} \int_0^{2\pi} v [(1 + 4u^2)^{3/2}]_0^{\sqrt{2}} dv = \frac{1}{12} (9^{3/2} - 1) \int_0^{2\pi} v dv = \frac{\pi}{6} (9^{3/2} - 1) \end{aligned}$$

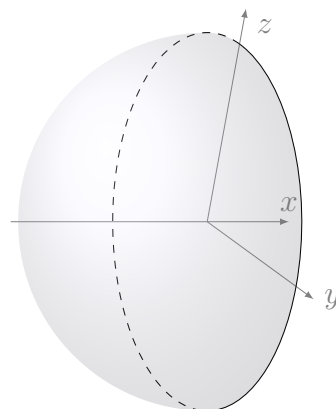
9. Vypočtete $\iint_S \frac{1}{z^2+9} dS$, kde \mathcal{S} je $x^2 + y^2 + z^2 = 9$, $x \leq 0$.

$$\vec{r}(u, v) = (3 \cos u \cos v, 3 \sin u \cos v, 3 \sin v),$$

$$u \in \left\langle \frac{\pi}{2}, \frac{3\pi}{2} \right\rangle, v \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle,$$

$$\vec{t}_u = (-3 \sin u \cos v, 3 \cos u \cos v, 0)$$

$$\vec{t}_v = (-3 \cos u \sin v, -3 \sin u \sin v, 3 \cos v)$$



$$\begin{aligned} \vec{n} = \vec{t}_u \times \vec{t}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 \sin u \cos v & 3 \cos u \cos v & 0 \\ -3 \cos u \sin v & -3 \sin u \sin v & 3 \cos v \end{vmatrix} = \\ &= (9 \cos u \cos^2 v, 9 \sin u \cos^2 v, 9 \sin^2 u \sin v \cos v + 9 \cos^2 u \sin v \cos v) = \\ &= (9 \cos u \cos^2 v, 9 \sin u \cos^2 v, 9 \sin v \cos v) \\ |\vec{n}| &= \sqrt{3^4 \cos^2 u \cos^4 v + 3^4 \sin^2 u \cos^4 v + 3^4 \sin^2 v \cos^2 v} = \\ &= \sqrt{3^4 \cos^4 v + 3^4 \sin^2 v \cos^2 v} = \sqrt{3^4 \cos^2 v} = 9 \cos v \end{aligned}$$

$$\begin{aligned} \iint_S \frac{1}{z^2+9} dS &= \iint_{\Omega} du dv = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{9 \cos v}{9 \sin^2 v + 9} du dv = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos v}{\sin^2 v + 1} dv = \\ &= \pi \left[\arctg(\sin v) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{\pi^2}{2}. \end{aligned}$$

10. Cvičení: Plošné integrály 2. druhu

Příklady: Vypočítejte dané plošné integrály 2. druhu.

1. Vypočítejte $\iint_{\mathcal{S}} \vec{f} d\vec{S}$, kde $\vec{f} = (x, y, z)$ a \mathcal{S} je část roviny $x + y + z = 2$ v prvním oktantu ($x > 0, y > 0, z > 0$) orientované směrem k počátku.

$$\vec{r}(u, v) = (u, v, 2 - u - v)$$

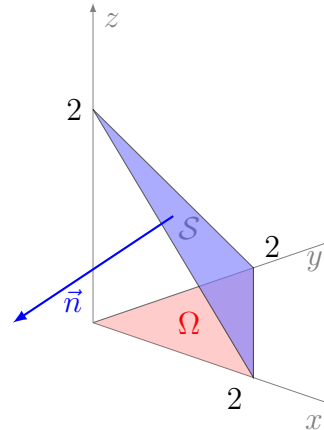
$$\Omega = \{[u, v] \in \mathbb{R}^2; u \in \langle 0, 2 \rangle \wedge v \in \langle 0, 2 - u \rangle\},$$

$$\vec{t}_u = (1, 0, -1)$$

$$\vec{t}_v = (0, 1, -1)$$

$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} =$$

$$= (1, 1, 1) \rightarrow \text{nesouhlasná orientace}$$



$$\iint_{\mathcal{S}} \vec{f} d\vec{S} = - \iint_{\Omega} (u, v, 2 - u - v) \cdot (1, 1, 1) du dv = - \iint_{\Omega} (u + v + 2 - u - v) du dv =$$

$$= -2 \int_0^2 \int_0^{2-u} dv du = -2 \int_0^2 (2 - u) du = [u^2 - 4u]_0^2 = -4$$

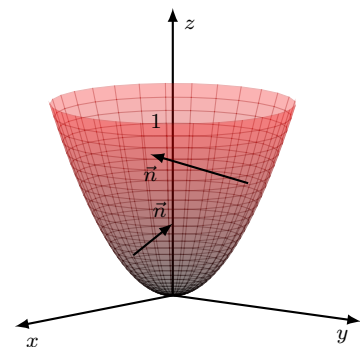
2. Vypočítejte $\iint_{\mathcal{S}} \vec{f} d\vec{S}$, kde $\vec{f} = (x, y, 2z)$ a \mathcal{S} je plášť rotačního paraboloidu $z = x^2 + y^2 \leq 1$ orientovaného dovnitř.

$$\vec{r}(u, v) = (u \cos v, u \sin v, u^2)$$

$$u \in \langle 0, 1 \rangle, v \in \langle 0, 2\pi \rangle,$$

$$\vec{t}_u = (\cos v, \sin v, 2u)$$

$$\vec{t}_v = (-u \sin v, u \cos v, 0)$$



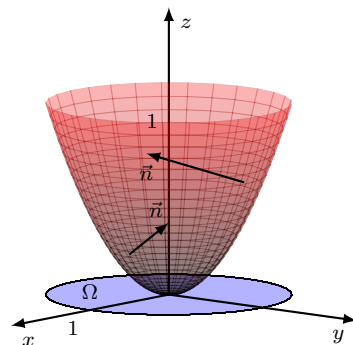
$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 2u \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (-2u^2 \cos v, -2u^2 \sin v, u) \rightarrow \text{souhlasná orientace}$$

$$\iint_{\mathcal{S}} \vec{f} d\vec{S} = \iint_{\Omega} (u \cos v, u \sin v, 2u^2) \cdot (-2u^2 \cos v, -2u^2 \sin v, u) du dv =$$

$$= \iint_{\Omega} (-2u^3 \cos^2 v - 2u^3 \sin^2 v + 2u^3) du dv = 0$$

3. Vypočítejte $\iint_{\mathcal{S}} \vec{f} d\vec{S}$, kde $\vec{f} = (x, y, z)$ a \mathcal{S} je plášť rotačního paraboloidu $z = x^2 + y^2 \leq 1$ orientovaného dovnitř.

$$\begin{aligned}\vec{r}(u, v) &= (u, v, u^2 + v^2) \\ \Omega &= \{[u, v] \in \mathbb{R}^2; u^2 + v^2 \leq 1\}, \\ \vec{t}_u &= (1, 0, 2u) \\ \vec{t}_v &= (0, 1, 2v)\end{aligned}$$

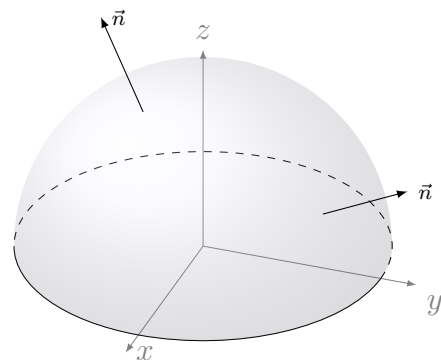


$$\vec{n} = \vec{t}_u \times \vec{t}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1) \rightarrow \text{souhlasná orientace}$$

$$\begin{aligned}\iint_{\mathcal{S}} \vec{f} d\vec{S} &= \iint_{\Omega} (u, v, u^2 + v^2) \cdot (-2u, -2v, 1) dudv = \iint_{\Omega} (-2u^2 - 2v^2 + u^2 + v^2) dudv = \\ &= - \iint_{\Omega} (u^2 + v^2) dudv = \begin{vmatrix} \text{polární s.:} \\ u = r \cos \varphi \\ v = r \sin \varphi \\ r \in \langle 0, 1 \rangle \\ \varphi \in \langle 0, 2\pi \rangle \\ |J| = r \end{vmatrix} = - \int_0^1 \int_0^{2\pi} r^3 d\varphi dr = -2\pi \int_0^1 r^3 dr = -2\pi \left[\frac{r^4}{4} \right]_0^1 = \\ &= -\frac{\pi}{2}\end{aligned}$$

4. Vypočítejte $\iint_{\mathcal{S}} \vec{f} d\vec{S}$, kde $\vec{f} = (y, -x, 1)$ a \mathcal{S} je $x^2 + y^2 + z^2 = 1, z > 0$ orientovaná ven.

$$\begin{aligned}\vec{r}(u, v) &= (\cos u \cos v, \sin u \cos v, \sin v), \\ u &\in \langle 0, 2\pi \rangle, v \in \langle 0, \frac{\pi}{2} \rangle, \\ \vec{t}_u &= (-\sin u \cos v, \cos u \cos v, 0) \\ \vec{t}_v &= (-\cos u \sin v, -\sin u \sin v, \cos v)\end{aligned}$$

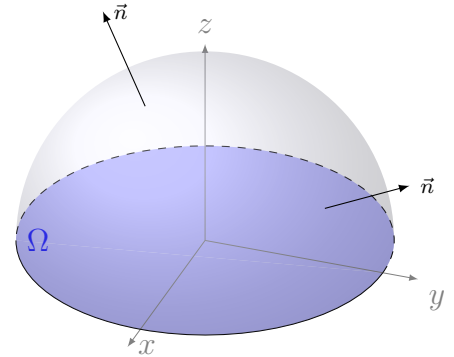


$$\begin{aligned}\vec{n} = \vec{t}_u \times \vec{t}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin u \cos v & \cos u \cos v & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{vmatrix} = \\ &= (\cos u \cos^2 v, \sin u \cos^2 v, \sin^2 u \sin v \cos v + \cos^2 u \sin v \cos v) = \\ &= (\cos u \cos^2 v, \sin u \cos^2 v, \sin v \cos v) \rightarrow \text{souhlasná orientace}\end{aligned}$$

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{f} d\vec{S} &= \iint_{\Omega} (\sin u \cos v, -\cos u \cos v, 1) \cdot (\cos u \cos^2 v, \sin u \cos^2 v, \sin v \cos v) du dv = \\
&\iint_{\Omega} (\cos u \sin u \cos^3 v - \cos u \sin u \cos^3 v + \sin v \cos v) du dv = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin v \cos v du dv = \\
&= 2\pi \int_0^{\frac{\pi}{2}} \sin v \cos v dv = \left| \begin{array}{l} t = \sin v \\ dt = \cos v dv \\ \int t dt = \frac{t^2}{2} + C \end{array} \right| = 2\pi \left[\frac{\sin^2 t}{2} \right]_0^{\frac{\pi}{2}} = \pi
\end{aligned}$$

5. Vypočtěte $\iint_{\mathcal{S}} \vec{f} d\vec{S}$, kde $\vec{f} = (x, y, z^2)$ a \mathcal{S} je část jednotkové sféry pro $z > 0$ orientované ven.

$$\begin{aligned}
\vec{r}(u, v) &= (u, v, \sqrt{1 - u^2 - v^2}), \\
\Omega &= \{[u, v] \in \mathbb{R}^2, u^2 + v^2 \leq 1\}, \\
\vec{t}_u &= \left(1, 0, -\frac{2u}{2\sqrt{1 - u^2 - v^2}}\right) \\
\vec{t}_v &= \left(0, 1, -\frac{2v}{2\sqrt{1 - u^2 - v^2}}\right)
\end{aligned}$$



$$\begin{aligned}
\vec{n} = \vec{t}_u \times \vec{t}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{2u}{2\sqrt{1 - u^2 - v^2}} \\ 0 & 1 & -\frac{2v}{2\sqrt{1 - u^2 - v^2}} \end{vmatrix} = \left(\frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}}, 1 \right) \\
&\rightarrow \text{souhlasná orientace}
\end{aligned}$$

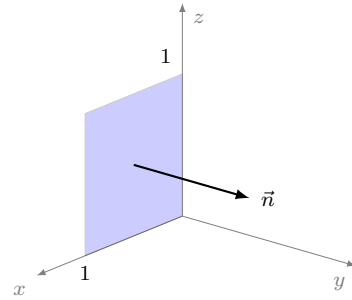
$$\begin{aligned}
\iint_{\mathcal{S}} \vec{f} d\vec{S} &= \iint_{\Omega} (u, v, 1 - u^2 - v^2) \cdot \left(\frac{u}{\sqrt{1 - u^2 - v^2}}, \frac{v}{\sqrt{1 - u^2 - v^2}}, 1 \right) du dv = \\
&\iint_{\Omega} \left(\frac{u^2}{\sqrt{1 - u^2 - v^2}} + \frac{v^2}{\sqrt{1 - u^2 - v^2}} + 1 - u^2 - v^2 \right) du dv = \left| \begin{array}{l} \text{polární s.:} \\ u = r \cos \varphi \\ v = r \sin \varphi \\ r \in \langle 0, 1 \rangle \\ \varphi \in \langle 0, 2\pi \rangle \\ |J| = r \end{array} \right| = \\
&= \int_0^1 \int_0^{2\pi} \left(\frac{r^2}{\sqrt{1 - r^2}} + 1 - r^2 \right) \cdot r d\varphi dr = 2\pi \int_0^1 \left(\frac{r^2}{\sqrt{1 - r^2}} + 1 - r^2 \right) \cdot r dr =
\end{aligned}$$

$$\begin{aligned}
&= \left| \begin{array}{l} t = 1 - r^2 \\ dt = -2r \, dr \\ -\pi \int \left(\frac{1-t}{\sqrt{t}} + t \right) dt = -\pi \left(2t^{1/2} - \frac{2}{3}t^{3/2} + \frac{t^2}{2} \right) + C \end{array} \right| = \\
&= -\pi \left[2\sqrt{1-r^2} - \frac{2}{3}(1-r^2)^{3/2} + \frac{(1-r^2)^2}{2} \right]_0^1 = \pi \left(2 - \frac{2}{3} + \frac{1}{2} \right) = \frac{11}{6}\pi
\end{aligned}$$

6. Vypočtěte tok vektorového pole $\vec{f} = (0, 1, 0)$ plochou $\mathcal{S} = \{[x, y, z] \in \mathbb{R}^3, x \in \langle 0, 1 \rangle, y = 0, z \in \langle 0, 1 \rangle\}$ orientované v kladném směru osy y .

$$\vec{n} = (0, 1, 0)$$

$$\begin{aligned}
\iint_{\mathcal{S}} \vec{f} \cdot \vec{n} \, dS &= \iint_{\mathcal{S}} (0, 1, 0) \cdot (0, 1, 0) \, dS = \\
&= \int_0^1 \int_0^1 1 \, dx \, dz = 1.
\end{aligned}$$



11. Cvičení: Integrální věty

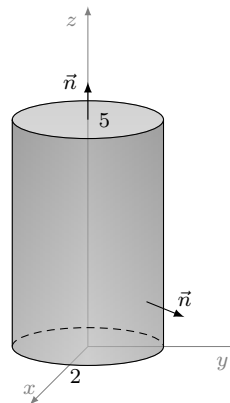
Příklady: Vypočtěte pomocí Gaussovy věty.

1. Vypočtěte tok vektorového pole $\vec{f} = (y^2, x^2, z^2)$ vně orientovaným povrchem válce $x^2 + y^2 = 4$, $0 \leq z \leq 5$.

$$\operatorname{div} \vec{f} = 2z$$

cylindrické souřadnice:

$$\begin{aligned} x &= r \cos \varphi & r &\in \langle 0, 2 \rangle \\ y &= r \sin \varphi & \varphi &\in \langle 0, 2\pi \rangle \\ z &= z & z &\in \langle 0, 5 \rangle \\ |J| &= r \end{aligned}$$



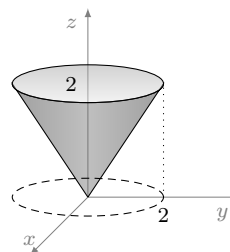
$$\begin{aligned} \iint_S \vec{f} d\vec{S} &= \iiint_V \operatorname{div} \vec{f} dx dy dz = \iiint_V 2z dx dy dz = \int_0^5 \int_0^{2\pi} \int_0^2 2z \cdot r dr d\varphi dz \\ &= \int_0^5 \int_0^{2\pi} z [r^2]_0^2 d\varphi dz = 4 \int_0^5 \int_0^{2\pi} z d\varphi dz = 8\pi \int_0^5 z dz = 8\pi \left[\frac{z^2}{2} \right]_0^5 = 100\pi \end{aligned}$$

2. Vypočtěte tok vektorového pole $\vec{f} = (x^2, (1 - 2x)y, 4z)$ vně orientovaným povrchem kužele $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 2$.

$$\operatorname{div} \vec{f} = 2x + 1 - 2x + 4 = 5$$

Polární souřadnice:

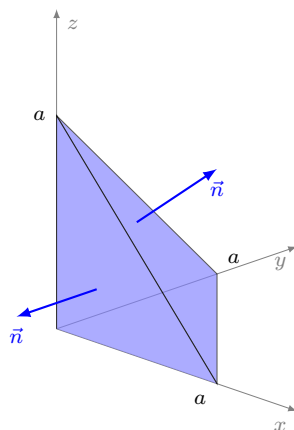
$$\begin{aligned} x &= r \cos \varphi & r &\in \langle 0, 2 \rangle \\ y &= r \sin \varphi & \varphi &\in \langle 0, 2\pi \rangle \\ |J| &= r \end{aligned}$$



$$\begin{aligned} \iint_S \vec{f} d\vec{S} &= \iiint_V \operatorname{div} \vec{f} dx dy dz = \iiint_V 5 dx dy dz = 5 \iint_{V_{xy}} \int_{\sqrt{x^2+y^2}}^2 dz dx dy = \\ &= 5 \iint_{V_{xy}} (2 - \sqrt{x^2 + y^2}) dx dy = 5 \int_0^{2\pi} \int_0^2 (2 - r) r dr d\varphi = 5 \int_0^{2\pi} \left[r^2 - \frac{r^3}{3} \right]_0^2 d\varphi = \\ &= \frac{40}{3} \pi \end{aligned}$$

3. Vypočítejte tok vektorového pole $\vec{f} = (x, z, y)$ vně orientovaným povrchem čtyřštěnu:
 $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq a, a > 0$.

$$\operatorname{div} \vec{f} = 1$$

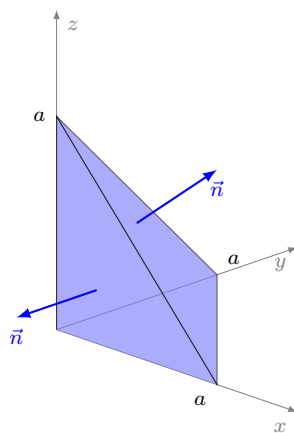


$$\begin{aligned} \iint_S \vec{f} d\vec{S} &= \iiint_V \operatorname{div} \vec{f} dx dy dz = \int_0^a \int_0^{a-x} \int_0^{a-x-y} dz dy dx = \int_0^a \int_0^{a-x} (a-x-y) dy dx \\ &= \int_0^a \left((a-x)[y]_0^{a-x} - \left[\frac{y^2}{2} \right]_0^{a-x} \right) dx = \int_0^a \left((a-x)^2 - \frac{(a-x)^2}{2} \right) dx = \frac{1}{2} \int_0^a (a-x)^2 dx \\ &= -\frac{1}{2} \left[\frac{(a-x)^3}{3} \right]_0^a = \frac{a^3}{6} \end{aligned}$$

4. Vypočítejte tok vektorového pole $\vec{f} = (y, z, x)$ vně orientovaným povrchem čtyřštěnu:
 $x \geq 0, y \geq 0, z \geq 0, x + y + z \leq a, a > 0$.

$$\operatorname{div} \vec{f} = 0$$

$$\iint_S \vec{f} d\vec{S} = 0$$

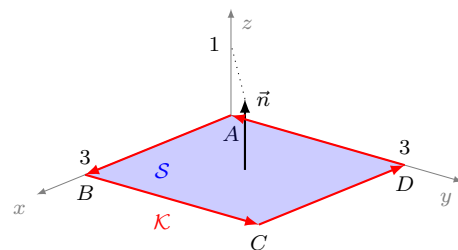


Příklady: Vypočtěte pomocí Stokesovy věty.

1. Vypočtěte práci, kterou vykoná vektorového pole $\vec{f} = (y - x, 2x - y, z)$ po obvodu čtverce $ABCD$, kde $A = [0, 0, 0]$, $B = [3, 0, 0]$, $C = [3, 3, 0]$, $D = [0, 3, 0]$, jehož orientace je dána pořadím bodů $ABCD$.

$$\text{rot } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - x & 2x - y & z \end{vmatrix} = (0, 0, 1)$$

$$\vec{n} = (0, 0, 1) \rightarrow \text{souhlasná orientace}$$

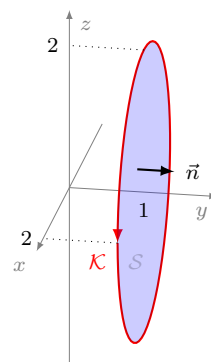


$$\oint_{\kappa} \vec{f} d\vec{r} = \iint_S \text{rot } \vec{f} d\vec{S} = \iint_S (0, 0, 1) \cdot (0, 0, 1) dS = \int_0^3 \int_0^3 1 dx dy = 9.$$

2. Vypočtěte práci, kterou vykoná vektorového pole $\vec{f} = (y - z, z - x, x - y)$ po kružnici $x^2 + z^2 = 4$, $y = 1$ orientovanou směrem z bodu $[0, 1, 2]$ do bodu $[2, 1, 0]$.

$$\text{rot } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z & z - x & x - y \end{vmatrix} = (-2, -2, -2)$$

$$\vec{n} = (0, 1, 0) \rightarrow \text{souhlasná orientace}$$



$$\oint_{\kappa} \vec{f} d\vec{r} = \iint_S \text{rot } \vec{f} d\vec{S} = \iint_S (-2, -2, -2) \cdot (0, 1, 0) dS = \iint_S -2 dS = \dots$$

$$\vec{r}(u, v) = (u \cos v, 1, u \sin v)$$

$$u \in \langle 0, 2 \rangle, v \in \langle 0, 2\pi \rangle$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & 0 & \sin v \\ -u \sin v & 0 & u \cos v \end{vmatrix} = (0, -u \sin^2 v - u \cos^2 v, 0) = (0, -u, 0)$$

$$|\vec{n}| = u$$

$$\dots = -2 \iint_{\Omega} u du dv = -2 \int_0^2 \int_0^{2\pi} u dv du = -4\pi \int_0^2 u du = -4\pi \left[\frac{u^2}{2} \right]_0^2 = -8\pi$$

3. Vypočítejte práci, kterou vykoná vektorového pole $\vec{f} = (y^2, z^2, x^2)$ po obvodu trojúhelníka ABC , kde $A = [3, 0, 0]$, $B = [0, 0, 3]$, $C = [0, 3, 0]$, orientace je dána pořadím bodů ABC .

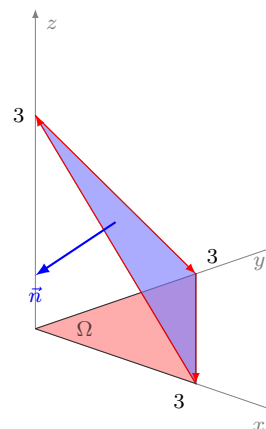
$$\text{rot } \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = (-2z, -2x, -2y)$$

$$\vec{r}(u, v) = (u, v, 3 - u - v)$$

$$\Omega = \{[u, v] \in \mathbb{R}^2 : u \in \langle 0, 3 \rangle, v \in \langle 0, 3 - u \rangle\}$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1, 1, 1)$$

→ nesouhlasná orientace



$$\begin{aligned} \oint_{\mathcal{K}} \vec{f} d\vec{r} &= \iint_S \text{rot } \vec{f} d\vec{S} = - \iint_S (-2(3 - u - v), -2u, -2v) \cdot (1, 1, 1) dS = \\ &= \iint_S (6 - 2u - 2v + 2u + 2v) dS = 6 \iint_{\Omega} dudv = 6 \int_0^3 \int_0^{3-u} dv du = 6 \int_0^3 (3 - u) du = \\ &= 6 \left[3u - \frac{u^2}{2} \right]_0^3 = 6 \left(9 - \frac{9}{2} \right) = 27 \end{aligned}$$