TUTORIAL No. 1

FLUID FLOW THEORY

In order to complete this tutorial you should already have completed level 1 or have a good basic knowledge of fluid mechanics equivalent to the Engineering Council part 1 examination 103.

When you have completed this tutorial, you should be able to do the following.

- Explain the meaning of viscosity.
- Define the units of viscosity.
- Describe the basic principles of viscometers.
- Describe non-Newtonian flow
- Explain and solve problems involving laminar flow though pipes and between parallel surfaces.
- Explain and solve problems involving drag force on spheres.
- Explain and solve problems involving turbulent flow.
- Explain and solve problems involving friction coefficient.

Throughout there are worked examples, assignments and typical exam questions. You should complete each assignment in order so that you progress from one level of knowledge to another.

Let us start by examining the meaning of viscosity and how it is measured.

1. VISCOSITY

1.1 BASIC THEORY

Molecules of fluids exert forces of attraction on each other. In liquids this is strong enough to keep the mass together but not strong enough to keep it rigid. In gases these forces are very weak and cannot hold the mass together.

When a fluid flows over a surface, the layer next to the surface may become attached to it (it wets the surface). The layers of fluid above the surface are moving so there must be shearing taking place between the layers of the fluid.

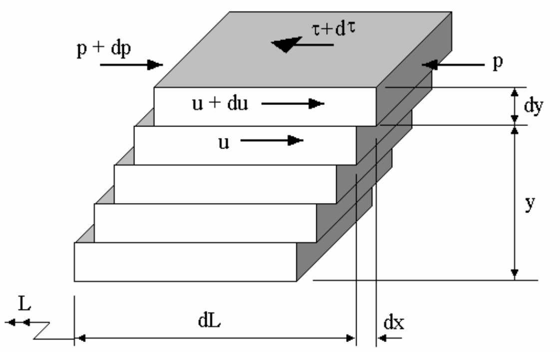


Fig.1.1

Let us suppose that the fluid is flowing over a flat surface in laminated layers from left to right as shown in figure 1.1.

y is the distance above the solid surface (no slip surface)
L is an arbitrary distance from a point upstream.
dy is the thickness of each layer.
dL is the length of the layer.
dx is the distance moved by each layer relative to the one below in a corresponding time dt.
u is the velocity of any layer.
du is the increase in velocity between two adjacent layers.

Each layer moves a distance dx in time dt relative to the layer below it. The ratio dx/dt must be the change in velocity between layers so du = dx/dt.

When any material is deformed sideways by a (shear) force acting in the same direction, a shear stress τ is produced between the layers and a corresponding shear strain γ is produced. Shear strain is defined as follows.

$$\gamma = \frac{\text{sideways deformation}}{\text{height of the layer being deformed}} = \frac{dx}{dy}$$

The rate of shear strain is defined as follows.

$$\dot{\gamma} = \frac{\text{shear strain}}{\text{time taken}} = \frac{\gamma}{dt} = \frac{dx}{dt\,dy} = \frac{du}{dy}$$

It is found that fluids such as water, oil and air, behave in such a manner that the shear stress between layers is directly proportional to the rate of shear strain.

$$\tau = \text{constant x } \dot{\gamma}$$

Fluids that obey this law are called ***NEWTONIAN FLUIDS***.

It is the constant in this formula that we know as the dynamic viscosity of the fluid.

DYNAMIC VISCOSITY $\mu = \dfrac{\text{shear stress}}{\text{rate of shear}} = \dfrac{\tau}{\dot{\gamma}} = \tau \dfrac{dy}{du}$

FORCE BALANCE AND VELOCITY DISTRIBUTION

A shear stress τ exists between each layer and this increases by dτ over each layer. The pressure difference between the downstream end and the upstream end is dp.

The pressure change is needed to overcome the shear stress. The total force on a layer must be zero so balancing forces on one layer (assumed 1 m wide) we get the following.

$$dp\,dy + d\tau\,dL = 0$$

$$\frac{d\tau}{dy} = -\frac{dp}{dL}$$

It is normally assumed that the pressure declines uniformly with distance downstream so the ***pressure gradient*** $\dfrac{dp}{dL}$ is assumed constant. The minus sign indicates that the pressure falls with distance. Integrating between the no slip surface (y = 0) and any height y we get

$$-\frac{dp}{dL} = \frac{d\tau}{dy} = \frac{d\left(\mu \dfrac{du}{dy}\right)}{dy}$$

$$-\frac{dp}{dL} = \mu \frac{d^2u}{dy^2} \text{.............}(1.1)$$

Integrating twice to solve u we get the following.

$$-y\frac{dp}{dL} = \mu\frac{du}{dy} + A$$

$$-\frac{y^2}{2}\frac{dp}{dL} = \mu u + Ay + B$$

A and B are constants of integration that should be solved based on the known conditions (boundary conditions). For the flat surface considered in figure 1.1 one boundary condition is that u = 0 when y = 0 (the no slip surface). Substitution reveals the following.

0 = 0 +0 +B hence B = 0

At some height δ above the surface, the velocity will reach the mainstream velocity u_o. This gives us the second boundary condition $u = u_o$ when y = δ.

Substituting we find the following.

$$-\frac{\delta^2}{2}\frac{dp}{dL} = \mu u_o + A\delta$$

$$A = -\frac{\delta}{2}\frac{dp}{dL} - \frac{\mu u_o}{\delta} \quad \text{hence}$$

$$-\frac{y^2}{2}\frac{dp}{dL} = \mu u + \left(-\frac{\delta}{2}\frac{dp}{dL} - \frac{\mu u_o}{\delta}\right)y$$

$$u = y\left(\frac{\delta}{2\mu}\frac{dp}{dL} + \frac{u_o}{\delta}\right)$$

Plotting u against y gives figure 1.2.

BOUNDARY LAYER.

The velocity grows from zero at the surface to a maximum at height δ. In theory, the value of δ is infinity but in practice it is taken as the height needed to obtain 99% of the mainstream velocity. This layer is called the boundary layer and δ is the boundary layer thickness. It is a very important concept and is discussed more fully in later work. The inverse gradient of the boundary layer is du/dy and this is the rate of shear strain γ.

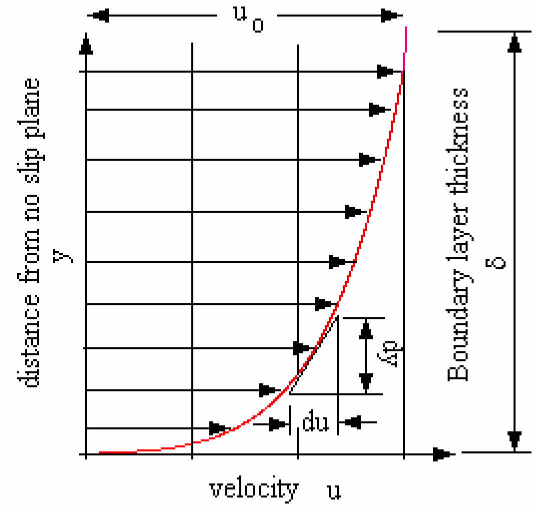


Fig.1.2

1.2. UNITS of VISCOSITY

1.2.1 DYNAMIC VISCOSITY μ

The units of dynamic viscosity μ are N s/m^2. It is normal in the international system (SI) to give a name to a compound unit. The old metric unit was a dyne.s/cm^2 and this was called a POISE after Poiseuille. The SI unit is related to the Poise as follows.

10 Poise = 1 Ns/m^2 which is not an acceptable multiple. Since, however, 1 Centi Poise (1cP) is 0.001 N s/m^2 then the cP is the accepted SI unit.

$$\mathbf{1cP = 0.001\ N\ s/m^2.}$$

The symbol η is also commonly used for dynamic viscosity.

There are other ways of expressing viscosity and this is covered next.

1.2.2 KINEMATIC VISCOSITY ν

This is defined as : ν = dynamic viscosity /density

$$\nu = \mu/\rho$$

The basic units are m^2/s. The old metric unit was the cm^2/s and this was called the STOKE after the British scientist. The SI unit is related to the Stoke as follows.

1 Stoke (St) = 0.0001 m^2/s and is not an acceptable SI multiple. The centi Stoke (cSt),however, is 0.000001 m^2/s and this is an acceptable multiple.

$$\mathbf{1cSt = 0.000001\ m^2/s = 1\ mm^2/s}$$

1.2.3 OTHER UNITS

Other units of viscosity have come about because of the way viscosity is measured. For example REDWOOD SECONDS comes from the name of the Redwood viscometer. Other units are Engler Degrees, SAE numbers and so on. Conversion charts and formulae are available to convert them into useable engineering or SI units.

1.2.4 VISCOMETERS

The measurement of viscosity is a large and complicated subject. The principles rely on the resistance to flow or the resistance to motion through a fluid. Many of these are covered in British Standards 188. The following is a brief description of some types.

U TUBE VISCOMETER

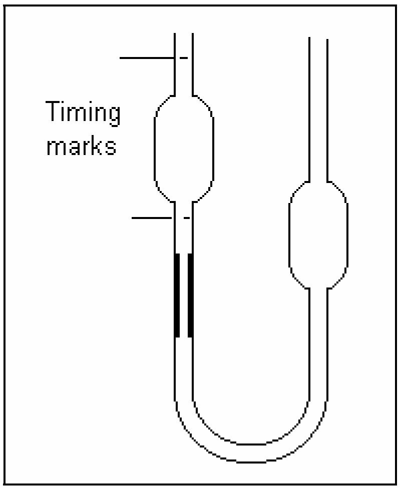


Fig.1.3

The fluid is drawn up into a reservoir and allowed to run through a capillary tube to another reservoir in the other limb of the U tube.

The time taken for the level to fall between the marks is converted into cSt by multiplying the time by the viscometer constant.

$$\nu = ct$$

The constant c should be accurately obtained by calibrating the viscometer against a master viscometer from a standards laboratory.

REDWOOD VISCOMETER

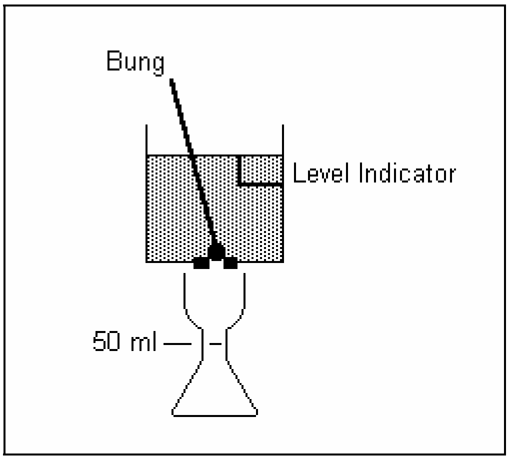


Fig.1.4

This works on the principle of allowing the fluid to run through an orifice of very accurate size in an agate block.

50 ml of fluid are allowed to fall from the level indicator into a measuring flask. The time taken is the viscosity in Redwood seconds. There are two sizes giving Redwood No.1 or No.2 seconds. These units are converted into engineering units with tables.

FALLING SPHERE VISCOMETER

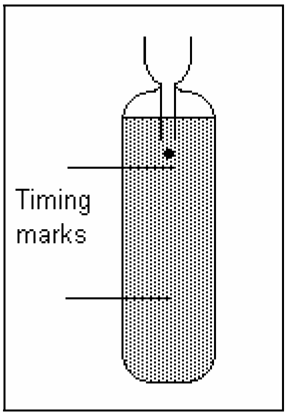


Fig.1.5

This viscometer is covered in BS188 and is based on measuring the time for a small sphere to fall in a viscous fluid from one level to another. The buoyant weight of the sphere is balanced by the fluid resistance and the sphere falls with a constant velocity. The theory is based on Stokes' Law and is only valid for very slow velocities. The theory is covered later in the section on laminar flow where it is shown that the terminal velocity (u) of the sphere is related to the dynamic viscosity (μ) and the density of the fluid and sphere (ρ_f and ρ_s) by the formula

$$\mu = F\ gd^2(\rho_s - \rho_f)/18u$$

F is a correction factor called the Faxen correction factor, which takes into account a reduction in the velocity due to the effect of the fluid being constrained to flow between the wall of the tube and the sphere.

ROTATIONAL TYPES

There are many types of viscometers, which use the principle that it requires a torque to rotate or oscillate a disc or cylinder in a fluid. The torque is related to the viscosity. Modern instruments consist of a small electric motor, which spins a disc or cylinder in the fluid. The torsion of the connecting shaft is measured and processed into a digital readout of the viscosity in engineering units.

You should now find out more details about viscometers by reading BS188, suitable textbooks or literature from oil companies.

ASSIGNMENT No. 1

1. Describe the principle of operation of the following types of viscometers.

a. Redwood Viscometers.

b. British Standard 188 glass U tube viscometer.

c. British Standard 188 Falling Sphere Viscometer.

d. Any form of Rotational Viscometer

Note that this covers the E.C. exam question 6a from the 1987 paper.

2. LAMINAR FLOW THEORY

The following work only applies to Newtonian fluids.

2.1 LAMINAR FLOW

A ***stream line*** is an imaginary line with no flow normal to it, only along it. When the flow is laminar, the streamlines are parallel and for flow between two parallel surfaces we may consider the flow as made up of parallel laminar layers. In a pipe these laminar layers are cylindrical and may be called ***stream tubes***. In laminar flow, no mixing occurs between adjacent layers and it occurs at low average velocities.

2.2 TURBULENT FLOW

The shearing process causes energy loss and heating of the fluid. This increases with mean velocity. When a certain critical velocity is exceeded, the streamlines break up and mixing of the fluid occurs. The diagram illustrates Reynolds coloured ribbon experiment. Coloured dye is injected into a horizontal flow. When the flow is laminar the dye passes along without mixing with the water. When the speed of the flow is increased turbulence sets in and the dye mixes with the surrounding water. One explanation of this transition is that it is necessary to change the pressure loss into other forms of energy such as angular kinetic energy as indicated by small eddies in the flow.

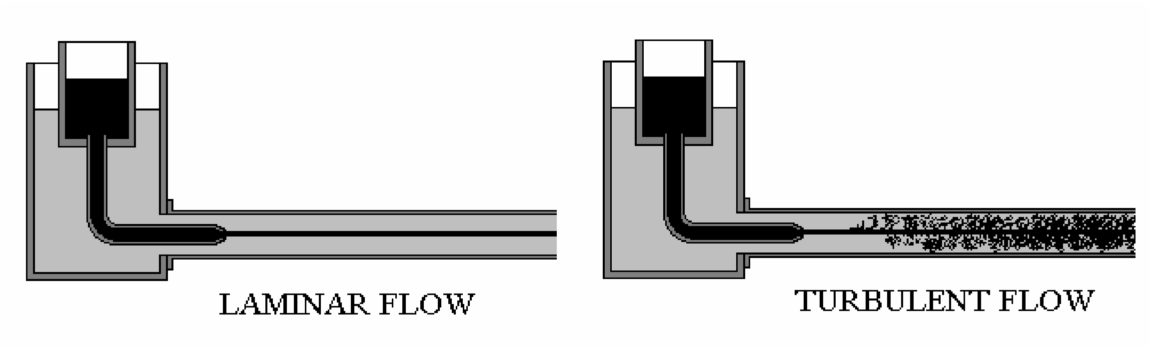


Fig.2.1

2.3 LAMINAR AND TURBULENT BOUNDARY LAYERS

In chapter 2 it was explained that a ***boundary layer*** is the layer in which the velocity grows from zero at the wall (no slip surface) to 99% of the maximum and the thickness of the layer is denoted δ. When the flow within the boundary layer becomes turbulent, the shape of the boundary layers waivers and when diagrams are drawn of turbulent boundary layers, the mean shape is usually shown. Comparing a laminar and turbulent boundary layer reveals that the turbulent layer is thinner than the laminar layer.

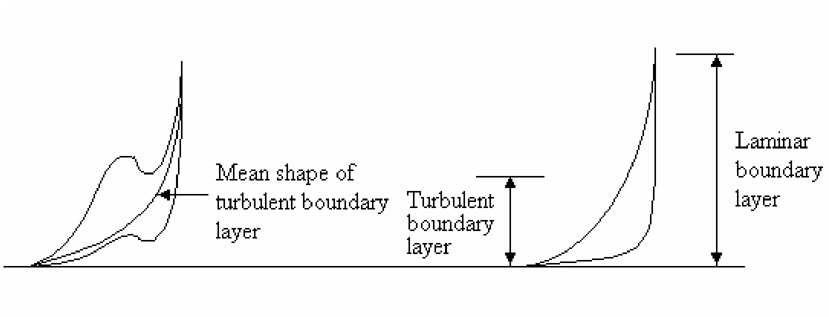


Fig.2.2

2.4 CRITICAL VELOCITY - REYNOLDS NUMBER

When a fluid flows in a pipe at a volumetric flow rate Q m^3/s the average velocity is defined $u_m = \frac{Q}{A}$ A is the cross sectional area.

The Reynolds number is defined as $R_e = \frac{\rho u_m D}{\mu} = \frac{u_m D}{\nu}$

If you check the units of R_e you will see that there are none and that it is a dimensionless number. You will learn more about such numbers in a later section.

Reynolds discovered that it was possible to predict the velocity or flow rate at which the transition from laminar to turbulent flow occurred for any Newtonian fluid in any pipe. He also discovered that the critical velocity at which it changed back again was different. He found that when the flow was gradually increased, the change from laminar to turbulent always occurred at a Reynolds number of 2500 and when the flow was gradually reduced it changed back again at a Reynolds number of 2000. Normally, 2000 is taken as the critical value.

WORKED EXAMPLE 2.1

Oil of density 860 kg/m^3 has a kinematic viscosity of 40 cSt. Calculate the critical velocity when it flows in a pipe 50 mm bore diameter.

SOLUTION

$$R_e = \frac{u_m D}{\nu}$$

$$u_m = \frac{R_e \nu}{D} = \frac{2000 \text{x} 40 \text{x} 10^{-6}}{0.05} = 1.6 \text{ m/s}$$

2.5 DERIVATION OF POISEUILLE'S EQUATION for LAMINAR FLOW

Poiseuille did the original derivation shown below which relates pressure loss in a pipe to the velocity and viscosity for LAMINAR FLOW. His equation is the basis for measurement of viscosity hence his name has been used for the unit of viscosity. Consider a pipe with laminar flow in it. Consider a stream tube of length ΔL at radius r and thickness dr.

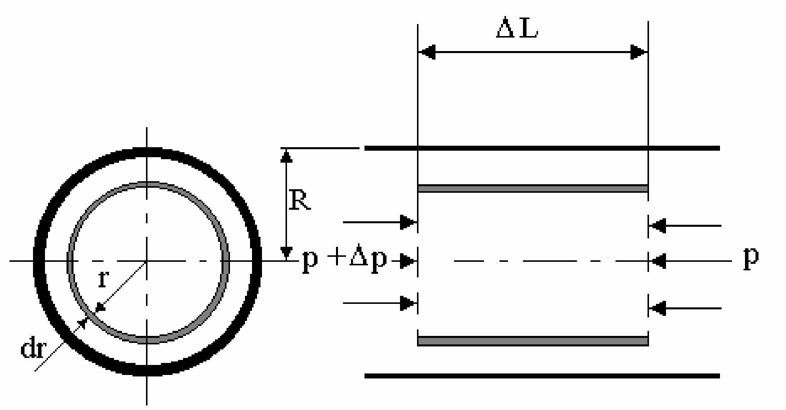


Fig.2.3

y is the distance from the pipe wall. $y = R - r$ $dy = -dr$ $\frac{du}{dy} = -\frac{du}{dr}$

The shear stress on the outside of the stream tube is τ. The force (F_s) acting from right to left is due to the shear stress and is found by multiplying τ by the surface area.

$Fs = \tau \times 2\pi r\, \Delta L$

For a Newtonian fluid , $\tau = \mu\frac{du}{dy} = -\mu\frac{du}{dr}$. Substituting for τ we get the following.

$$F_s = -2\pi r \Delta L \mu \frac{du}{dr}$$

The pressure difference between the left end and the right end of the section is Δp. The force due to this (F_p) is Δp x circular area of radius r.

$F_p = \Delta p \times \pi r^2$

Equating forces we have $-2\pi r \mu \Delta L \frac{du}{dr} = \Delta p \pi r^2$

$$du = -\frac{\Delta p}{2\mu \Delta L} r dr$$

In order to obtain the velocity of the streamline at any radius r we must integrate between the limits u = 0 when r = R and u = u when r = r.

$$\int_0^u du = -\frac{\Delta p}{2\mu\Delta L}\int_R^r r dr$$

$$u = -\frac{\Delta p}{4\mu \Delta L}\left(r^2 - R^2\right)$$

$$u = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right)$$

This is the equation of a Parabola so if the equation is plotted to show the boundary layer, it is seen to extend from zero at the edge to a maximum at the middle.

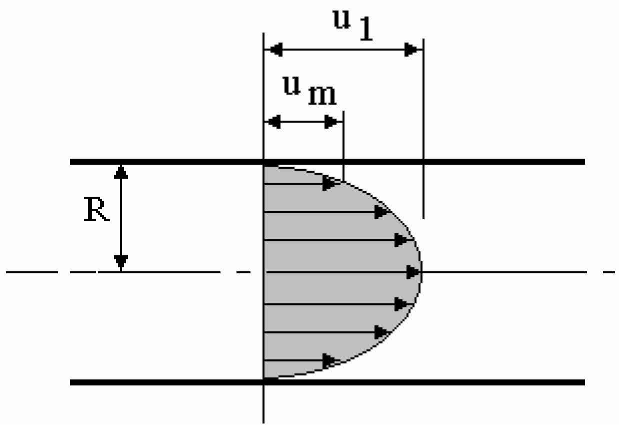


Fig.2.4

For maximum velocity put r = 0 and we get $u_1 = \dfrac{\Delta p R^2}{4\mu\Delta L}$

The average height of a parabola is half the maximum value so the average velocity is

$$u_m = \frac{\Delta p R^2}{8\mu\Delta L}$$

Often we wish to calculate the pressure drop in terms of diameter D. Substitute R=D/2 and rearrange.

$$\Delta p = \frac{32\mu\Delta L u_m}{D^2}$$

The volume flow rate is average velocity x cross sectional area.

$$Q = \frac{\pi R^2 \Delta p R^2}{8\mu\Delta L} = \frac{\pi R^4 \Delta p}{8\mu\Delta L} = \frac{\pi D^4 \Delta p}{128\mu\Delta L}$$

This is often changed to give the pressure drop as a friction head.

The friction head for a length L is found from $h_f = \Delta p/\rho g$

$$h_f = \frac{32\mu L u_m}{\rho g D^2}$$

This is Poiseuille's equation that applies only to laminar flow.

WORKED EXAMPLE 2.2

A capillary tube is 30 mm long and 1 mm bore. The head required to produce a flow rate of 8 mm^3/s is 30 mm. The fluid density is 800 kg/m^3.
Calculate the dynamic and kinematic viscosity of the oil.

SOLUTION

Rearranging Poiseuille's equation we get

$$\mu = \frac{h_f \rho g D^2}{32 L u_m}$$

$$A = \frac{\pi d^2}{4} = \frac{\pi \times 1^2}{4} = 0.785 \text{ mm}^2$$

$$u_m = \frac{Q}{A} = \frac{8}{0.785} = 10.18 \text{ mm/s}$$

$$\mu = \frac{0.03 \times 800 \times 9.81 \times 0.001^2}{32 \times 0.03 \times 0.01018} = 0.0241 \text{ N s/m or } 24.1 \text{ cP}$$

$$\nu = \frac{\mu}{\rho} = \frac{0.0241}{800} = 30.11 \times 10^{-6} \text{ m}^2/\text{s or } 30.11 \text{ cSt}$$

WORKED EXAMPLE No.2.3

Oil flows in a pipe 100 mm bore with a Reynolds number of 250. The dynamic viscosity is 0.018 Ns/m^2. The density is 900 kg/m^3.

Determine the pressure drop per metre length, the average velocity and the radius at which it occurs.

SOLUTION

$Re = \rho u_m D/\mu$.

Hence $u_m = Re\ \mu/ \rho D$

$u_m = (250 \times 0.018)/(900 \times 0.1) = 0.05$ m/s

$\Delta p = 32\mu L\ u_m /D^2$

$\Delta p = 32 \times 0.018 \times 1 \times 0.05/0.1^2$

$\Delta p = 2.88$ Pascals.

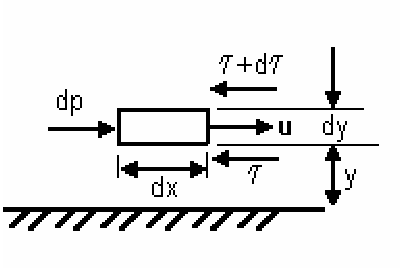
$u = \{\Delta p/4L\mu\}(R^2 - r^2)$ which is made equal to the average velocity 0.05 m/s

$0.05 = (2.88/4 \times 1 \times 0.018)(0.05^2 - r^2)$

$r = 0.035$ m or 35.3 mm.

2.6. FLOW BETWEEN FLAT PLATES

Consider a small element of fluid moving at velocity u with a length dx and height dy at distance y above a flat surface. The shear stress acting on the element increases by dτ in the y direction and the pressure decreases by dp in the x direction. It was shown earlier that $-\frac{dp}{dx} = \mu\frac{d^2u}{dy^2}$



It is assumed that dp/dx does not vary with y so it may be regarded as a fixed value in the following work.

Fig.2.5

Integrating once $-y\frac{\mathrm{dp}}{\mathrm{dx}} = \mu\frac{du}{dy} + A$

Integrating again $-\frac{\mathrm{y}^2}{2}\frac{dp}{dx} = \mu u + Ay + B............(2.6A)$

A and B are constants of integration. The solution of the equation now depends upon the boundary conditions that will yield A and B.

WORKED EXAMPLE No.2.4

Derive the equation linking velocity u and height y at a given point in the x direction when the flow is laminar between two stationary flat parallel plates distance h apart. Go on to derive the volume flow rate and mean velocity.

SOLUTION

When a fluid touches a surface, it sticks to it and moves with it. The velocity at the flat plates is the same as the plates and in this case is zero. The boundary conditions are hence

u = 0 when y = 0

Substituting into equation 2.6A yields that B = 0

u=0 when y=h

Substituting into equation 2.6A yields that A = (dp/dx)h/2

Putting this into equation 2.6A yields

$$\mathbf{u = (dp/dx)(1/2\mu)\{y^2 - hy\}}$$

(The student should do the algebra for this). The result is a parabolic distribution similar that given by Poiseuille's equation earlier only this time it is between two flat parallel surfaces.

FLOW RATE

To find the flow rate we consider flow through a small rectangular slit of width B and height dy at height y.

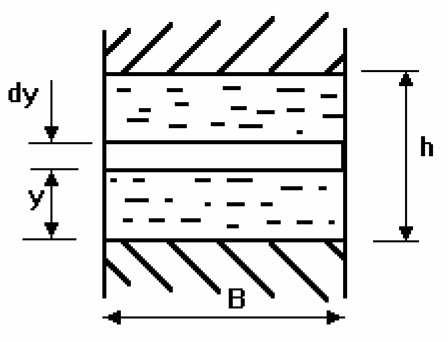


Fig.2.6

The flow through the slit is $dQ = u\,Bdy = (dp/dx)(1/2\mu)\{y^2 - hy\}\,Bdy$

Integrating between y = 0 and y = h to find Q yields

$$\mathbf{Q = -B(dp/dx)(h^3/12\mu)}$$

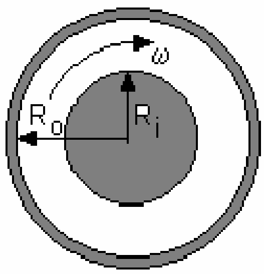
The mean velocity is $u_m = Q/Area = Q/Bh$

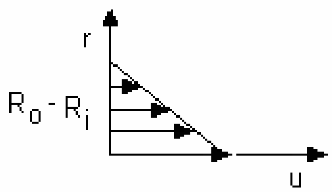
hence $$\mathbf{u_m = -(dp/dx)(h^2/12\mu)}$$

(The student should do the algebra)

2.7 CONCENTRIC CYLINDERS

This could be a shaft rotating in a bush filled with oil or a rotational viscometer. Consider a shaft rotating in a cylinder with the gap between filled with a Newtonian liquid. There is no overall flow rate so equation 2.A does not apply.





Due to the stickiness of the fluid, the liquid sticks to both surfaces and has a velocity $u = \omega R_i$ at the inner layer and zero at the outer layer.

Fig 2.7

If the gap is small, it may be assumed that the change in the velocity across the gap changes from u to zero linearly with radius r.

$$\tau = \mu \, du/dy$$

But since the change is linear $du/dy = u/(R_o-R_i) = \omega R_i /(R_o-R_i)$

$\tau = \mu \omega R_i /(R_o-R_i)$

Shear force on cylinder F = shear stress x surface area

$$F = 2\pi R_i h \tau = \frac{2\pi R_i^2 h \mu \omega}{R_o - R_i}$$

Torque = F x R_i

$$T = Fr = \frac{2\pi R_i^3 h \mu \omega}{R_o - R_i}$$

In the case of a rotational viscometer we rearrange so that

$$\mu = \frac{T(R_o - R)}{2\pi R_i^3 h \omega}$$

In reality, it is unlikely that the velocity varies linearly with radius and the bottom of the cylinder would have an affect on the torque.

<u>2.8 FALLING SPHERES</u>

This theory may be applied to particle separation in tanks and to a falling sphere viscometer. When a sphere falls, it initially accelerates under the action of gravity. The resistance to motion is due to the shearing of the liquid passing around it. At some point, the resistance balances the force of gravity and the sphere falls at a constant velocity. This is the terminal velocity. For a body immersed in a liquid, the buoyant weight is W and this is equal to the viscous resistance R when the terminal velocity is reached.

R = W = volume x density difference x gravity

$$R = W = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

ρ_s = density of the sphere material
ρ_f = density of fluid
d = sphere diameter

The viscous resistance is much harder to derive from first principles and this will not be attempted here. In general, we use the concept of DRAG and define the DRAG COEFFICIENT as

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}}$$

The dynamic pressure of a flow stream is $\frac{\rho u^2}{2}$

The projected area of a sphere is $\frac{\pi d^2}{4}$

$$C_D = \frac{8R}{\rho u^2 \pi d^2}$$

Research shows the following relationship between C_D and R_e for a sphere.

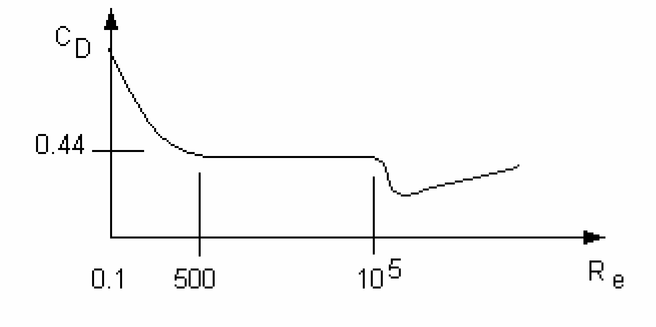


Fig. 2.8

For $R_e<0.2$ the flow is called Stokes flow and Stokes showed that $R = 3\pi d\mu u$ hence $C_D=24\mu/\rho_f ud = 24/R_e$

For $0.2 < R_e < 500$ the flow is called Allen flow and $C_D=18.5R_e^{-0.6}$

For $500 < R_e < 10^5$ C_D is constant $C_D = 0.44$

An empirical formula that covers the range $0.2 < R_e < 10^5$ is as follows.

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

For a falling sphere viscometer, Stokes flow applies. Equating the drag force and the buoyant weight we get

$$3\pi d\mu u = (\pi d^3/6)(\rho_s - \rho_f)\, g$$

$\mu = gd^2(\rho_s - \rho_f)/18u$ for a falling sphere vicometer

The terminal velocity for Stokes flow is $u = d^2g(\rho_s - \rho_f)18\mu$

This formula assumes a fluid of infinite width but in a falling sphere viscometer, the liquid is squeezed between the sphere and the tube walls and additional viscous resistance is produced. The Faxen correction factor F is used to correct the result.

2.9 THRUST BEARINGS

Consider a round flat disc of radius R rotating at angular velocity ω rad/s on top of a flat surface and separated from it by an oil film of thickness t.

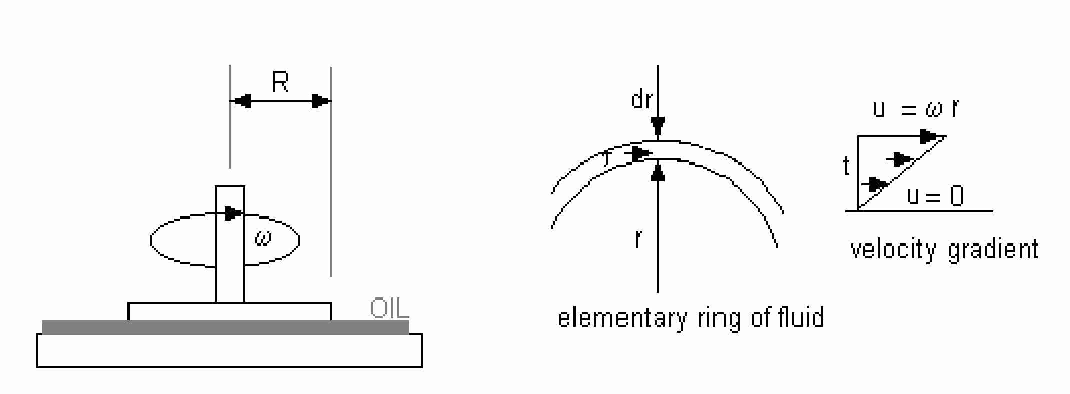


Fig.2.9

Assume the velocity gradient is linear in which case du/dy = u/t = ωr/t at any radius r.

The shear stress on the ring is $\tau = \mu \frac{du}{dy} = \mu \frac{\omega r}{t}$

The shear force is $dF = 2\pi r^2 dr \mu \frac{\omega}{t}$

The torque is $dT = rdF = 2\pi r^3 dr \mu \frac{\omega}{t}$

The total torque is found by integrating with respect to r.

$$T = \int_0^R 2\pi r^3 dr \mu \frac{\omega}{t} = \pi R^4 \mu \frac{\omega}{2t}$$

In terms of diameter D this is $T = \frac{\mu\pi\omega D^4}{32t}$

There are many variations on this theme that you should be prepared to handle.

2.10 MORE ON FLOW THROUGH PIPES

Consider an elementary thin cylindrical layer that makes an element of flow within a pipe. The length is δx , the inside radius is r and the radial thickness is dr. The pressure difference between the ends is δp and the shear stress on the surface increases by dτ from the inner to the outer surface. The velocity at any point is u and the dynamic viscosity is μ.

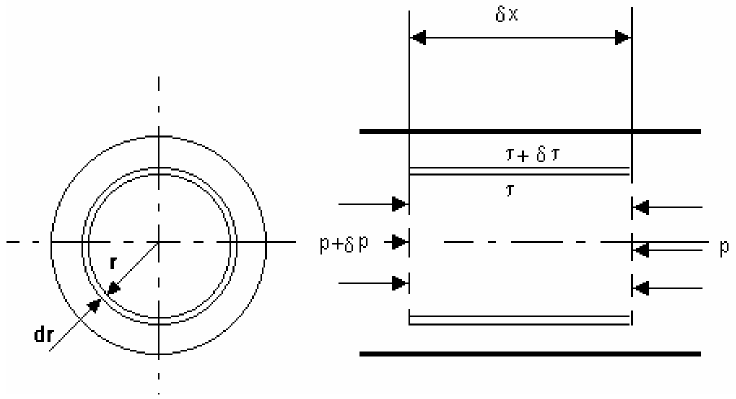


Fig.2.10

The pressure force acting in the direction of flow is $\{\pi(r+dr)^2-\pi r^2\}\delta p$

The shear force opposing is $\{(\tau+\delta\tau)(2\pi)(r+dr) - \tau 2\pi r\}\delta x$

Equating, simplifying and ignoring the product of two small quantities we have the following result.

$$\frac{\delta p}{\delta x} = \frac{\tau}{r} + \frac{d\tau}{dr} \qquad \tau = \mu\frac{du}{dy} \text{ for Newtonian fluids.}$$

If y is measured from the inside of the pipe then r = -y and dy = - dr so $\tau = -\mu\frac{du}{dr}$

$$\frac{\delta p}{\delta x} = -\frac{\mu}{r}\frac{du}{dr} - \mu\frac{d^2u}{dr^2}$$

$$\frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dr^2} = -\frac{1}{\mu}\frac{\delta p}{\delta x}$$

$$\frac{du}{dr} + \frac{rd^2u}{dr^2} = -\frac{r}{\mu}\frac{\delta p}{\delta x}$$

Using partial differentiation to differentiate $\frac{d\left(r\frac{du}{dr}\right)}{dr}$ yields the result $\frac{du}{dr} + \frac{rd^2u}{dr^2}$

hence $\frac{d\left(r\frac{du}{dr}\right)}{dr} = -\frac{r}{\mu}\frac{\delta p}{\delta x}$

Integrating we get $r\frac{du}{dr} = -\frac{r^2}{2\mu}\frac{\delta p}{\delta x} + A$

$$\frac{du}{dr} = -\frac{r}{2\mu}\frac{\delta p}{\delta x} + \frac{A}{r} \quad \text{...........(A)}$$

where A is a constant of integration.

Integrating again we get

$$u = -\frac{r^2}{4\mu}\frac{\delta p}{\delta x} + A\ln r + B ....(B)$$

where B is another constant of integration.

Equations (A) and (B) may be used to derive Poiseuille's equation or it may be used to solve flow through an annular passage.

2.10.1 PIPE

At the middle r=0 so from equation (A) it follows that A=0

At the wall, u=0 and r=R. Putting this into equation B yields

$$0 = -\frac{R^2}{4\mu}\frac{\delta p}{\delta x} + A\ln R + B \quad \text{where A = 0}$$

$$B = \frac{R^2}{4\mu}\frac{\delta p}{\delta x}$$

$$u = -\frac{r^2}{4\mu}\frac{\delta p}{\delta x} + \frac{R^2}{4\mu}\frac{\delta p}{\delta x} = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left\{R^2 - r^2\right\} \text{ and this isPoiseuille's equation again.}$$

2.10.2 ANNULUS

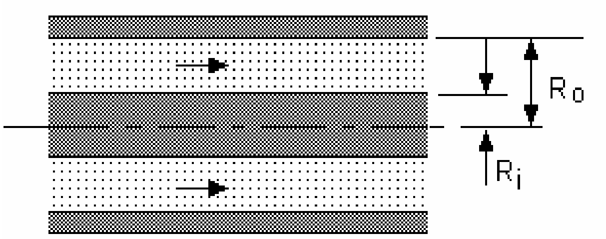


Fig.2.11

$$u = -\frac{r^2}{4\mu}\frac{\delta p}{\delta x} + A\ln r + B$$

The boundary conditions are $u = 0$ at $r = R_i$ and $r = R_o$.

$$0 = -\frac{R_o^2}{4\mu}\frac{\delta p}{\delta x} + A\ln R_o + B..........................(C)$$

$$0 = -\frac{R_i^2}{4\mu}\frac{\delta p}{\delta x} + A\ln R_i + B ....(D)$$

subtract D from C

$$0 = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left\{-R_o^2 + R_i^2\right\} + A\left\{\ln R_o - \ln R_i\right\}$$

$$0 = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left\{R_i^2 - R_0^2\right\} + A\ln\left\{\frac{R_o}{R_i}\right\}$$

$$A = \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}$$

This may be substituted back into equation D. The same result will be obtained from C.

$$0 = -\frac{R_i^2}{4\mu}\frac{\delta p}{\delta x} + \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln R_i + B$$

$$B = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[R_i^2 - \left\{\frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right\}\ln R_i\right]$$ This is put into equation B

$$u = \frac{-r^2}{4\mu}\frac{\delta p}{\delta x} + \frac{1}{4\mu}\frac{\delta p}{\delta x}\frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln r + \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[R_i^2 - \left\{\frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\right\}\ln R_i\right]$$

$$u = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[-r^2 + \frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln r + R_i^2 - \frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln R_i\right]$$

$$u = \frac{1}{4\mu}\frac{\delta p}{\delta x}\left[\frac{\{R_o^2 - R_i^2\}}{\ln\left\{\frac{R_o}{R_i}\right\}}\ln\frac{r}{R_i} + R_i^2 - r^2\right]$$

For given values the velocity distribution is similar to this.

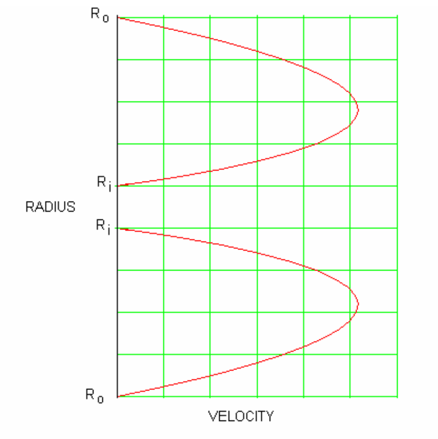


Fig. 2.12

ASSIGNMENT 2

1. Oil flows in a pipe 80 mm bore diameter with a mean velocity of 0.4 m/s. The density is 890 kg/m^3 and the viscosity is 0.075 Ns/m^2. Show that the flow is laminar and hence deduce the pressure loss per metre length.
   (150 Pa per metre).

2. Oil flows in a pipe 100 mm bore diameter with a Reynolds' Number of 500. The density is 800 kg/m^3. Calculate the velocity of a streamline at a radius of 40 mm. The viscosity μ = 0.08 Ns/m^2. (0.36 m/s)

3. A liquid of dynamic viscosity 5 x 10^{-3} Ns/m^2 flows through a capillary of diameter 3.0 mm under a pressure gradient of 1800 N/m^3. Evaluate the volumetric flow rate, the mean velocity, the centre line velocity and the radial position at which the velocity is equal to the mean velocity.
   (u_{av} = 0.101 m/s, u_{max} = 0.202 m/s r = 1.06 mm)

4. Similar to Q6 1998

a. Explain the term Stokes flow and terminal velocity.

b. Show that a spherical particle with Stokes flow has a terminal velocity given by
   $u = d^2g(\rho_s - \rho_f)/18\mu$
   Go on to show that $C_D=24/R_e$

c. For spherical particles, a useful empirical formula relating the drag coefficient and the Reynold's number is

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

   Given ρ_f = 1000 kg/m^3, μ= 1 cP and ρ_s= 2630 kg/m^3 determine the maximum size of spherical particles that will be lifted upwards by a vertical stream of water moving at 1 m/s.

d. If the water velocity is reduced to 0.5 m/s, show that particles with a diameter of less than 5.95 mm will fall downwards.

5. Similar to Q5 1998
   A simple fluid coupling consists of two parallel round discs of radius R separated by a a gap h. One disc is connected to the input shaft and rotates at ω_1 rad/s. The other disc is connected to the output shaft and rotates at ω_2 rad/s. The discs are separated by oil of dynamic viscosity μ and it may be assumed that the velocity gradient is linear at all radii.

   Show that the Torque at the input shaft is given by $T = \frac{\pi D^4 \mu(\omega_1 - \omega_2)}{32h}$

   The input shaft rotates at 900 rev/min and transmits 500W of power. Calculate the output speed, torque and power. (747 rev/min, 5.3 Nm and 414 W)
   Show by application of max/min theory that the output speed is half the input speed when maximum output power is obtained.

6. Show that for fully developed laminar flow of a fluid of viscosity μ between horizontal parallel plates a distance h apart, the mean velocity u_m is related to the pressure gradient dp/dx by $u_m = -(h^2/12\mu)(dp/dx)$

   Fig.2.11 shows a flanged pipe joint of internal diameter d_i containing viscous fluid of viscosity μ at gauge pressure p. The flange has an outer diameter d_o and is imperfectly tightened so that there is a narrow gap of thickness h. Obtain an expression for the leakage rate of the fluid through the flange.

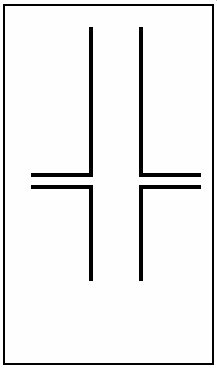


Fig.2.13

   Note that this is a radial flow problem and B in the notes becomes $2\pi r$ and dp/dx becomes -dp/dr. An integration between inner and outer radii will be required to give flow rate Q in terms of pressure drop p.

   The answer is $Q = (2\pi h^3 p/12\mu)/\{\ln(d_o/d_i)\}$

3. TURBULENT FLOW

3.1 FRICTION COEFFICIENT

The friction coefficient is a convenient idea that can be used to calculate the pressure drop in a pipe. It is defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}}$$

3.1.1 DYNAMIC PRESSURE

Consider a fluid flowing with mean velocity u_m. If the kinetic energy of the fluid is converted into flow or fluid energy, the pressure would increase. The pressure rise due to this conversion is called the dynamic pressure.

KE = ½ mu_m^2

Flow Energy = p Q   Q is the volume flow rate and $\rho = m/Q$

Equating ½ mu_m^2 = p Q $p = mu^2/2Q = ½ \rho u_m^2$

3.1.2 WALL SHEAR STRESS τ_o

The wall shear stress is the shear stress in the layer of fluid next to the wall of the pipe.

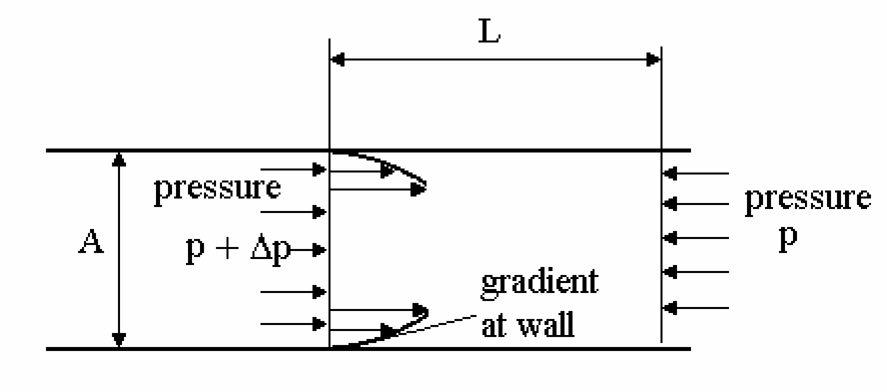


Fig.3.1

The shear stress in the layer next to the wall is $\tau_o = \mu\left(\frac{du}{dy}\right)_{wall}$

The shear force resisting flow is $F_s = \tau_o \pi L D$

The resulting pressure drop produces a force of $F_p = \frac{\Delta p \pi D^2}{4}$

Equating forces gives $\tau_o = \frac{D\Delta p}{4L}$

3.1.3 FRICTION COEFFICIENT for LAMINAR FLOW

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

From Poiseuille's equation $\Delta p = \frac{32\mu L u_m}{D^2}$ Hence $C_f = \left(\frac{2D}{4L\rho u_m^2}\right)\left(\frac{32\mu L u}{D^2}\right) = \frac{16\mu}{\rho u_m^2 D} = \frac{16}{R_e}$

3.1.4 DARCY FORMULA

This formula is mainly used for calculating the pressure loss in a pipe due to turbulent flow but it can be used for laminar flow also.

Turbulent flow in pipes occurs when the Reynolds Number exceeds 2500 but this is not a clear point so 3000 is used to be sure. In order to calculate the frictional losses we use the concept of friction coefficient symbol C_f. This was defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

Rearranging equation to make Δp the subject

$$\Delta p = \frac{4C_f L\rho u_m^2}{2D}$$

This is often expressed as a friction head h_f

$$h_f = \frac{\Delta p}{\rho g} = \frac{4C_f L u_m^2}{2gD}$$

This is the Darcy formula. In the case of laminar flow, Darcy's and Poiseuille's equations must give the same result so equating them gives

$$\frac{4C_f L u_m^2}{2gD} = \frac{32\mu L u_m}{\rho g D^2}$$

$$C_f = \frac{16\mu}{\rho u_m D} = \frac{16}{R_e}$$

This is the same result as before for laminar flow.

Turbulent flow may be safely assumed in pipes when the Reynolds' Number exceeds 3000. In order to calculate the frictional losses we use the concept of friction coefficient symbol C_f. Note that in older textbooks C_f was written as f but now the symbol f represents $4C_f$.

3.1.5 FLUID RESISTANCE

Fluid resistance is an alternative approach to solving problems involving losses. The above equations may be expressed in terms of flow rate Q by substituting u = Q/A

$$h_f = \frac{4C_f L u_m^2}{2gD} = \frac{4C_f L Q^2}{2gDA^2}$$

Substituting $A = \pi D^2/4$ we get the following.

$$h_f = \frac{32C_f L Q^2}{g\pi^2 D^5} = RQ^2$$

R is the fluid resistance or restriction. $R = \frac{32C_f L}{g\pi^2 D^5}$

If we want pressure loss instead of head loss the equations are as follows.

$$p_f = \rho g h_f = \frac{32\rho C_f L Q^2}{\pi^2 D^5} = RQ^2$$ R is the fluid resistance or restriction. $R = \frac{32\rho C_f L}{\pi^2 D^5}$

It should be noted that R contains the friction coefficient and this is a variable with velocity and surface roughness so R should be used with care.

3.2 MOODY DIAGRAM AND RELATIVE SURFACE ROUGHNESS

In general the friction head is some function of u_m such that $h_f = \phi u_m^n$. Clearly for laminar flow, n =1 but for turbulent flow n is between 1 and 2 and its precise value depends upon the roughness of the pipe surface. Surface roughness promotes turbulence and the effect is shown in the following work.

Relative surface roughness is defined as $\varepsilon = k/D$ where k is the mean surface roughness and D the bore diameter.

An American Engineer called Moody conducted exhaustive experiments and came up with the Moody Chart. The chart is a plot of C_f vertically against R_e horizontally for various values of ε. In order to use this chart you must know two of the three co-ordinates in order to pick out the point on the chart and hence pick out the unknown third co-ordinate. For smooth pipes, (the bottom curve on the diagram), various formulae have been derived such as those by Blasius and Lee.

BLASIUS $C_f = 0.0791\ R_e^{0.25}$

LEE $C_f = 0.0018 + 0.152\ R_e^{0.35}$.

The Moody diagram shows that the friction coefficient reduces with Reynolds number but at a certain point, it becomes constant. When this point is reached, the flow is said to be fully developed turbulent flow. This point occurs at lower Reynolds numbers for rough pipes.

A formula that gives an approximate answer for any surface roughness is that given by Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

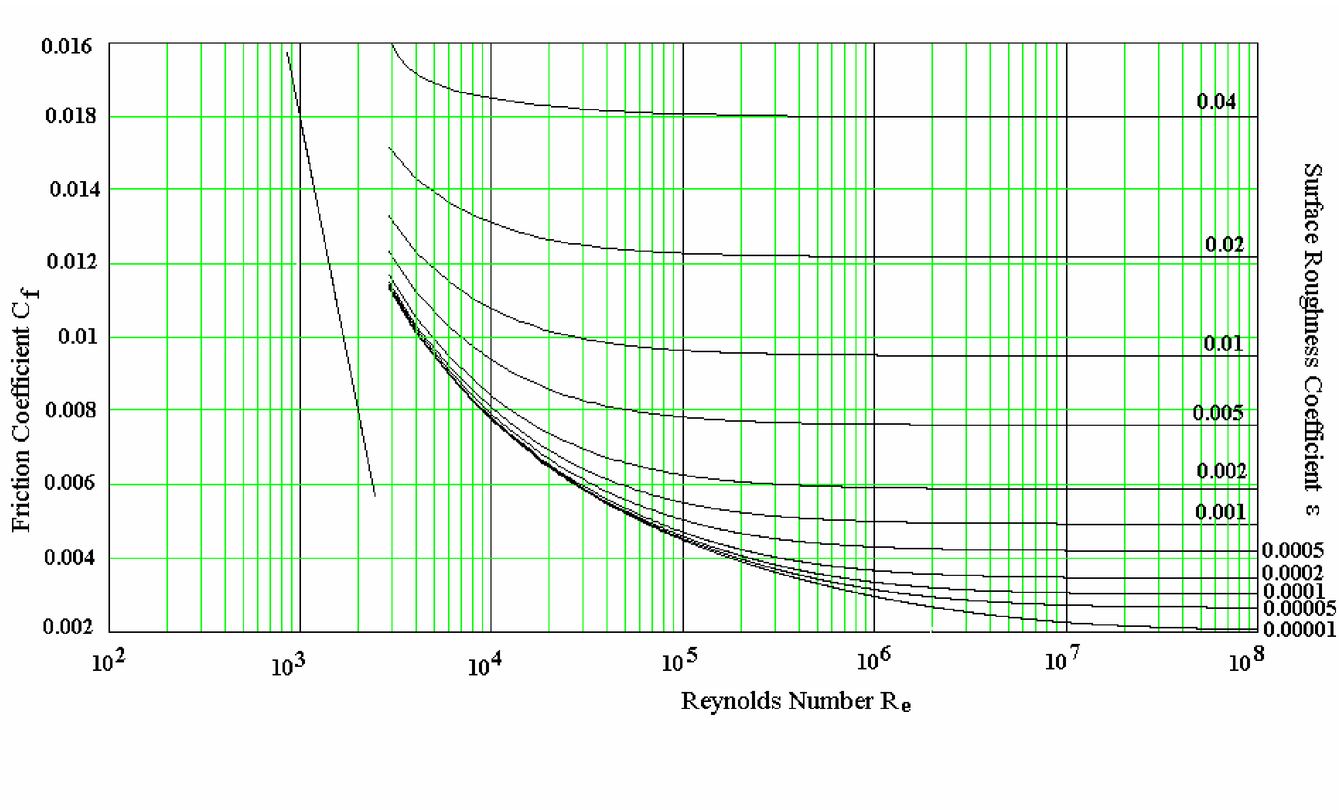

0.016
0.018
0.014
0.012
0.01
0.008
0.006
0.004
0.002
Friction Coefficient C_f
0.04
0.02
0.01
0.005
0.002
0.001
0.0005
0.0002
0.0001
0.00005
0.00001
Surface Roughness Coefficient ε
10^2
10^3
10^4
10^5
10^6
10^7
10^8
Reynolds Number R_e

Fig. 3.2 CHART

WORKED EXAMPLE 3.1

Determine the friction coefficient for a pipe 100 mm bore with a mean surface roughness of 0.06 mm when a fluid flows through it with a Reynolds number of 20 000.

SOLUTION

The mean surface roughness ε = k/d = 0.06/100 = 0.0006
Locate the line for ε = k/d = 0.0006.
Trace the line until it meets the vertical line at Re = 20 000. Read of the value of C_f horizontally on the left. Answer C_f = 0.0067.Check using the formula from Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = 12.206$$

$$C_f = 0.0067$$

WORKED EXAMPLE 3.2

Oil flows in a pipe 80 mm bore with a mean velocity of 4 m/s. The mean surface roughness is 0.02 mm and the length is 60 m. The dynamic viscosity is 0.005 N s/m^2 and the density is 900 kg/m^3. Determine the pressure loss.

SOLUTION

Re = ρud/μ = (900 x 4 x 0.08)/0.005 = 57600

ε= k/d = 0.02/80 = 0.00025

From the chart C_f = 0.0052

h_f = $4C_fLu2/2dg$ = (4 x 0.0052 x 60 x 4^2)/(2 x 9.81 x 0.08) = 12.72 m

Δp = $\rho g h_f$ = 900 x 9.81 x 12.72 = 112.32 kPa.

ASSIGNMENT 3

1. A pipe is 25 km long and 80 mm bore diameter. The mean surface roughness is 0.03 mm. It carries oil of density 825 kg/m^3 at a rate of 10 kg/s. The dynamic viscosity is 0.025 N s/m^2.

   Determine the friction coefficient using the Moody Chart and calculate the friction head. (Ans. 3075 m.)

2. Water flows in a pipe at 0.015 m^3/s. The pipe is 50 mm bore diameter. The pressure drop is 13 420 Pa per metre length. The density is 1000 kg/m^3 and the dynamic viscosity is 0.001 N s/m^2.

   Determine
   i. the wall shear stress (167.75 Pa)
   ii. the dynamic pressure (29180 Pa).
   iii. the friction coefficient (0.00575)
   iv. the mean surface roughness (0.0875 mm)

3. Explain briefly what is meant by fully developed laminar flow. The velocity u at any radius r in fully developed laminar flow through a straight horizontal pipe of internal radius r_o is given by

   $$u = (1/4\mu)(r_o^2 - r^2)dp/dx$$

   dp/dx is the pressure gradient in the direction of flow and μ is the dynamic viscosity.

   Show that the pressure drop over a length L is given by the following formula.

   $$\Delta p = 32\mu L u_m/D^2$$

   The wall skin friction coefficient is defined as $C_f = 2\tau_o/(\rho u_m^2)$.

   Show that $C_f = 16/R_e$ where $R_e = \rho u_m D/\mu$ and ρ is the density, u_m is the mean velocity and τ_o is the wall shear stress.

4. Oil with viscosity 2 x 10^{-2} Ns/m^2 and density 850 kg/m^3 is pumped along a straight horizontal pipe with a flow rate of 5 dm^3/s. The static pressure difference between two tapping points 10 m apart is 80 N/m^2. Assuming laminar flow determine the following.

   i. The pipe diameter.
   ii. The Reynolds number.

   Comment on the validity of the assumption that the flow is laminar

4. NON-NEWTONIAN FLUIDS

A Newtonian fluid as discussed so far in this tutorial is a fluid that obeys the law $\tau = \mu \frac{du}{dy} = \mu\dot{\gamma}$

A Non – Newtonian fluid is generally described by the non-linear law $\tau = \tau_y + k\dot{\gamma}^n$

τ_y is known as the yield shear stress and $\dot{\gamma}$ is the rate of shear strain. Figure 4.1 shows the principle forms of this equation.

Graph A shows an ideal fluid that has no viscosity and hence has no shear stress at any point. This is often used in theoretical models of fluid flow.

Graph B shows a Newtonian Fluid. This is the type of fluid with which this book is mostly concerned, fluids such as water and oil. The graph is hence a straight line and the gradient is the viscosity μ.

There is a range of other liquid or semi-liquid materials that do not obey this law and produce strange flow characteristics. Such materials include various foodstuffs, paints, cements and so on. Many of these are in fact solid particles suspended in a liquid with various concentrations.

Graph C shows the relationship for a ***Dilatent fluid***. The gradient and hence viscosity increases with $\dot{\gamma}$ and such fluids are also called ***shear-thickening.*** This phenomenon occurs with some solutions of sugar and starches.

Graph D shows the relationship for a ***Pseudo-plastic***. The gradient and hence viscosity reduces with $\dot{\gamma}$ and they are called ***shear-thinning.*** Most foodstuffs are like this as well as clay and liquid cement..

Other fluids behave like a ***plastic*** and require a minimum stress before it shears τ_y. This is plastic behaviour but unlike plastics, there may be no elasticity prior to shearing.

Graph E shows the relationship for a ***Bingham plastic.*** This is the special case where the behaviour is the same as a Newtonian fluid except for the existence of the yield stress. Foodstuffs containing high level of fats approximate to this model (butter, margarine, chocolate and Mayonnaise).

Graph F shows the relationship for a ***plastic*** fluid that exhibits shear thickening characteristics.

Graph G shows the relationship for a ***Casson fluid***. This is a plastic fluid that exhibits shear-thinning characteristics. This model was developed for fluids containing rod like solids and is often applied to molten chocolate and blood.

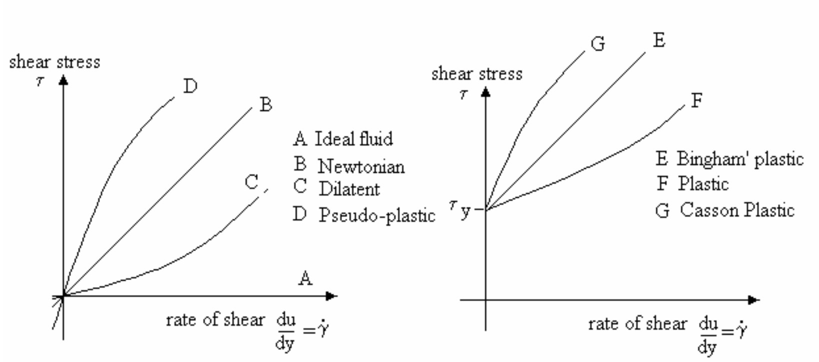


Fig.4.1

MATHEMATICAL MODELS

The graphs that relate shear stress τ and rate of shear strain γ are based on models or equations. Most are mathematical equations created to represent empirical data.

Hirschel and Bulkeley developed the power law for non-Newtonian equations. This is as follows.

$\tau = \tau_y + K\dot{\gamma}^n$ K is called the consistency coefficient and n is a power.

In the case of a Newtonian fluid n = 1 and $\tau_y = 0$ and K = μ (the dynamic viscosity) $\tau = \mu\dot{\gamma}$

For a Bingham plastic, n = 1 and K is also called the plastic viscosity μ_p. The relationship reduces to

$$\tau = \tau_y + \mu_p\dot{\gamma}$$

For a ***dilatent fluid,*** $\tau_y = 0$ and n>1

For a ***pseudo-plastic,*** $\tau_y = 0$ and n<1

The model for both is $\tau = K\dot{\gamma}^n$

The ***Herchel-Bulkeley*** model is as follows. $\tau = \tau_y + K\dot{\gamma}^n$

This may be developed as follows.

$$\tau = \tau_y + K\dot{\gamma}^n$$

$\tau - \tau_y = K\dot{\gamma}^n$ sometimes written as $\tau - \tau_y = \mu_p\dot{\gamma}^n$ where μ_p is called the plastic viscosity.

dividing by $\dot{\gamma}$

$$\frac{\tau}{\dot{\gamma}} - \frac{\tau_y}{\dot{\gamma}} = K\frac{\dot{\gamma}^n}{\dot{\gamma}} = K\dot{\gamma}^{n-1}$$

$\frac{\tau}{\dot{\gamma}} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$ The ratio is called the apparent viscosity μ_{app}

$$\mu_{app} = \frac{\tau}{\dot{\gamma}} = \frac{\tau_y}{\dot{\gamma}} + K\dot{\gamma}^{n-1}$$

For a Bingham plastic n = 1 so $\mu_{app} = \frac{\tau_y}{\dot{\gamma}} + K$

For a Fluid with no yield shear value $\tau_y = 0$ so $\mu_{app} = K\dot{\gamma}^{n-1}$

The ***Casson*** fluid model is quite different in form from the others and is as follows.

$$\tau^{\frac{1}{2}} = \tau_y^{\frac{1}{2}} + K\dot{\gamma}^{\frac{1}{2}}$$

THE FLOW OF A PLASTIC FLUID

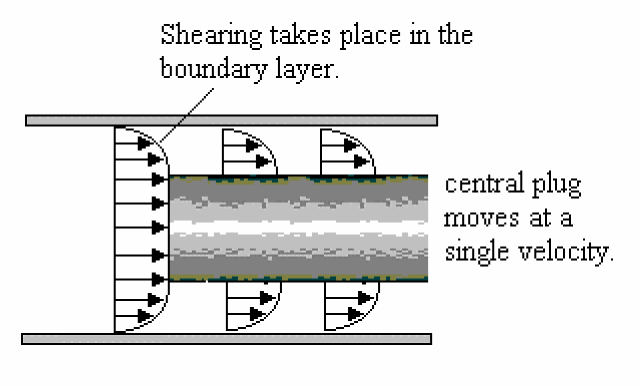


Fig.4.2

Note that fluids with a shear yield stress will flow in a pipe as a plug. Within a certain radius, the shear stress will be insufficient to produce shearing so inside that radius the fluid flows as a solid plug. Fig. 4.2 shows a typical situation for a Bingham Plastic.

MINIMUM PRESSURE

The shear stress acting on the surface of the plug is the yield value. Let the plug be diameter d.
The pressure force acting on the plug is $\Delta p \times \pi d^2/4$
The shear force acting on the surface of the plug is $\tau_y \times \pi d L$
Equating we find $\Delta p \times \pi d^2/4 = \tau_y \times \pi d L$

$$d = \tau_y \times 4 L/\Delta p \quad \text{or} \quad \Delta p = \tau_y \times 4 L/d$$

The minimum pressure required to produce flow must occur when d is largest and equal to the bore of the pipe. $\Delta p \text{ (minimum)} = \tau_y \times 4 L/D$

The diameter of the plug at any greater pressure must be given by $d = \tau_y \times 4 L/\Delta p$

For a Bingham Plastic, the boundary layer between the plug and the wall must be laminar and the velocity must be related to radius by the formula derived earlier.

$$u = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right) = \frac{\Delta p}{16\mu L}\left(D^2 - d^2\right)$$

FLOW RATE

The flow rate should be calculated in two stages. The plug moves at a constant velocity so the flow rate for the plug is simply $Q_p = u \times \text{cross sectional area} = u \times \pi d^2/4$

The flow within the boundary layer is found in the usual way as follows. Consider an elementary ring radius r and width dr.

$$dQ = u \times 2\pi r\, dr = \frac{\Delta p}{4\mu L}\left(R^2 - r^2\right) \times 2\pi r\, dr$$

$$Q = \frac{\Delta p \pi}{2\mu L}\int_R^r \left(rR^2 - r^3\right) dr$$

$$Q = \frac{\Delta p \pi}{2\mu L}\left[\frac{r^2 R^2}{2} - \frac{r^4}{4}\right]_r^R = \frac{\Delta p \pi}{2\mu L}\left[\left(\frac{R^4}{2} - \frac{R^4}{4}\right) - \left(\frac{r^2 R^2}{2} - \frac{r^4}{4}\right)\right]$$

$$Q = \frac{\Delta p \pi}{2\mu L}\left[\left(\frac{R^4}{4}\right) - \frac{r^2 R^2}{2} + \frac{r^4}{4}\right]$$

The mean velocity as always is defined as u_m = Q/Cross sectional area.

WORKED EXAMPLE 4.1

The Herchel-Bulkeley model for a non-Newtonian fluid is as follows. $\tau = \tau_y + K\dot{\gamma}^n$.

Derive an equation for the minimum pressure required drop per metre length in a straight horizontal pipe that will produce flow.

Given that the pressure drop per metre length in the pipe is 60 Pa/m and the yield shear stress is 0.2 Pa, calculate the radius of the slug sliding through the middle.

SOLUTION

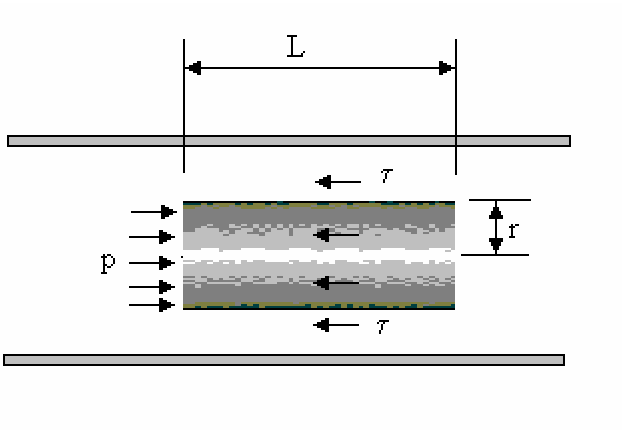


Fig. 3.3

The pressure difference p acting on the cross sectional area must produce sufficient force to overcome the shear stress τ acting on the surface area of the cylindrical slug. For the slug to move, the shear stress must be at least equal to the yield value τy. Balancing the forces gives the following.

$p \times \pi r^2 = \tau_y \times 2\pi rL$

$p/L = 2\tau_y /r$

$60 = 2 \times 0.2/r$

$r = 0.4/60 = 0.0066$ m or 6.6 mm

WORKED EXAMPLE 4.2

A Bingham plastic flows in a pipe and it is observed that the central plug is 30 mm diameter when the pressure drop is 100 Pa/m.

Calculate the yield shear stress.

Given that at a larger radius the rate of shear strain is 20 s^{-1} and the consistency coefficient is 0.6 Pa s, calculate the shear stress.

SOLUTION

For a Bingham plastic, the same theory as in the last example applies.

$p/L = 2\tau_y /r$
$100 = 2\ \tau_y/0.015$
$\tau_y = 100 \times 0.015/2 = 0.75$ Pa

A mathematical model for a Bingham plastic is

$\tau = \tau_y + K\dot{\gamma} = 0.75 + 0.6 \times 20 = 12.75$ Pa

ASSIGNMENT 4

1. Research has shown that tomato ketchup has the following viscous properties at 25°C.

Consistency coefficient K = 18.7 Pa s^n
Power n = 0.27
Shear yield stress = 32 Pa

Calculate the apparent viscosity when the rate of shear is 1, 10, 100 and 1000 s^{-1} and conclude on the effect of the shear rate on the apparent viscosity.

Answers

$\gamma = 1$ $\mu_{app} = 50.7$
$\gamma = 10$ $\mu_{app} = 6.682$
$\gamma = 100$ $\mu_{app} = 0.968$
$\gamma = 1000$ $\mu_{app} = 0.153$

2. A Bingham plastic fluid has a viscosity of 0.05 N s/m^2 and yield stress of 0.6 N/m^2. It flows in a tube 15 mm bore diameter and 3 m long.

(i) Evaluate the minimum pressure drop required to produce flow. (480 N/m^2)

The actual pressure drop is twice the minimum value. Sketch the velocity profile and calculate the following.

(ii) The radius of the solid core. (3.75 mm)
(iii) The velocity of the core. (67.5 mm/s)
(iv) The volumetric flow rate. (7.46 cm^3/s)

3. A non-Newtonian fluid is modelled by the equation $\tau = K\left(\frac{du}{dr}\right)^n$ where n = 0.8 and K = 0.05 N $s^{0.8}$/m^2. It flows through a tube 6 mm bore diameter under the influence of a pressure drop of 6400 N/m^2 per metre length. Obtain an expression for the velocity profile and evaluate the following.

(i) The centre line velocity. (0.953 m/s)
(ii) The mean velocity. (0.5 m/s)

FLUID MECHANICS 203
TUTORIAL No.2

APPLICATIONS OF BERNOULLI

On completion of this tutorial you should be able to

- derive Bernoulli's equation for liquids.
- find the pressure losses in piped systems due to fluid friction.
- find the minor frictional losses in piped systems.
- match pumps of known characteristics to a given system.
- derive the basic relationship between pressure, velocity and force..
- solve problems involving flow through orifices.
- solve problems involving flow through Venturi meters.
- understand orifice meters.
- understand nozzle meters.
- understand the principles of jet pumps
- solve problems from past papers.

Let's start by revising basics. The flow of a fluid in a pipe depends upon two fundamental laws, the conservation of mass and energy.

1. PIPE FLOW

The solution of pipe flow problems requires the applications of two principles, the law of conservation of mass (continuity equation) and the law of conservation of energy (Bernoulli's equation)

1.1 CONSERVATION OF MASS

When a fluid flows at a constant rate in a pipe or duct, the mass flow rate must be the same at all points along the length. Consider a liquid being pumped into a tank as shown (fig.1).

The mass flow rate at any section is $m = \rho A u_m$

ρ = density (kg/m^3)
u_m = mean velocity (m/s)
A = Cross Sectional Area (m^2)

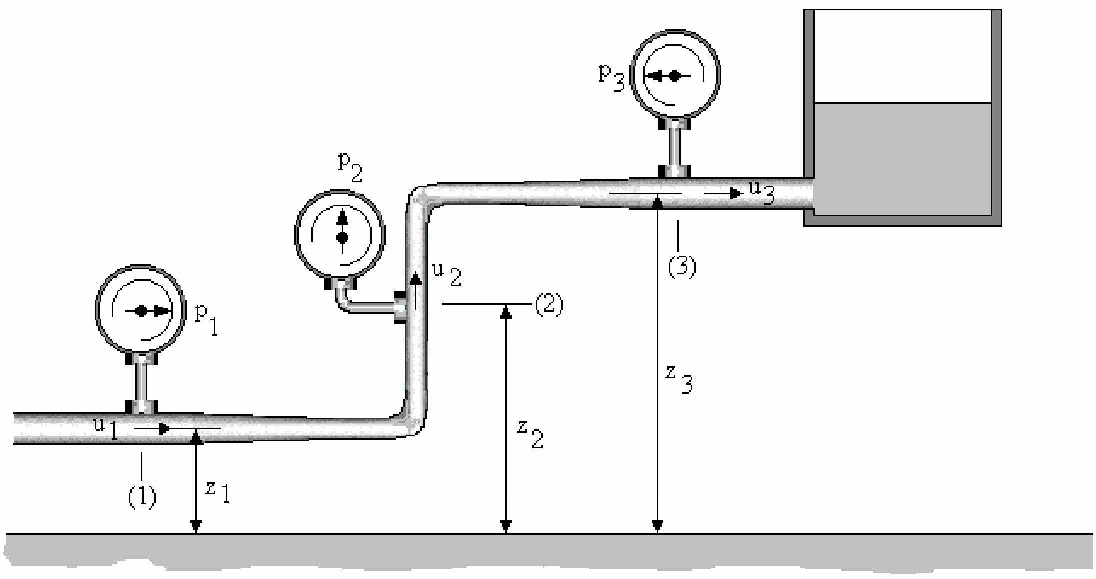


Fig.1.1

For the system shown the mass flow rate at (1), (2) and (3) must be the same so

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 = \rho_3 A_3 u_3$$

In the case of liquids the density is equal and cancels so

$$A_1 u_1 = A_2 u_2 = A_3 u_3 = Q$$

1.2 CONSERVATION OF ENERGY

ENERGY FORMS

FLOW ENERGY

This is the energy a fluid possesses by virtue of its pressure.

The formula is ***F.E. = pQ Joules***

p is the pressure (Pascals)
Q is volume rate (m^3)

POTENTIAL OR GRAVITATIONAL ENERGY

This is the energy a fluid possesses by virtue of its altitude relative to a datum level.

The formula is ***P.E. = mgz Joules***

m is mass (kg)
z is altitude (m)

KINETIC ENERGY

This is the energy a fluid possesses by virtue of its velocity.

The formula is $K.E. = \frac{1}{2} m u_m^2$ ***Joules***
u_m is mean velocity (m/s)

INTERNAL ENERGY

This is the energy a fluid possesses by virtue of its temperature. It is usually expressed relative to 0°C. The formula is $U = mc\theta$
c is the specific heat capacity (J/kg °C)
θ is the temperature in °C

In the following work, internal energy is not considered in the energy balance.

SPECIFIC ENERGY

Specific energy is the energy per kg so the three energy forms as specific energy are as follows.

$F.E./m = pQ/m = p/\rho$ ***Joules/kg***
$P.E/m. = gz$ ***Joules/kg***
$K.E./m = \frac{1}{2} u^2$ ***Joules/kg***

ENERGY HEAD

If the energy terms are divided by the weight mg, the result is energy per Newton. Examining the units closely we have J/N = N m/N = metres.

It is normal to refer to the energy in this form as the energy head. The three energy terms expressed this way are as follows.

$$F.E./mg = p/\rho g = h$$
$$P.E./mg = z$$
$$K.E./mg = u^2/2g$$

The flow energy term is called the pressure head and this follows since earlier it was shown that $p/\rho g = h$. This is the height that the liquid would rise to in a vertical pipe connected to the system.

The potential energy term is the actual altitude relative to a datum.

The term $u^2/2g$ is called the kinetic head and this is the pressure head that would result if the velocity is converted into pressure.

1.3 BERNOULLI'S EQUATION

Bernoulli's equation is based on the conservation of energy. If no energy is added to the system as work or heat then the total energy of the fluid is conserved. Remember that internal (thermal energy) has not been included.

The total energy E_T at (1) and (2) on the diagram (fig.3.1) must be equal so :

$$E_T = p_1 Q_1 + mgz_1 + m\frac{u_1^2}{2} = p_2 Q_2 + mgz_2 + m\frac{u_2^2}{2}$$

Dividing by mass gives the specific energy form

$$\frac{E_T}{m} = \frac{p_1}{\rho_1} + gz_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + gz_2 + \frac{u_2^2}{2}$$

Dividing by g gives the energy terms per unit weight

$$\frac{E_T}{mg} = \frac{p_1}{g\rho_1} + z_1 + \frac{u_1^2}{2g} = \frac{p_2}{g\rho_2} + z_2 + \frac{u_2^2}{2g}$$

Since $p/\rho g$ = pressure head h then the total head is given by the following.

$$h_T = h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g}$$

This is the head form of the equation in which each term is an energy head in metres. z is the potential or gravitational head and $u^2/2g$ is the kinetic or velocity head.

For liquids the density is the same at both points so multiplying by ρg gives the pressure form. The total pressure is as follows.

$$p_T = p_1 + \rho gz_1 + \frac{\rho u_1^2}{2} = p_2 + \rho gz_2 + \frac{\rho u_2^2}{2}$$

In real systems there is friction in the pipe and elsewhere. This produces heat that is absorbed by the liquid causing a rise in the internal energy and hence the temperature. In fact the temperature rise will be very small except in extreme cases because it takes a lot of energy to raise the temperature. If the pipe is long, the energy might be lost as heat transfer to the surroundings. Since the equations did not include internal energy, the balance is lost and we need to add an extra term to the right side of the equation to maintain the balance. This term is either the head lost to friction h_L or the pressure loss p_L.

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$

The pressure form of the equation is as follows.

$$p_1 + \rho gz_1 + \frac{\rho u_1^2}{2} = p_2 + \rho gz_2 + \frac{\rho u_2^2}{2} + p_L$$

The total energy of the fluid (excluding internal energy) is no longer constant.

Note that if a point is a ***free surface*** the pressure is normally atmospheric but if gauge pressures are used, the pressure and pressure head becomes zero. Also, if the surface area is large (say a large tank), the velocity of the surface is small and when squared becomes negligible so the kinetic energy term is neglected (made zero).

WORKED EXAMPLE No. 1

The diagram shows a pump delivering water through as pipe 30 mm bore to a tank. Find the pressure at point (1) when the flow rate is 1.4 dm^3/s. The density of water is 1000 kg/m^3. The loss of pressure due to friction is 50 kPa.

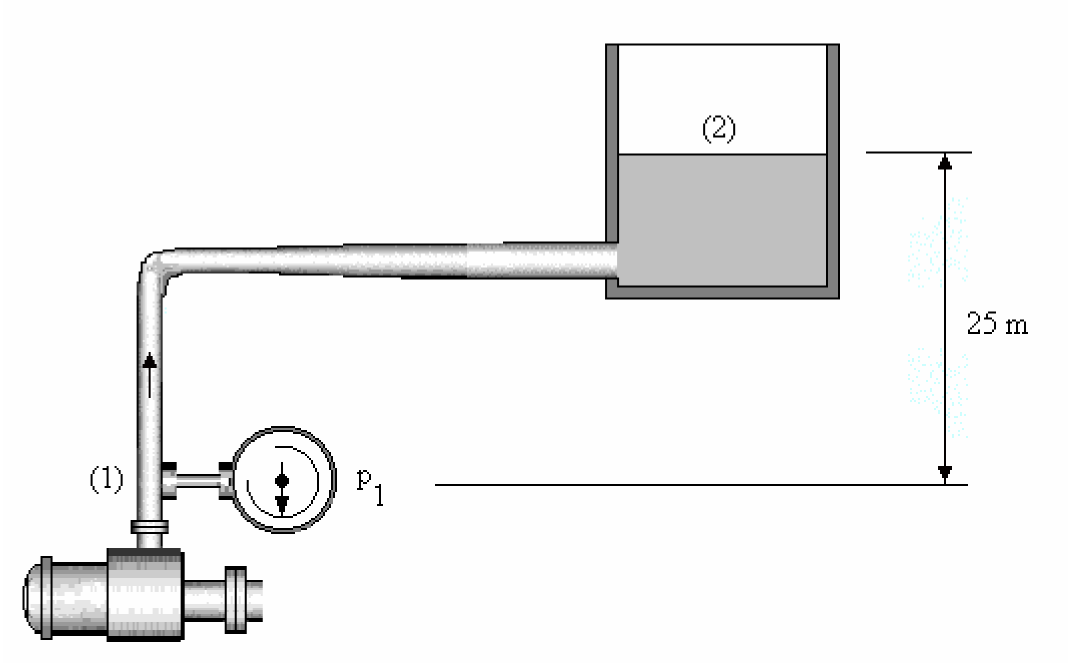


Fig.1.2

SOLUTION

Area of bore A = π x $0.03^2/4 = 706.8 \times 10^{-6}$ m^2.
Flow rate Q = 1.4 dm^3/s = 0.0014 m^3/s
Mean velocity in pipe = Q/A = 1.98 m/s

Apply Bernoulli between point (1) and the surface of the tank.

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

Make the low level the datum level and $z_1 = 0$ and $z_2 = 25$.

The pressure on the surface is zero gauge pressure.

P_L = 50 000 Pa

The velocity at (1) is 1.98 m/s and at the surface it is zero.

$$p_1 + 0 + \frac{1000x1.98^2}{2} = 0 + 1000x9.9125 + 0 + 50000$$
$$p_1 = 293.29kPa \text{ gauge pressure}$$

WORKED EXAMPLE 2

The diagram shows a tank that is drained by a horizontal pipe. Calculate the pressure head at point (2) when the valve is partly closed so that the flow rate is reduced to 20 dm^3/s. The pressure loss is equal to 2 m head.

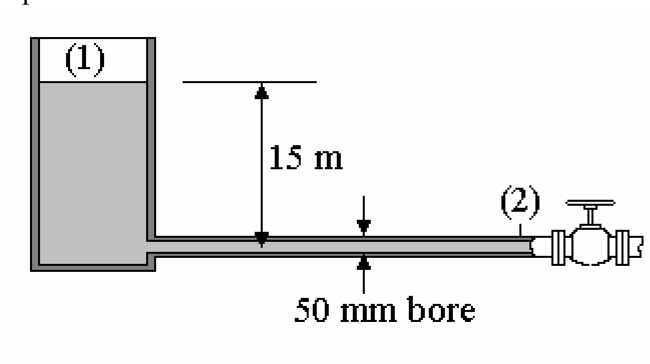


Fig.1.3

SOLUTION

Since point (1) is a free surface, $h_1 = 0$ and u_1 is assumed negligible.

The datum level is point (2) so $z_1 = 15$ and $z_2 = 0$.
Q = 0.02 m3/s
$A_2 = \pi d^2/4 = \pi \text{ x } (0.05^2)/4 = 1.963 \text{ x } 10^{-3} \text{ m}^2$.
$u_2 = Q/A = 0.02/1.963 \text{ x } 10^{-3} = 10.18$ m/s

Bernoulli's equation in head form is as follows.

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$

$$0 + 15 + 0 = h_2 + 0 + \frac{10.18^2}{2 \text{ x } 9.81} + 2$$

$$h_2 = 7.72\text{m}$$

WORKED EXAMPLE 3

The diagram shows a horizontal nozzle discharging into the atmosphere. The inlet has a bore area of 600 mm^2 and the exit has a bore area of 200 mm^2. Calculate the flow rate when the inlet pressure is 400 Pa. Assume there is no energy loss.

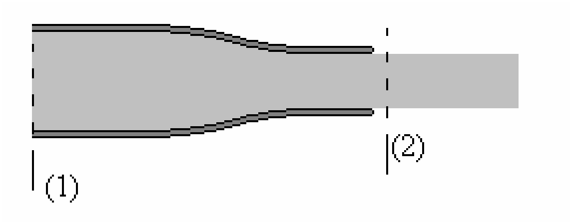


Fig. 1.4

SOLUTION

Apply Bernoulli between (1) and (2)

$$p_1 + \rho g z_1 + \frac{\rho u_1^2}{2} = p_2 + \rho g z_2 + \frac{\rho u_2^2}{2} + p_L$$

Using gauge pressure, p2 = 0 and being horizontal the potential terms cancel. The loss term is zero so the equation simplifies to the following.

$$p_1 + \frac{\rho u_1^2}{2} = \frac{\rho u_2^2}{2}$$

From the continuity equation we have

$$u_1 = \frac{Q}{A_1} = \frac{Q}{600 \text{ x } 10^{-6}} = 1\,666.7Q$$

$$u_2 = \frac{Q}{A_2} = \frac{Q}{200 \text{ x } 10^{-6}} = 5\,000\,Q$$

Putting this into Bernoulli's equation we have the following.

$$400 + 1000 \text{ x } \frac{(1666.7Q)^2}{2} = 1000 \text{ x } \frac{(5000Q)^2}{2}$$

$$400 + 1.389 \text{ x} 10^9 Q^2 = 12.5 \text{ x } 10^9 Q^2$$

$$400 = 11.11 \text{ x} 10^9 Q^2$$

$$Q^2 = \frac{400}{11.11 \text{ x} 10^9} = 36 \text{ x} 10^{-9}$$

$$Q = 189.7 \text{ x } 10^{-6} \text{ m}^3\text{/s or } 189.7 \text{ cm}^3\text{/s}$$

1.4 HYDRAULIC GRADIENT

Consider a tank draining into another tank at a lower level as shown. There are small vertical tubes at points along the length to indicate the pressure head (h). Relative to a datum, the total energy head is $h_T = h + z + u^2/2g$ and this is shown as line A.

The hydraulic grade line is the line joining the free surfaces in the tubes and represents the sum of h and z only. This is shown as line B and it is always below the line of h_T by the velocity head $u^2/2g$. Note that at exit from the pipe, the velocity head is not recovered but lost as friction as the emerging jet collides with the static liquid. The free surface of the tank does not rise.

The only reason why the hydraulic grade line is not horizontal is because there is a frictional loss h_f. The actual gradient of the line at any point is the rate of change with length $i = \delta h_f/\delta L$

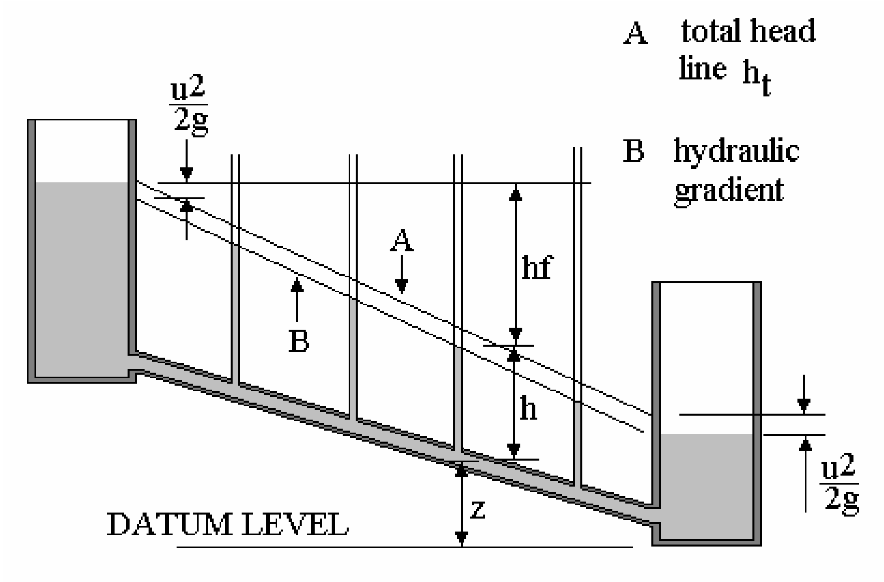


Fig.1.5

SELF ASSESSMENT EXERCISE 1

1. A pipe 100 mm bore diameter carries oil of density 900 kg/m^3 at a rate of 4 kg/s. The pipe reduces to 60 mm bore diameter and rises 120 m in altitude. The pressure at this point is atmospheric (zero gauge). Assuming no frictional losses, determine:

   i. The volume/s (4.44 dm^3/s)
   ii. The velocity at each section (0.566 m/s and 1.57 m/s)
   iii. The pressure at the lower end. (1.06 MPa)

2. A pipe 120 mm bore diameter carries water with a head of 3 m. The pipe descends 12 m in altitude and reduces to 80 mm bore diameter. The pressure head at this point is 13 m. The density is 1000 kg/m^3. Assuming no losses, determine

   i. The velocity in the small pipe (7 m/s)
   ii. The volume flow rate. (35 dm^3/s)

3. A horizontal nozzle reduces from 100 mm bore diameter at inlet to 50 mm at exit. It carries liquid of density 1000 kg/m^3 at a rate of 0.05 m^3/s. The pressure at the wide end is 500 kPa (gauge). Calculate the pressure at the narrow end neglecting friction. (196 kPa)

4. A pipe carries oil of density 800 kg/m^3. At a given point (1) the pipe has a bore area of 0.005 m^2 and the oil flows with a mean velocity of 4 m/s with a gauge pressure of 800 kPa. Point (2) is further along the pipe and there the bore area is 0.002 m^2 and the level is 50 m above point (1). Calculate the pressure at this point (2). Neglect friction. (374 kPa)

5. A horizontal nozzle has an inlet velocity u_1 and an outlet velocity u_2 and discharges into the atmosphere. Show that the velocity at exit is given by the following formulae.

$$u_2 = \{2\Delta p/\rho + u_1^2\}^{½}$$

and

$$u_2 = \{2g\Delta h + u_1^2\}^{½}$$

2 PRESSURE LOSSES IN PIPE SYSTEMS

2 .1 REVIEW OF EARLIER WORK

FRICTION COEFFICIENT

The friction coefficient is a convenient idea that can be used to calculate the pressure drop in a pipe. It is defined as follows.

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}}$$

$$p = \tfrac{1}{2}\rho u_m^2$$

$$\tau_o = \frac{D\Delta p}{4L}$$

$$C_f = \frac{\text{Wall Shear Stress}}{\text{Dynamic Pressure}} = \frac{2D\Delta p}{4L\rho u_m^2}$$

From Poiseuille's equation $\Delta p = \dfrac{32\mu L u_m}{D^2}$ Hence $C_f = \left(\dfrac{2D}{4L\rho u_m^2}\right)\left(\dfrac{32\mu L u}{D^2}\right) = \dfrac{16\mu}{\rho u_m^2 D} = \dfrac{16}{R_e}$

DARCY FORMULA

$$\Delta p = \frac{4C_f L\rho u_m^2}{2D}$$

This is often expressed as a friction head h_f

$$h_f = \frac{\Delta p}{\rho g} = \frac{4C_f L u_m^2}{2gD}$$

This is the Darcy formula. In the case of laminar flow, Darcy's and Poiseuille's equations must give the same result so equating them gives

$$\frac{4C_f L u_m^2}{2gD} = \frac{32\mu L u_m}{\rho g D^2}$$

$$C_f = \frac{16\mu}{\rho u_m D} = \frac{16}{R_e}$$

This is the same result as before for laminar flow.

A formula that gives an approximate answer for any surface roughness is that given by Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

This gives a very close model of the Moody chart covered earlier.

WORKED EXAMPLE 4

Determine the friction coefficient for a pipe 100 mm bore with a mean surface roughness of 0.06 mm when a fluid flows through it with a Reynolds number of 20 000.

SOLUTION

The mean surface roughness ε = k/d = 0.06/100 = 0.0006
Locate the line for ε = k/d = 0.0006.
Trace the line until it meets the vertical line at Re = 20 000. Read of the value of C_f horizontally on the left. Answer $C_f = 0.0067$

Check using the formula from Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\frac{6.9}{20000} + \left(\frac{0.0006}{3.71}\right)^{1.11}\right\}$$

$$\frac{1}{\sqrt{C_f}} = 12.206$$

$$C_f = 0.0067$$

WORKED EXAMPLE 5

Oil flows in a pipe 80 mm bore with a mean velocity of 4 m/s. The mean surface roughness is 0.02 mm and the length is 60 m. The dynamic viscosity is 0.005 N s/m^2 and the density is 900 kg/m^3. Determine the pressure loss.

SOLUTION

$Re = \rho ud/\mu = (900 \times 4 \times 0.08)/0.005 = 57600$

$\varepsilon = k/d = 0.02/80 = 0.00025$

From the chart $C_f = 0.0052$

$h_f = 4C_fLu2/2dg = (4 \times 0.0052 \times 60 \times 4^2)/(2 \times 9.81 \times 0.08) = 12.72$ m

$\Delta p = \rho g h_f = 900 \times 9.81 \times 12.72 = 112.32$ kPa.

2.2 MINOR LOSSES

Minor losses occur in the following circumstances.

i. Exit from a pipe into a tank.
ii. Entry to a pipe from a tank.
iii. Sudden enlargement in a pipe.
iv. Sudden contraction in a pipe.
v. Bends in a pipe.
vi. Any other source of restriction such as pipe fittings and valves.

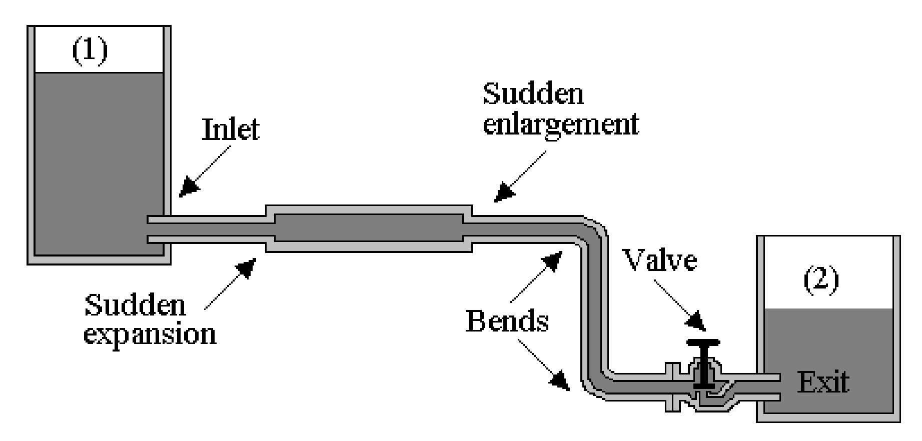


Fig.2.1

In general, minor losses are neglected when the pipe friction is large in comparison but for short pipe systems with bends, fittings and changes in section, the minor losses are the dominant factor.

In general, the minor losses are expressed as a fraction of the kinetic head or dynamic pressure in the smaller pipe.

Minor head loss = $k\,u^2/2g$ Minor pressure loss = $\frac{1}{2}k\rho u^2$

Values of k can be derived for standard cases but for items like elbows and valves in a pipeline, it is determined by experimental methods.

Minor losses can also be expressed in terms of fluid resistance R as follows.

$$h_L = k\frac{u^2}{2} = k\frac{Q^2}{2A^2} = k\frac{8Q^2}{\pi^2 D^4} = RQ^2 \quad \text{Hence } R = \frac{8k}{\pi^2 D^4}$$

$$p_L = k\frac{8\rho g Q^2}{\pi^2 D^4} = RQ^2 \quad \text{hence } R = \frac{8k\rho g}{\pi^2 D^4}$$

Before you go on to look at the derivations, you must first learn about the coefficients of contraction and velocity.

COEFFICIENT OF CONTRACTION Cc

The fluid approaches the entrance from all directions and the radial velocity causes the jet to contract just inside the pipe. The jet then spreads out to fill the pipe. The point where the jet is smallest is called the *VENA CONTRACTA*.

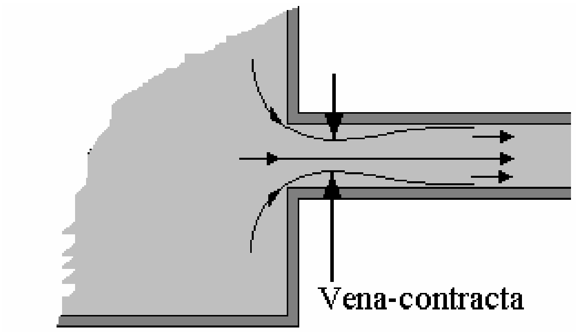


Fig.2.2

The coefficient of contraction C_c is defined as $C_c = A_j/A_o$

A_j is the cross sectional area of the jet and A_o is the c.s.a. of the pipe. For a round pipe this becomes $C_c = d_j^2/d_o^2$.

COEFFICIENT OF VELOCITY C_v

The coefficient of velocity is defined as **C_v = actual velocity/theoretical velocity**

In this instance it refers to the velocity at the vena-contracta but as you will see later on, it applies to other situations also.

EXIT FROM A PIPE INTO A TANK.

The liquid emerges from the pipe and collides with stationary liquid causing it to swirl about before finally coming to rest. All the kinetic energy is dissipated by friction. It follows that all the kinetic head is lost so k = 1.0

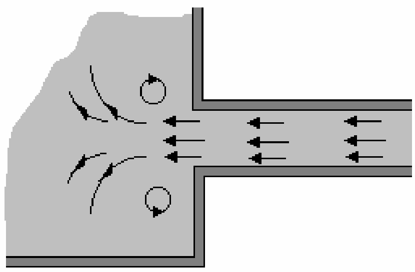


Fig.2.3

ENTRY TO A PIPE FROM A TANK

The value of k varies from 0.78 to 0.04 depending on the shape of the inlet. A good rounded inlet has a low value but the case shown is the worst.

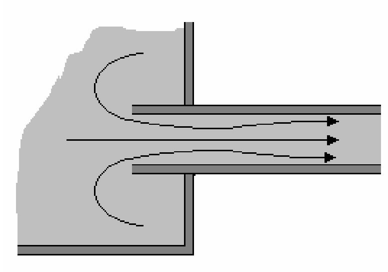


Fig.2.4

SUDDEN ENLARGEMENT

This is similar to a pipe discharging into a tank but this time it does not collide with static fluid but with slower moving fluid in the large pipe. The resulting loss coefficient is given by the following expression.

$$k = \left\{1 - \left(\frac{d_1}{d_2}\right)^2\right\}^2$$

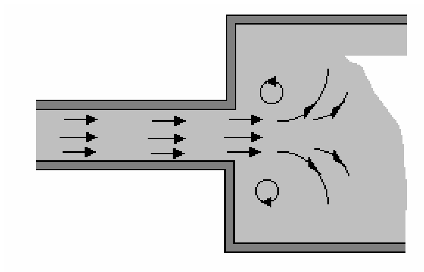


Fig.2.5

SUDDEN CONTRACTION

This is similar to the entry to a pipe from a tank. The best case gives k = 0 and the worse case is for a sharp corner which gives k = 0.5.

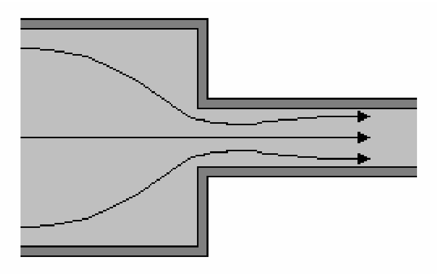


Fig.2.6

BENDS AND FITTINGS

The k value for bends depends upon the radius of the bend and the diameter of the pipe. The k value for bends and the other cases is on various data sheets. For fittings, the manufacturer usually gives the k value. Often instead of a k value, the loss is expressed as an equivalent length of straight pipe that is to be added to L in the Darcy formula.

WORKED EXAMPLE 6

A tank of water empties by gravity through a horizontal pipe into another tank. There is a sudden enlargement in the pipe as shown. At a certain time, the difference in levels is 3 m. Each pipe is 2 m long and has a friction coefficient $C_f = 0.005$. The inlet loss constant is K = 0.3.

Calculate the volume flow rate at this point.

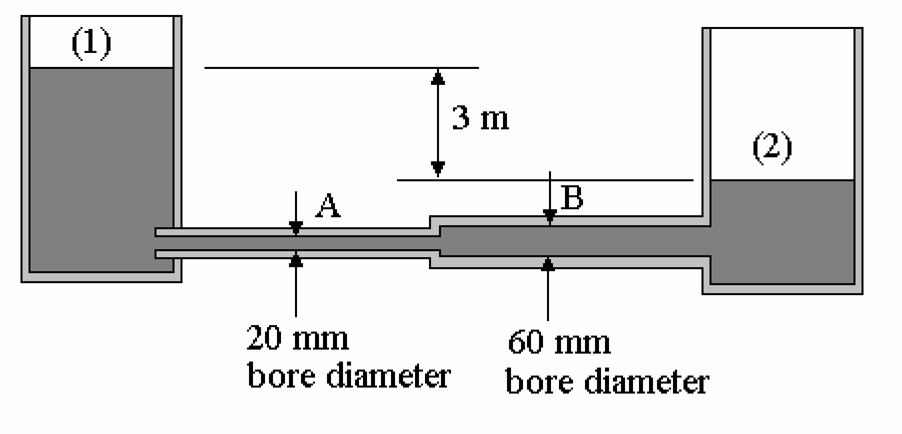


Fig.2.7

SOLUTION

There are five different sources of pressure loss in the system and these may be expressed in terms of the fluid resistance as follows.

The head loss is made up of five different parts. It is usual to express each as a fraction of the kinetic head as follows.

Resistance pipe A $$R_1 = \frac{32C_fL}{gD_A{}^5\pi^2} = \frac{32 \text{ x } 0.005 \text{ x } 2}{g \text{ x } 0.02^5\pi^2} = 1.0328 \text{ x } 10^6 s^2m^{-5}$$

Resistance in pipe B $$R_2 = \frac{32C_fL}{gD_B^5\pi^2} = \frac{32 \text{ x } 0.005 \text{ x } 2}{g \text{ x } 0.06^5\pi^2} = 4.250\text{x}10^3 s^2m^{-5}$$

Loss at entry K=0.3 $$R_3 = \frac{8K}{g\pi^2D_A{}^4} = \frac{8 \text{ x } 0.3}{g\,\pi^2 \text{x } 0.02^4} = 158\, s^2m^{-5}$$

Loss at sudden enlargement. $$k = \left\{1 - \left(\frac{d_A}{d_B}\right)^2\right\}^2 = \left\{1 - \left(\frac{20}{60}\right)^2\right\}^2 = 0.79$$

$$R_4 = \frac{8K}{g\pi^2D_A{}^4} = \frac{8\text{x}0.79}{g\pi^2\text{x } 0.02^4} = 407.7\, s^2m^{-5}$$

Loss at exit K=1 $$R_5 = \frac{8K}{g\pi^2D_B{}^4} = \frac{8\text{x}1}{g\pi^2\text{x } 0.06^4} = 63710\, s^2m^{-5}$$

Total losses.

$$h_L = R_1Q^2 + R_2Q^2 + R_3Q^2 + R_4Q^2 + R_5Q^2$$
$$h_L = (R_1 + R_2 + R_2 + R_4 + R_5)Q^2$$
$$h_L = 1.101 \text{ x } 10^6 Q^2$$

BERNOULLI'S EQUATION

Apply Bernoulli between the free surfaces (1) and (2)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$

On the free surface the velocities are small and about equal and the pressures are both atmospheric so the equation reduces to the following.

$z_1 - z_2 = h_L = 3$

$3 = 1.101 \text{ x } 10^6 Q^2$

$Q^2 = 2.724 \text{ x } 10^{-6}$

$Q = 1.65 \text{ x } 10^{-3} m^3/s$

2.3 SIPHONS

Liquid will siphon from a tank to a lower level even if the pipe connecting them rises above the level of both tanks as shown in the diagram. Calculation will reveal that the pressure at point (2) is lower than atmospheric pressure (a vacuum) and there is a limit to this pressure when the liquid starts to turn into vapour. For water about 8 metres is the practical limit that it can be sucked (8 m water head of vacuum).

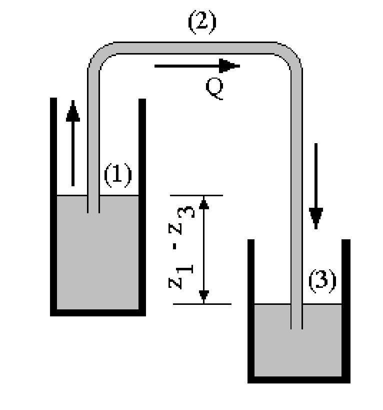


Fig.2.8

WORKED EXAMPLE 7

A tank of water empties by gravity through a siphon. The difference in levels is 3 m and the highest point of the siphon is 2 m above the top surface level and the length of pipe from inlet to the highest point is 2.5 m. The pipe has a bore of 25 mm and length 6 m. The friction coefficient for the pipe is 0.007.The inlet loss coefficient K is 0.7.

Calculate the volume flow rate and the pressure at the highest point in the pipe.

SOLUTION

There are three different sources of pressure loss in the system and these may be expressed in terms of the fluid resistance as follows.

Pipe Resistance $$R_1 = \frac{32C_fL}{gD^5\pi^2} = \frac{32 \text{ x } 0.007 \text{ x } 6}{g \text{ x } 0.025^5 \pi^2} = 1.422 \text{ x } 10^6 \text{ s}^2\text{m}^{-5}$$

Entry Loss Resistance $$R_2 = \frac{8K}{g\pi^2 D^4} = \frac{8 \text{ x } 0.7}{g\pi^2 \text{x } 0.025^4} = 15.1 \text{ x } 10^3 \text{ s}^2\text{m}^{-5}$$

Exit loss Resistance $$R_3 = \frac{8K}{g\pi^2 D^4} = \frac{8\text{x}1}{g\pi^2 \text{x } 0.025^4} = 21.57 \text{ x } 10^3 \text{ s}^2\text{m}^{-5}$$

Total Resistance $R_T = R_1 + R_2 + R_3 = 1.458 \text{ x } 10^6 \text{ s}^2 \text{ m}^{-5}$

Apply Bernoulli between the free surfaces (1) and (3)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_3 + z_3 + \frac{u_3^2}{2g} + h_L$$
$$0_1 + z_1 + 0 = 0 + z_3 + 0 + h_L$$
$$z_1 - z_3 = h_L = 3$$

Flow rate $$Q = \sqrt{\frac{z_1 - z_3}{R_T}} = \sqrt{\frac{3}{1.458 \text{ x } 10^6}} = 1.434\text{x}10^{-3}\text{ m}^3/\text{s}$$

Bore Area $A=\pi D^2/4 = \pi \text{ x } 0.025^2/4 = 490.87 \text{ x } 10^{-6} \text{ m}^2$

Velocity in Pipe $u = Q/A = 1.434 \text{ x } 10^{-3}/490.87 \text{ x } 10^{-6} = 2.922$ m/s

Apply Bernoulli between the free surfaces (1) and (2)

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g} + h_L$$
$$0 + 0 + 0 = h_2 + 2 + \frac{2.922^2}{2g} + h_L$$
$$h_2 = -2 - h_L \frac{2.922^2}{2g} - h_L$$
$$h_2 = -2 + 0.435 - h_L = -2.435 - h_L$$

Calculate the losses between (1) and (2)

Pipe friction Resistance is proportionally smaller by the length ratio.

$R_1 = (2.5/6) \text{ x}1.422 \text{ x } 10^6 = 0.593 \text{ x } 10^6$

Entry Resistance $R_2 = 15.1 \text{ x } 10^3$ as before

Total resistance $R_T = 608.1 \text{ x } 10^3$

Head loss $h_L = R_T Q^2 = 1.245\text{m}$

The pressure head at point (2) is hence $h_2 = -2.435 - 1.245 = -3.68$ m head

3. MATCHING PUMPS TO A PIPE SYSTEM.

The ideal pump for any given pipe system will produce the required flow rate at the required pressure. The maximum efficiency of the pump will occur at these conditions. These points are considered in detail in a later tutorial.

The relationship between flow rate Q, pressure head H and efficiency η depend upon the speed but most of all, they depend upon the type of pump. The diagram below shows typical relationships.

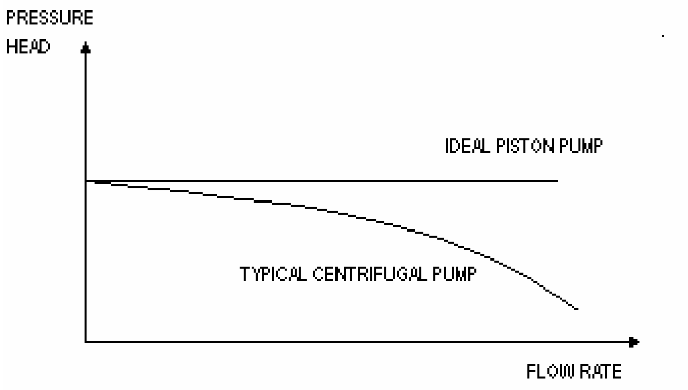


Figure 3.1

The relationship between pressure head and flow rate for a given pipe system is generally one that requires a bigger head for a bigger flow rate. The exact relationship depends upon the losses. If the pump is required to raise the level of the flow, then the required head h is the change in level (lift) plus the losses. The losses are due to pipe friction (and hence the friction factor C_f) , the losses at entry, exit, bends, sudden changes in section and fittings such as valves. The relationship is typically as shown.

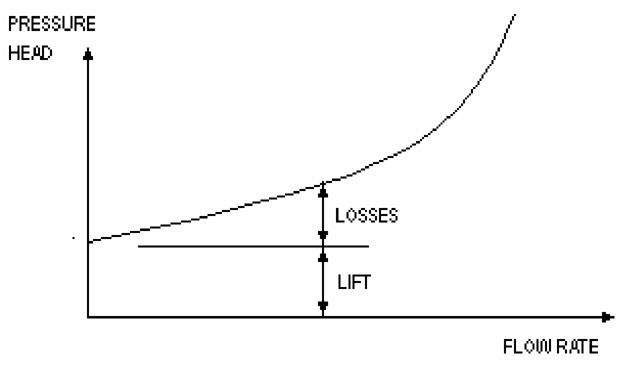


Figure 3.2

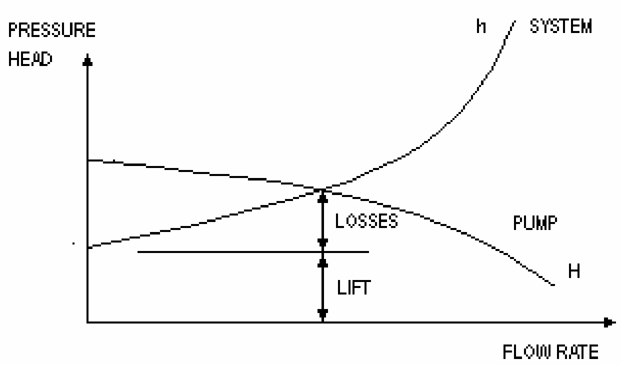


Figure 3.3

If a given pump is to work with a given system, the operating point must be common to each. In other words H = h at the required flow rate.

The solution of problems depends upon finding the relationship between head and flow rate for both the pump and the system and finding the point where the graphs cross.

SELF ASSESSMENT EXERCISE 2

1. A pipe carries oil at a mean velocity of 6 m/s. The pipe is 5 km long and 1.5 m diameter. The surface roughness is 0.8 mm. The density is 890 kg/m^3 and the dynamic viscosity is 0.014 N s/m^2. Determine the friction coefficient from the Moody chart and go on to calculate the friction head h_f.
(Ans. $C_f = 0.0045$ $h_f = 110.1$ m)

2. The diagram shows a tank draining into another lower tank through a pipe. Note the velocity and pressure is both zero on the surface on a large tank. Calculate the flow rate using the data given on the diagram. (Ans. 7.16 dm^3/s)

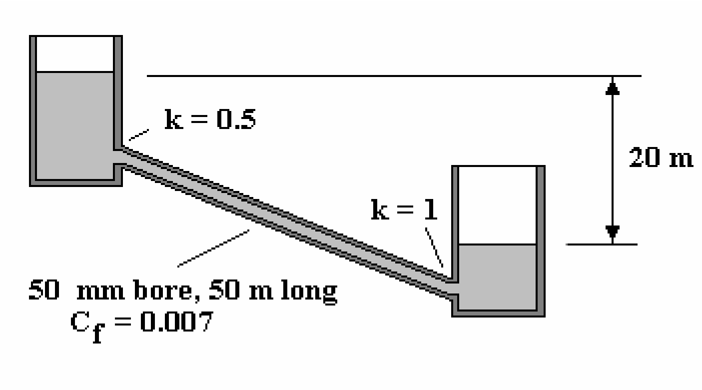


Fig. 3.4

3. Water flows through the sudden pipe expansion shown below at a flow rate of 3 dm^3/s. Upstream of the expansion the pipe diameter is 25 mm and downstream the diameter is 40 mm. There are pressure tappings at section (1), about half a diameter upstream, and at section (2), about 5 diameters downstream. At section (1) the gauge pressure is 0.3 bar.

   Evaluate the following.
   (i) The gauge pressure at section (2) (0.387 bar)
   (ii) The total force exerted by the fluid on the expansion. (-23 N)

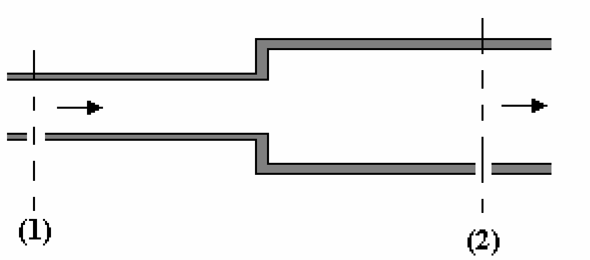


Fig. 3.5

4. A tank of water empties by gravity through a siphon into a lower tank. The difference in levels is 6 m and the highest point of the siphon is 2 m above the top surface level. The length of pipe from the inlet to the highest point is 3 m. The pipe has a bore of 30 mm and length 11 m. The friction coefficient for the pipe is 0.006.The inlet loss coefficient K is 0.6.

   Calculate the volume flow rate and the pressure at the highest point in the pipe.
   (Answers 2.378 dm^3/s and –4.31 m)

5. A domestic water supply consists of a large tank with a loss free-inlet to a 10 mm diameter pipe of length 20 m, that contains 9 right angles bends. The pipe discharges to atmosphere 8.0 m below the free surface level of the water in the tank.

   Evaluate the flow rate of water assuming that there is a loss of 0.75 velocity heads in each bend and that friction in the pipe is given by the Blasius equation $C_f=0.079(Re)^{-0.25}$
   The dynamic viscosity is 0.89×10^{-3} and the density is 997 kg/m^3.
   (0.118 dm^3/s).

6. A pump A whose characteristics are given in table 1, is used to pump water from an open tank through 40 m of 70 mm diameter pipe of friction factor $C_f=0.005$ to another open tank in which the surface level of the water is 5.0 m above that in the supply tank.

   Determine the flow rate when the pump is operated at 1450 rev/min.
   (7.8 dm^3/s)

   It is desired to increase the flow rate and 3 possibilities are under investigation.
   (i) To install a second identical pump in series with pump A.

   (ii) To install a second identical pump in parallel with pump A.

   (iii) To increase the speed of the pump by 10%.

   Predict the flow rate that would occur in each of these situations.

Head-Flow Characteristics of pump A when operating at 1450 rev/min

Head/m		9.75	8.83	7.73	6.90	5.50	3.83
Flow Rate/(l/s)	4.73	6.22	7.57	8.36	9.55	10.75	

Table 1

7. A steel pipe of 0.075 m inside diameter and length 120 m is connected to a large reservoir. Water is discharged to atmosphere through a gate valve at the free end, which is 6 m below the surface level in the reservoir. There are four right angle bends in the pipe line. Find the rate of discharge when the valve is fully open. (ans. 8.3 dm^3/s).The kinematic viscosity of the water may be taken to be 1.14 x 10^{-6} m^2/s. Use a value of the friction factor C_f taken from table 2 which gives C_f as a function of the Reynolds number Re and allow for other losses as follows.

   at entry to the pipe 0.5 velocity heads.
   at each right angle bend 0.9 velocity heads.
   for a fully open gate valve 0.2 velocity heads.

Re x 10^5	0.987	1.184	1.382
C_f	0.00448	0.00432	0.00419

Table 2

8. (i) Sketch diagrams showing the relationship between Reynolds number, Re, and friction factor, C_f , for the head lost when oil flows through pipes of varying degrees of roughness. Discuss the importance of the information given in the diagrams when specifying the pipework for a particular system.

   (ii) The connection between the supply tank and the suction side of a pump consists of 0.4 m of horizontal pipe , a gate valve one elbow of equivalent pipe length 0.7 m and a vertical pipe down to the tank.

   If the diameter of the pipes is 25 mm and the flow rate is 30 l/min, estimate the maximum distance at which the supply tank may be placed below the pump inlet in order that the pressure there is no less than 0.8 bar absolute. (Ans. 1.78 m)

   The fluid has kinematic viscosity 40×10^{-6} m^2/s and density 870 kg/m^3.

   Assume
   (a) for laminar flow $C_f = 16/(Re)$ and for turbulent flow $C_f = 0.08/(Re)^{0.25}$.

   (b) head loss due to friction is $4C_f V^2L/2gD$ and due to fittings is $KV^2/2g$.

   where K=0.72 for an elbow and K=0.25 for a gate valve.

   What would be a suitable diameter for the delivery pipe ?

4. DIFFERENTIAL PRESSURE DEVICES

Differential pressure devices produce differential pressure as a result of changes in fluid velocity. They have many uses but mainly they are used for flow measurement. In this section you will apply Bernoulli's equation to such devices. You will also briefly examine forces produced by momentum changes.

4.1 GENERAL RELATIONSHIP

Many devices make use of the transition of flow energy into kinetic energy. Consider a flow of liquid which is constrained to flow from one sectional area into a smaller sectional area as shown below.

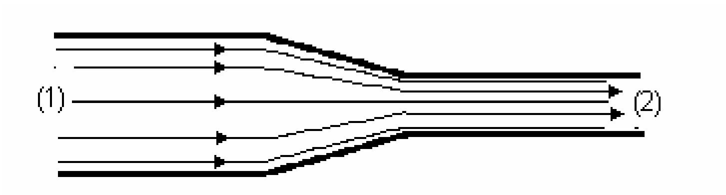


Fig.4.1

The velocity in the smaller bore u_2 is given by the continuity equation as

$$u_2 = u_1A_1/A_2$$

Let $A_1/A_2 = r$ $\qquad u_2 = ru_1$

In BS1042 the symbol used is m but r is used here to avoid confusion with mass.

If we apply Bernoulli (head form) between (1) and (2) and ignoring energy losses we have

$$h_1 + z_1 + \frac{u_1^2}{2g} = h_2 + z_2 + \frac{u_2^2}{2g}$$

For a horizontal system $z_1=z_2$ so

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g}$$

$$2g(h_1 - h_2) = (u_2^2 - u_1^2) = u_1^2(r^2 - 1)$$

$$u_1 = \sqrt{\frac{2g(h_1 - h_2)}{(r^2 - 1)}}$$

$$Vol/s = Q = A_1u_1 = A_1\sqrt{\frac{2g(h_1 - h_2)}{(r^2 - 1)}}$$

In terms of pressure rather than head we get, by substituting $p= \rho gh$

$$Q = A_1\sqrt{\frac{2\Delta p}{\rho(r^2 - 1)}}$$

To find the mass flow remember $m = \rho Au = \rho Q$

Because we did not allow for energy loss, we introduce a coefficient of discharge C_d to correct the answer resulting in

$$Q = C_d A_1 \sqrt{\frac{2\Delta p}{\rho\left(r^2 - 1\right)}}$$

The value of C_d depends upon many factors and is not constant over a wide range of flows. BS1042 should be used to determine suitable values. It will be shown later that if there is a contraction of the jet, the formula needs further modification.

For a given device, if we regard C_d as constant then the equation may be reduced to :

$$Q = K(\Delta p)^{0.5}$$

where K is the meter constant.

4.2 MOMENTUM and PRESSURE FORCES

Changes in velocities mean changes in momentum and Newton's second law tells us that this is accompanied by a force such that

Force = rate of change of momentum.

Pressure changes in the fluid must also be considered as these also produce a force. Translated into a form that helps us solve the force produced on devices such as those considered here, we use the equation $F = \Delta(pA) + m\,\Delta u$.

When dealing with devices that produce a change in direction, such as pipe bends, this has to be considered more carefully and this is covered in chapter 4. In the case of sudden changes in section, we may apply the formula

$$F = (p_1 A_1 + m u_1) - (p_2 A_2 + m u_2)$$

point 1 is upstream and point 2 is downstream.

WORKED EXAMPLE 8

A pipe carrying water experiences a sudden reduction in area as shown. The area at point (1) is 0.002 m^2 and at point (2) it is 0.001 m^2. The pressure at point (2) is 500 kPa and the velocity is 8 m/s. The loss coefficient K is 0.4. The density of water is 1000 kg/m^3. Calculate the following.

i. The mass flow rate.
ii. The pressure at point (1)
iii. The force acting on the section.

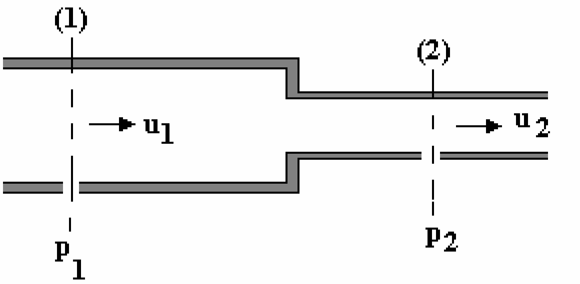


Fig.4.2

SOLUTION

$u_1 = u_2A_2/A_1 = (8 \text{ x } 0.001)/0.002 = 4$ m/s
$m = \rho A_1u_1 = 1000 \text{ x } 0.002 \text{ x } 4 = 8$ kg/s.
$Q = A_1u_1 = 0.002 \text{ x } 4 = 0.008\ m^3/s$
Pressure loss at contraction = $\frac{1}{2}\rho k u_1^2 = \frac{1}{2} \text{ x } 1000 \text{ x } 0.4 \text{ x } 4^2 = 3200$ Pa
Apply Bernoulli between (1) and (2)

$$p_1 + \frac{\rho u_1^2}{2} = p_2 + \frac{\rho u_2^2}{2} + p_L$$

$$p_1 + \frac{1000 \text{ x } 4^2}{2} = 500 \text{ x } 10^3 + \frac{1000 \text{ x } 8^2}{2} + 3200$$

$$p_1 = 527.2 \text{ kPa}$$

$F = (p_1A_1 + mu_1) - (p_2 A_2 + mu_2)$

$F = [(527.2 \text{ x } 10^3 \text{ x } 0.002) + (8 \text{ x } 4)] – [500 \text{ x } 10^3 \text{ x } 0.001) + (8 \text{ x } 8)]$

$F = 1054.4 + 32 – 500 – 64$

$F = 522.4$ N

5. SPECIFIC DEVICES

We will now examine specific d.p. devices starting with an orifice. All these devices appear in BS1042

5.1. ORIFICE METERS

When a liquid flows through an orifice it experiences frictional energy loss and a contraction in the diameter of the jet, both of which affect the value of C_d. The diagram below shows this contraction which is due to the fluid approaching the orifice from radial directions and not along the centre line. This makes the velocity of the jet greater than it would otherwise be because of the reduction in area. In addition to this, there is a 2 or 3 % reduction in velocity due to friction. The value of C_d depends upon the sharpness of the orifice edge. In a sharp edged orifice C_d is typically 0.62 but is slightly larger if the sharp edge is replaced by a square edge.

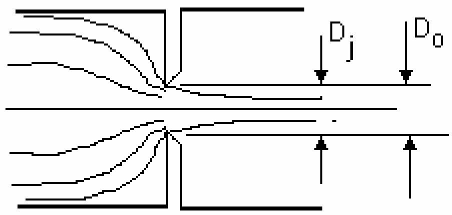


Figure 5.1

5.1.1 COEFFICIENT OF CONTRACTION

The coefficient of contraction is defined as

$$C_c = \text{Area of Jet/Area of Orifice} = A_j/A_o = D_j^2/D_o^2$$

5.1.2 COEFFICIENT OF VELOCITY

The coefficient of velocity is defined as

$$C_v = \text{Actual velocity of jet/theoretical velocity}$$

The theoretical velocity = $(2\Delta p/\rho)^{½}$

It follows that the actual velocity is :

$$u = C_v(2\Delta p/\rho)^{½}$$

5.1.3 COEFFICIENT OF DISCHARGE

The flow rate through the orifice is the product of area and velocity so

$$Q = A_j u = C_c C_v A_o (2\Delta p/\rho)^{½}$$

The product of $C_c C_v$ must be the coefficient of discharge so it follows that

$$C_d = C_c C_v$$

and $$Q = C_d A_o (2\Delta p/\rho)^{½}$$

This formula neglects the approach velocity. The kinetic energy up stream of the orifice is not usually neglected. Let's do the derivation of the flow formula again.

FLOW THROUGH AN ORIFICE

Referring to fig.21, applying Bernoulli between point (1) upstream and the vena-contracta (2) we have

$$p_1 + ½ \rho u_1^2 = p_2 + ½ \rho u_2^2$$
$$p_1 - p_2 = ½ \rho(u_2^2 - u_1^2)$$

$u_1A_1 = u_2A_2$ $\quad$ $u_1 = u_2A_2/A_1 = u_2d_2^2/d_1^2$

$A_2/A_0 = C_c = d_2^2/d_0^2$ $\quad$ $d_2^2 = C_c d_0^2$

$$u_1 = u_2C_cd_0^2/d_1^2 = u_2C_c\beta^2$$

$\beta = d_0/d_1$

$$p_1 - p_2 = \Delta p = \frac{1}{2}\rho u_2^2\left(1 - C_c^2\beta^4\right)$$

$$u_2 = \sqrt{\frac{2\Delta p}{\rho\left(1 - C_c^2\beta^4\right)}}$$

This is the velocity at the vena contracta. If friction is taken into account a coefficient of velocity must be used to correct it.

$$u_2 = C_V\sqrt{\frac{2\Delta p}{\rho\left(1 - C_c^2\beta^4\right)}}$$

$Q = A_2u_2 \quad A_2 = C_cA_0$

$$Q = C_v C_c A_o\sqrt{\frac{2\Delta p}{\rho\left(1 - C_c^2\beta^4\right)}}$$

$$Q = C_d A_o\sqrt{\frac{2\Delta p}{\rho\left(1 - C_c^2\beta^4\right)}}$$

This formula may be rearranged to give the pressure drop if the flow is known.

$$\Delta p = \left(\frac{Q}{C_d A_o}\right)^2\left(1 - C_c^2\beta^4\right)\frac{\rho}{2}$$

The pressure tapping points are normally placed at one pipe diameter upstream and one half pipe diameter downstream in order to get the maximum d.p. However if the maximum value is not important, the d.p. is more easily obtained by the use of corner or flange tappings. The results are still valid but less d.p. is obtained.

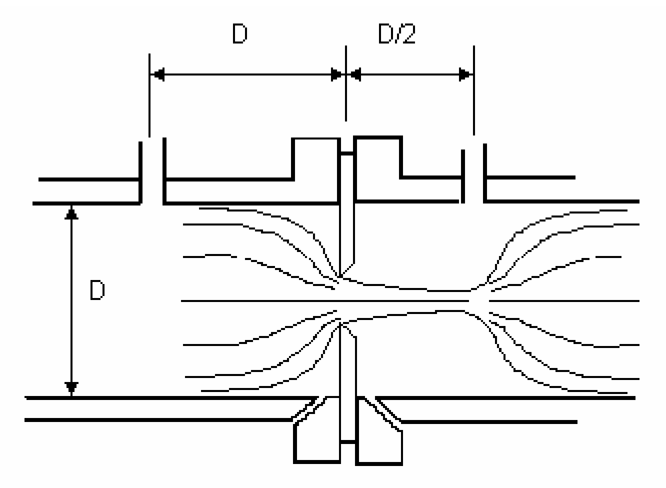


Fig.5.2 showing tapping positions

Figure 5.2 shows how the flow after the orifice must expand to the full bore of the pipe. The velocity in the full bore is less than the jet so the jet must be slowed down. It can only do this by colliding with the slower moving fluid downstream and consequently there is a lot of friction and energy loss in the turbulent mixing taking place. The result is that only a small amount of kinetic energy is reconverted into pressure downstream and the overall pressure loss for the system is high. The loss from the vena contracta (2) to the point downstream where the flow has settled (3) is the loss due to sudden expansion covered earlier and is given by

$$\text{pressure loss due to expansion} = \tfrac{1}{2}\,\rho(u_2 - u_3)^2$$

Further pressure losses are produced by skin friction and could be estimated. The problem is that the mean velocity is uncertain in the areas near the orifice so it is difficult to apply Darcy's formula.

Figure 5.3 shows the way that pressure changes on approach to and departure from the orifice.

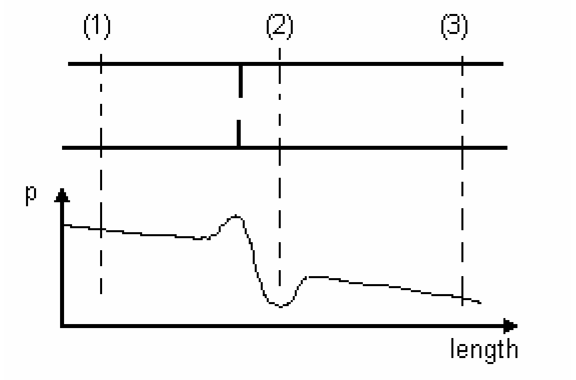


Figure 5.3

WORKED EXAMPLE No.9

The figure shows a sharp edged orifice plate of diameter 20 mm in a horizontal pipe of diameter 25 mm. There are three pressure tappings as follows.

(1) at about 3 pipe diameters upstream of the orifice plate. (2) at half a pipe diameter downstream of the orifice plate and (3) at about 5 pipe diameters downstream of the orifice plate. The tappings read pressures p_1, p_2 and p_3 respectively.

If there is a flow rate of 0.8 x 10^{-3} m^3/s of water at 25^oC, evaluate the pressure differences p_1-p_2 and p_1-p_3. Calculate the % of pressure recovered downstream of the orifice. It may be assumed that the discharge coefficient is 0.64 and the contraction coefficient is 0.74. The density and viscosity for water are usually given on the front of the exam paper. The density is 998 kg/m^3.

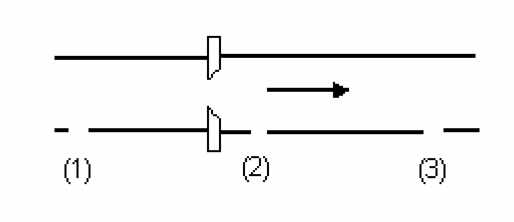


Fig.5.4

SOLUTION

First the pressure drop from 1 to 2. There is friction in the jet so the formula to be used is

$$\Delta p = (Q/C_dA_0)^2(1 - C_c^2\beta^4)\rho/2$$

$$A_0 = p \text{ x } 0.02^2/4 = 0.0003142 \text{ m}^2 \quad b = 20/25 = 0.8$$

$$\Delta p = \{0.0008/(0.64\text{x } 0.0003142)^2\}(1 - 0.74^2\text{x}0.8^4)998/2$$

$$\Delta p = p_1 - p_2 = 6.126 \text{ kPa}$$

This includes the pressure loss due to friction in the jet as well as due to the change in velocity.

$$u_1 = u_3 = 0.0008/(p\text{x}0.025^2/4\ ) = 1.63 \text{ m/s}$$

$$A_2 = C_c \text{ x } p \text{ x } 0.02^2/4 = 0.000232 \text{ m}^2$$

$$u_2 = 0.0008/0.000232 = 3.44 \text{ m/s}$$

loss due to sudden expansion = $\rho(u_2 - u_3)^2/2 = 998(3.44 - 1.63)^2/2 = 1.63$ kPa

Now we must find the pressure loss due to friction in the jet.

Ideal jet velocity = u_2/C_v $\qquad C_v = C_d/C_c = 0.64/0.74 = 0.865$

Ideal jet velocity = 3.44/0.865 = 3.98 m/s

Loss of kinetic energy as pressure = $(\rho/2)(3.98^2 - 3.44^2)$ = 1.99 kPa

$p_1 + \rho u_1^2/2 = p_3 + \rho u_3^2/2$

$p_1 - p_3 = (\rho/2)(u_3^2 - u_1^2)$ + losses

$u_3 = u_3$

$p_1 - p_3$ = losses = 1.63 kPa + 1.99 kPa = 3.62 kPa

The pressure regained downstream = 6.126 - 3.62 = 2.5 kPa

The diffuser efficiency = 2.5/6.126 = 41%

5.2. VENTURI METERS

The Venturi Meter is designed to taper down to the throat gradually and then taper out again. No contraction occurs in the flow so $C_c = 1$. The outlet (diffuser) is designed to expand the flow gradually so that the kinetic energy at the throat is reconverted into pressure with little friction. Consequently the coefficient of discharge is much better than for an orifice meter. The overall pressure loss is much better than for an orifice meter.

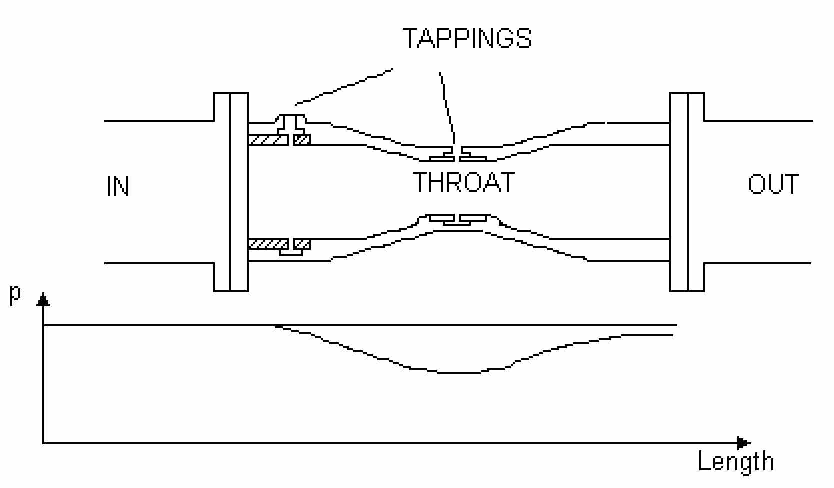


Fig.5.5 showing pressure distribution

If there is no vena-contracta then the flow rate is given by the formula

$$Q = C_d A_1 \sqrt{\frac{2\Delta p}{\rho(r^2 - 1)}}$$

and $C_d = C_v$ and is about 0.97 for a good meter.

The draw back of the Venturi is the expense involved in the design. The pressure tappings have special inserts in the bore to gather the pressure from around the circumference.

5.3 NOZZLE METER

The nozzle meter is a compromise between the orifice and the venturi. It may be easily fitted in a pipe between flanges with flange or corner tappings. There is no contraction of the jet but there is little pressure recovery downstream. The loss due to sudden expansion occurs down stream. The flow formula is the same as before.

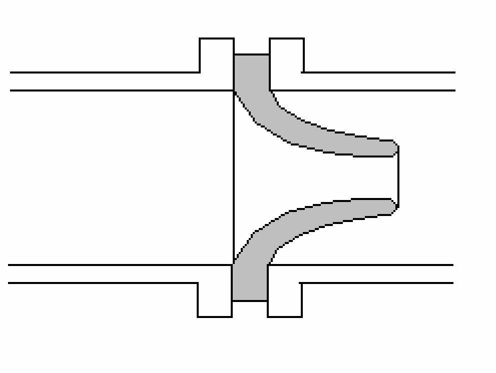


Fig.5.6 Nozzle Meter

WORKED EXAMPLE No.10

A nozzle is 100 mm diameter at inlet and 20 mm diameter at outlet. The coefficient of velocity is 0.97 and there is no contraction of the jet. The jet discharges into the atmosphere. The static pressure at inlet is 300 kPa gauge. The density is 1000 kg/m^3.

Calculate:

a. the velocity at exit.

b. the flow rate.

c. the pressure loss due to friction expressed as a fraction of the dynamic pressure at outlet.

d. the force on the nozzle.

SOLUTION

The velocity at exit when the inlet velocity is not negligible is
$Q = A_1C_d[(2\Delta p/\rho)/(r^2 - 1)]^{0.5}$

$r = A_1/A_2 = d_1^2/d_2^2 = (100/20)^2 = 25$

$C_d = C_v\, C_c = 0.97 \times 1 = 0.97$
$A_1 = (p \times 0.1^2)/4 = 0.00785\ m^2$

hence $Q = 0.97 \times 0.00785\ [(2 \times 300 \times 10^3/1000)/(25^2 - 1)]^{0.5}$
$Q = 0.00747\ m^3/s$

The velocity at inlet = $Q/A_1 = 0.00747/0.00785 = 0.951$ m/s
The velocity at outlet = $Q/A_2 = 0.00747 \times 4/(p \times 0.02^2) = 23.8$ m/s

The dynamic pressure of the jet is $\rho u_2^2/2 = 1000 \times 23.8^2/2 = 282.7$ kPa.

Applying Bernoulli between the inlet (1) and outlet (2) using the pressure form we have
$p_1 - p_2 = \rho u_2^2/2 - \rho u_1^2/2$ + pressure loss to friction
$3 \times 10^5 = (1000/2)(23.8^2 - 0.951^2)$ + pressure loss
$3 \times 10^5 = 2.827 \times 10^5$ + pressure loss

pressure loss = 17.3 kPa

Expressed as a fraction of the dynamic pressure of the jet this is 17.3/ 282.7 or 6.1%.

The force exerted on the water is given by
$F = p_1A_1 + - p_2\, A_2 + mu_1 - mu_2$

We must use gauge pressures to solve this problem because the atmosphere acts on the outer surface of the nozzle. The mass flow is 7.47 kg/s.

$F = 300 \times 10^3 \times 0.00785 - 0 + 7.47(0.951 - 23.8) = 2.18$ kN

The figure is positive which indicates the force is accelerating the water out of the nozzle. The force on the nozzle is the reaction to this and is opposite in direction. Think of a fireman's hose. The force on the nozzle pushes it away from the water like a rocket. The force to accelerate the water must be supplied by those holding it.

6. JET PUMPS

Jet pumps are devices that suck up liquid by the use of a jet discharging into an annular area as shown.

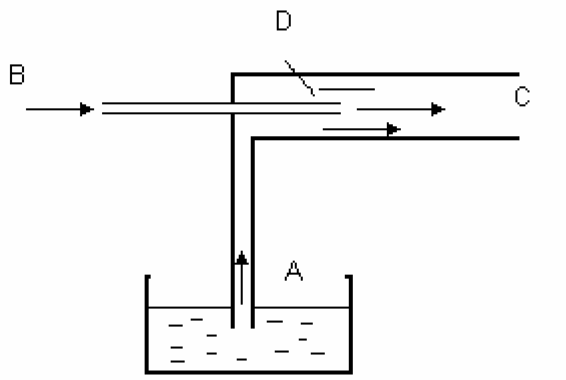


Fig.6.1 A Typical jet Pump.

The solution of jet pump problems requires the use of momentum as well as energy considerations. First apply Bernoulli between A and D and assume no frictional losses. Note that D is a annular area and $u_D = 4Q/\{p(d_1{}^2 - d_2{}^2)\}$ where d_1 is the diameter of the large pipe and d_2 the diameter of the small pipe.

$$h_A + \frac{u_A^2}{2g} + z_A = h_D + \frac{u_D^2}{2g} + z_D$$

Making A the datum and using gauge pressures we find $h_A = 0$ $u_A = 0$ $z_A = 0$

$$0 = h_D + \frac{u_D^2}{2g} + z_D$$

$$h_D = -\frac{u_D^2}{2g} - z_D$$

From this the head at the point where pipes B and D meet is found.

Next apply the conservation of momentum between the points where B and D join and the exit at C.

$$p_B A_B + \rho Q_B u_B + p_D A_D + \rho Q_D u_D = p_C A_C + \rho Q_C u_C$$

but $p_C = 0$ gauge and $p_B = p_D = p_{(BD)}$ so

$$p_{(BD)} A_{(BD)} + \rho Q_B u_B + \rho Q_D u_D = \rho Q_C u_C$$

where (BD) refers to the area of the large pipe and is the same as A_C.

Next apply conservation of mass $\rho Q_B + \rho Q_D = \rho Q_C$ $Q_B + Q_D = Q_C$

With these equations it is possible to solve the velocity and flow rate in pipe B. The resulting equation is:

$$Q_B^2\left\{\frac{1}{A_B}-\frac{1}{A_C}\right\}-\frac{2Q_BQ_D}{A_C}+\frac{p_BA_C}{\rho}+Q_D^2\left\{\frac{1}{A_D}-\frac{1}{A_C}\right\}=0$$

$aQ_B^2+bQ_B+c=0$ This is a quadratic equation whence

$$Q_B=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

SELF ASSESSMENT EXERCISE 3

Take the density of water to be 997 kg/m^3 throughout unless otherwise stated.

1. A Venturi meter is 50 mm bore diameter at inlet and 10 mm bore diameter at the throat. Oil of density 900 kg/m^3 flows through it and a differential pressure head of 80 mm is produced. Given C_d = 0.92, determine the flow rate in kg/s.
(ans. 0.0815 kg/s).

2. A Venturi meter is 60 mm bore diameter at inlet and 20 mm bore diameter at the throat. Water of density 1000 kg/m^3 flows through it and a differential pressure head of 150 mm is produced. Given C_d = 0.95, determine the flow rate in dm^3/s.
(ans. 0.515 dm^3/s).

3. Calculate the differential pressure expected from a Venturi meter when the flow rate is 2 dm^3/s of water. The area ratio is 4 and C_d is 0.94. The inlet c.s.a. is 900 mm^2. (ans. 41.916 kPa).

4. Calculate the mass flow rate of water through a Venturi meter when the differential pressure is 980 Pa given C_d = 0.93, the area ratio is 5 and the inlet c.s.a. is 1000 mm^2. (ans. 0.266 kg/s).

5. Calculate the flow rate of water through an orifice meter with an area ratio of 4 given C_d is 0.62, the pipe area is 900 mm^2 and the d.p. is 586 Pa. (ans. 0.156 dm^3/3).

6. Water flows at a mass flow rate 0f 0.8 kg/s through a pipe of diameter 30 mm fitted with a 15 mm diameter .sharp edged orifice.

   There are pressure tappings (a) 60 mm upstream of the orifice, (b) 15 mm downstream of the orifice and (c) 150 mm downstream of the orifice, recording pressure p_a, p_b and p_c respectively. Assuming a contraction coefficient 0f 0.68, evaluate

   (i) the pressure difference (p_a - p_b) and hence the discharge coefficient.
   (21.6 kPa, 0.67)

   (ii)the pressure difference (p_b - p_c) and hence the diffuser efficiency.
   (-6.4 kPa, 29.5%)

   (iii) the net force on the orifice plate.
   (10.8 N)

   State any assumption made in your analysis.

7. The figure shows an ejector (or jet pump) which extracts 2 x 10^{-3} m^3/s of water from tank A which is situated 2.0 m below the centre-line of the ejector. The diameter of the outer pipe of the ejector is 40 mm and water is supplied from a reservoir to the thin-walled inner pipe which is of diameter 20 mm. The ejector discharges to atmosphere at section C.

   Evaluate the pressure p at section D, just downstream of the end of pipe B, the velocity in pipe B and the required height of the free water level in the reservoir supplying pipe B. (-21.8 kPa gauge, 12.922 m/s, 6.28 m).

   It may be assumed that both supply pipes are loss free.

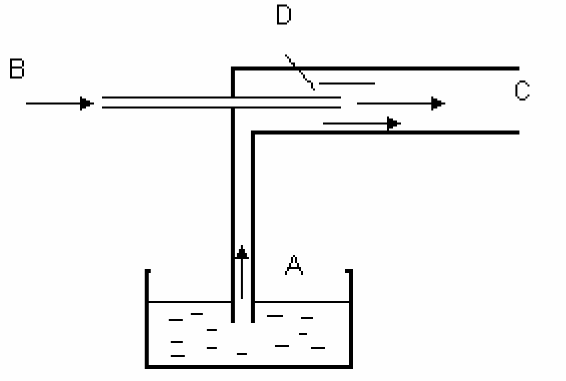


Figure 6.2

8. Discuss the use of orifice plates and venturi-meters for the measurement of flow rates in pipes.

   Water flows with a mean velocity of 0.6 m/s in a 50 mm diameter pipe fitted with a sharp edged orifice of diameter 30 mm. Assuming the contraction coefficient is 0.64, find the pressure difference between tappings at the vena contracta and a few diameters upstream of the orifice, and hence evaluate the discharge coefficient.
   Estimate also the overall pressure loss caused by the orifice plate.
   It may be assumed that there is no loss of energy upstream of the vena contracta.

9. Fig.28 shows an ejector pump BDC designed to lift 2 x 10^{-3} m^3/s of water from an open tank A, 3.0 m below the level of the centre-line of the pump. The pump discharges to atmosphere at C.

   The diameter of thin-walled inner pipe 12 mm and the internal diameter of the outer pipe of the is 25 mm. Assuming that there is no energy loss in pipe AD and there is no shear stress on the wall of pipe DC, calculate the pressure at point D and the required velocity of the water in pipe BD.
   (-43.3 kPa and 20.947 m/s)

   Derive all the equations used and state your assumptions.

FLUID MECHANICS
TUTORIAL No. 3

BOUNDARY LAYER THEORY

In order to complete this tutorial you should already have completed tutorial 1 and 2 in this series. This tutorial examines boundary layer theory in some depth.

When you have completed this tutorial, you should be able to do the following.

- Discuss the drag on bluff objects including long cylinders and spheres.
- Explain skin drag and form drag.
- Discuss the formation of wakes.
- Explain the concept of momentum thickness and displacement thickness.
- Solve problems involving laminar and turbulent boundary layers.

Throughout there are worked examples, assignments and typical exam questions. You should complete each assignment in order so that you progress from one level of knowledge to another.

Let us start by examining how drag is created on objects.

1. DRAG

When a fluid flows around the outside of a body, it produces a force that tends to drag the body in the direction of the flow. The drag acting on a moving object such as a ship or an aeroplane must be overcome by the propulsion system. Drag takes two forms, skin friction drag and form drag.

1.1 SKIN FRICTION DRAG

Skin friction drag is due to the viscous shearing that takes place between the surface and the layer of fluid immediately above it. This occurs on surfaces of objects that are long in the direction of flow compared to their height. Such bodies are called ***STREAMLINED***. When a fluid flows over a solid surface, the layer next to the surface may become attached to it (it wets the surface). This is called the 'no slip condition'. The layers of fluid above the surface are moving so there must be shearing taking place between the layers of the fluid. The shear stress acting between the wall and the first moving layer next to it is called ***the wall shear stress*** and denoted τ_w.

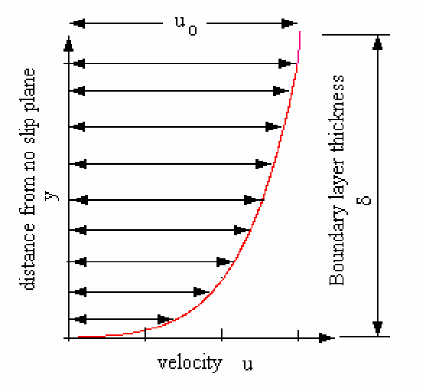


Fig.1.1

The result is that the velocity of the fluid u increases with height y. The boundary layer thickness δ is taken as the distance required for the velocity to reach 99% of u_o. This layer is called the ***BOUNDARY LAYER*** and δ is the boundary layer thickness. Fig. 1.1 Shows how the velocity "u" varies with height "y" for a typical boundary layer.

In a pipe, this is the only form of drag and it results in a pressure and energy lost along the length. A thin flat plate is an example of a streamlined object. Consider a stream of fluid flowing with a uniform velocity u_o. When the stream is interrupted by the plate (fig. 1.2), the boundary layer forms on both sides. The diagram shows what happens on one side only.

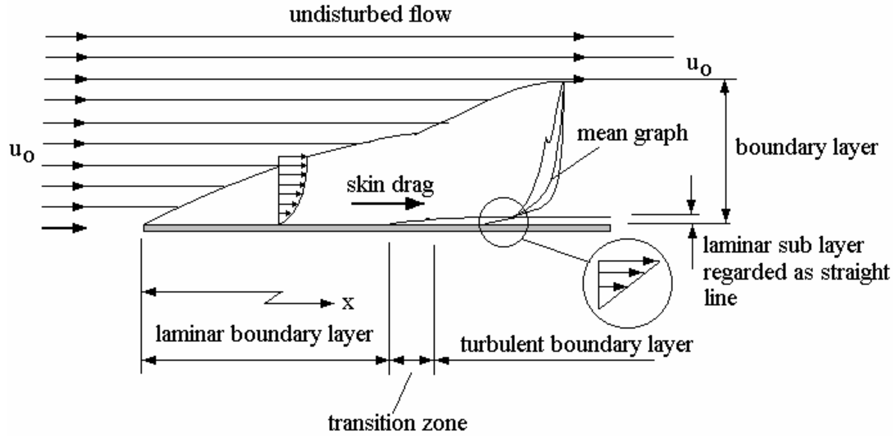


Fig. 1.2

The boundary layer thickness δ grows with distance from the leading edge. At some distance from the leading edge, it reaches a constant thickness. It is then called a ***FULLY DEVELOPED BOUNDARY LAYER***.

The Reynolds number for these cases is defined as: $(R_e)_x = \dfrac{\rho u_o x}{\mu}$

x is the distance from the leading edge. At low Reynolds numbers, the boundary layer may be laminar throughout the entire thickness. At higher Reynolds numbers, it is turbulent. This means that at some distance from the leading edge the flow within the boundary layer becomes turbulent. A turbulent boundary layer is very unsteady and the streamlines do not remain parallel. The boundary layer shape represents an average of the velocity at any height. There is a region between the laminar and turbulent section where transition takes place

The turbulent boundary layer exists on top of a thin laminar layer called the ***LAMINAR SUB LAYER***. The velocity gradient within this layer is linear as shown. A deeper analysis would reveal that for long surfaces, the boundary layer is turbulent over most of the length. Many equations have been developed to describe the shape of the laminar and turbulent boundary layers and these may be used to estimate the skin friction drag.

Note that for this ideal example, it is assumed that the velocity is the undisturbed velocity u_o everywhere outside the boundary layer and that there is no acceleration and hence no change in the static pressure acting on the surface. There is hence no drag force due to pressure changes.

CALCULATING SKIN DRAG

The skin drag is due to the wall shear stress τ_w and this acts on the surface area (wetted area). The drag force is hence: **R = τ_w x wetted area.** The dynamic pressure is the pressure resulting from the conversion of the kinetic energy of the stream into pressure and is defined by the expression $\dfrac{\rho u_o^2}{2}$ .The drag coefficient is defined as

$$C_{Df} = \frac{\text{Drag force}}{\text{dynamic pressure x wetted area}}$$

$$C_{Df} = \frac{2R}{\rho u_0^2 \text{ x wetted area}} = \frac{2\tau_w}{\rho u_0^2}$$

Note that this is the same definition for the pipe friction coefficient C_f and it is in fact the same thing. It is used in the Darcy formula to calculate the pressure lost in pipes due to friction. For a smooth surface, it can be shown that $C_{Df} = 0.074\ (R_e)_x^{-1/5}$

$(Re)_x$ is the Reynolds number based on the length. $(R_e)_x = \dfrac{\rho u_o L}{\mu}$

WORKED EXAMPLE 1.1

Calculate the drag force on each side of a thin smooth plate 2 m long and 1 m wide with the length parallel to a flow of fluid moving at 30 m/s. The density of the fluid is 800 kg/m^3 and the dynamic viscosity is 8 cP.

SOLUTION

$$(R_e)_x = \frac{\rho u_o L}{\mu} = \frac{800 \text{ x } 30 \text{ x } 2}{0.008} = 6 \text{ x } 10^6$$

$$C_{Df} = 0.074 \text{ x } (6 \text{ x} 10^6)^{-\frac{1}{5}} = 0.00326$$

$$\text{Dynamic pressure} = \frac{\rho u_0^2}{2} = \frac{800 \text{ x } 30^2}{2} = 360 \text{ kPa}$$

$$\tau_w = C_{Df} \text{ x dynamic pressure} = 0.00326 \text{ x } 360 \text{ x } 10^3 = 1173.6 \text{ Pa}$$

$$R = \tau_w \text{ x Wetted Area} = 1173.6 \text{ x } 2 \text{ x } 1 = 2347.2 \text{ N}$$

On a small area the drag is $dR = \tau_w\, dA$. If the body is not a thin plate and has an area inclined at an angle θ to the flow direction, the drag force in the direction of flow is $\tau_w\, dA \cos\theta$.

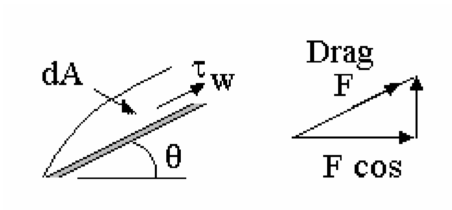


Fig.1.3

The drag force acting on the entire surface area is found by integrating over the entire area.

$$R = \oint \tau_w \cos\theta \, dA$$

Solving this equation requires more advanced studies concerning the boundary layer and students should refer to the classic textbooks on this subject.

SELF ASSESSMENT EXERCISE No. 1

1. A smooth thin plate 5 m long and 1 m wide is placed in an air stream moving at 3 m/s with its length parallel with the flow. Calculate the drag force on each side of the plate. The density of the air is 1.2 kg/m^3 and the kinematic viscosity is 1.6 x 10^{-5} m^2/s.
(0.128 N)

2. A pipe bore diameter D and length L has fully developed laminar flow throughout the entire length with a centre line velocity u_o. Given that the drag coefficient is given as C_{Df} = 16/Re where $Re = \frac{\rho u_o D}{\mu}$, show that the drag force on the inside of the pipe is given as $R=8\pi\mu u_o L$ and hence the pressure loss in the pipe due to skin friction is
$p_L = 32\mu u_o L/D^2$

1.2 FORM DRAG and WAKES

Form or pressure drag applies to bodies that are tall in comparison to the length in the direction of flow. Such bodies are called ***BLUFF BODIES***.

Consider the case below that could for example, be the pier of a bridge in a river. The water speeds up around the leading edges and the boundary layer quickly breaks away from the surface. Water is sucked in from behind the pier in the opposite direction. The total effect is to produce eddy currents or whirl pools that are shed in the wake. There is a build up of positive pressure on the front and a negative pressure at the back. The pressure force resulting is the form drag. When the breakaway or separation point is at the front corner, the drag is almost entirely due to this effect but if the separation point moves along the side towards the back, then a boundary layer forms and skin friction drag is also produced. In reality, the drag is always a combination of skin friction and form drag. The degree of each depends upon the shape of the body.

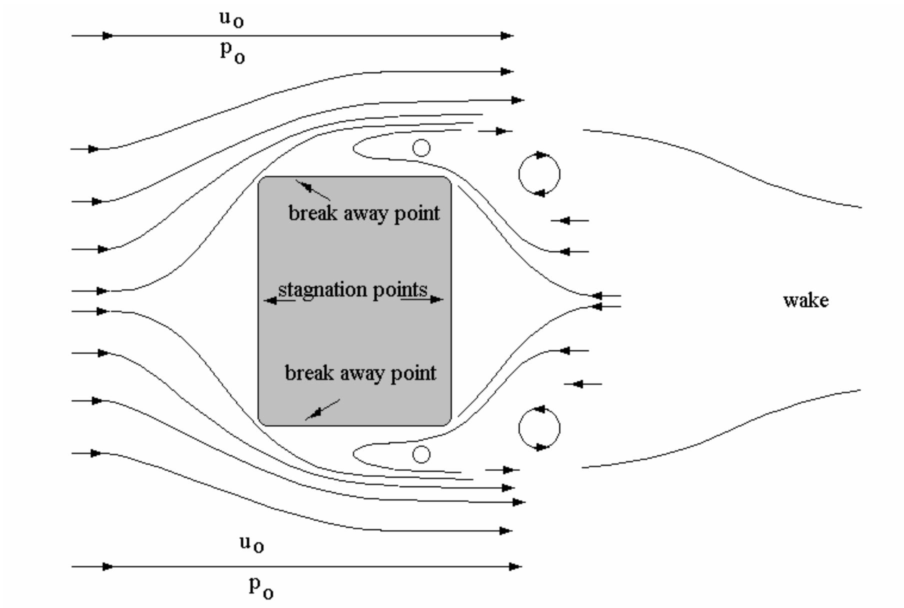


Fig.1.4

The next diagram typifies what happens when fluid flows around a bluff object. The fluid speeds up around the front edge. Remember that the closer the streamlines, the faster the velocity. The line representing the maximum velocity is shown but also remember that this is the maximum at any point and this maximum value also increases as the stream lines get closer together.

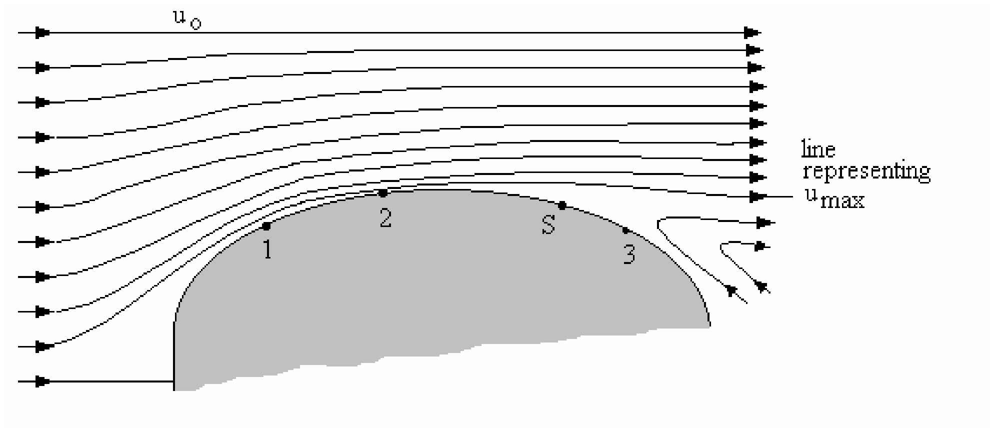


Fig.1.5

Two important effects affect the drag.

Outside the boundary layer, the velocity increases up to point 2 so the pressure acting on the surface goes down. The boundary layer thickness δ gets smaller until at point S it is reduced to zero and the flow separates from the surface. At point 3, the pressure is negative. ***This change in pressure is responsible for the form drag.***

Inside the boundary layer, the velocity is reduced from u_{max} to zero and skin friction drag results.

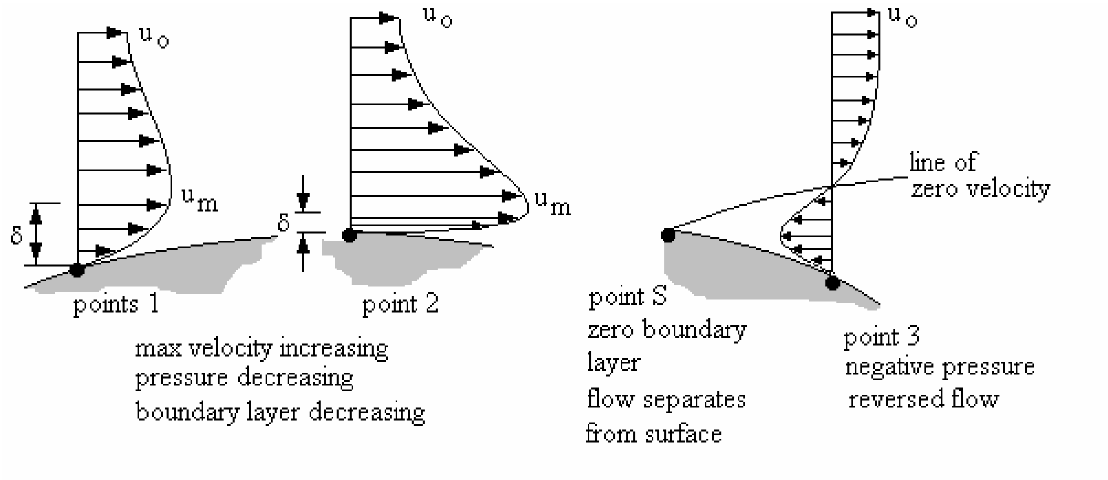


Fig.1.6

In problems involving liquids with a free surface, a negative pressure shows up as a drop in level and the pressure build up on the front shows as a rise in level. If the object is totally immersed, the pressure on the front rises and a vacuum is formed at the back. This results in a pressure force opposing movement (form drag). The swirling flow forms ***vortices*** and the wake is an area of great turbulence behind the object that takes some distance to settle down and revert to the normal flow condition.

Here is an outline of the mathematical approach needed to solve the form drag.

Form drag is due to pressure changes only. The drag coefficient due to pressure only is denoted C_{Dp} and defined as

$$C_{Dp} = \frac{\text{Drag force}}{\text{dynamic pressure x projected area}}$$

$$C_{Dp} = \frac{2R}{\rho u_0^2 \text{ x projected area}}$$

The projected area is the area of the outline of the shape projected at right angles to the flow. The pressure acting at any point on the surface is p. The force exerted by the pressure on a small surface area is p dA. If the surface is inclined at an angle θ to the general direction of flow, the force is p cosθ dA. The total force is found by integrating all over the surface.

$$R = \oint p \cos\theta \, dA$$

The pressure distribution over the surface is often expressed in the form of a pressure coefficient defined as follows.

$$C_p = \frac{2(p - p_o)}{\rho u_o^2}$$

p_o is the static pressure of the undisturbed fluid, u_o is the velocity of the undisturbed fluid and $\frac{\rho u_o^2}{2}$ is the dynamic pressure of the stream.

Consider any streamline that is affected by the surface. Applying Bernoulli between an undisturbed point and another point on the surface, we have the following.

$$p_0 + \frac{\rho u_o^2}{2} = p + \frac{\rho u^2}{2}$$

$$p - p_o = \frac{\rho}{2}(u_o^2 - u^2)$$

$$C_p = \frac{2(p - p_o)}{\rho u_o^2} = \frac{2\left(\frac{\rho}{2}(u_o^2 - u^2)\right)}{\rho u_o^2} = \frac{(u_o^2 - u^2)}{u_o^2} = 1 - \frac{u^2}{u_o^2}$$

In order to calculate the drag force, further knowledge about the velocity distribution over the object would be needed and students are again recommended to study the classic textbooks on this subject. The equation shows that if $u<u_o$ then the pressure is positive and if $u>u_o$ the pressure is negative.

1.3 <u>TOTAL DRAG</u>

It has been explained that a body usually experiences both skin friction drag and form drag. The total drag is the sum of both. This applies to aeroplanes and ships as well as bluff objects such as cylinders and spheres. The drag force on a body is very hard to predict by purely theoretical methods. Much of the data about drag forces is based on experimental data and the concept of a drag coefficient is widely used.

The ***DRAG COEFFICIENT*** is denoted $\mathbf{C_D}$ and is defined by the following expression.

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}}$$

$$C_D = \frac{2R}{\rho u_o^2 \text{ x projected Area}}$$

WORKED EXAMPLE 1.2

A cylinder 80 mm diameter and 200 mm long is placed in a stream of fluid flowing at 0.5 m/s. The axis of the cylinder is normal to the direction of flow. The density of the fluid is 800 kg/m^3. The drag force is measured and found to be 30 N.

Calculate the drag coefficient.

At a point on the surface the pressure is measured as 96 Pa above the ambient level.

Calculate the velocity at this point.

SOLUTION

Projected area = 0.08 x 0.2 =0.016 m^2
R = 30 N
u_o = 0.5 m/s
ρ = 800 kg/m^3
dynamic pressure = $\rho u_o^2/2 = 800 \times 0.5^2/2 = 100$ Pa

$$C_D = \frac{\text{Resistance force}}{\text{Dynamic pressure x projected Area}} = \frac{30}{100 \times 0.016} = 18.75$$

$$p - p_o = \frac{\rho}{2}(u_o^2 - u^2)$$

$$96 = \frac{800}{2}(0.5^2 - u^2)$$

$$\frac{96 \times 2}{800} = (0.5^2 - u^2)$$

$$0.24 = 0.25 - u^2$$

$$u^2 = 0.01$$

$$u = 0.1 \text{ m/s}$$

1.4 APPLICATION TO A CYLINDER

The drag coefficient is defined as : $C_D = \dfrac{2R}{\rho u_o^2 \text{ x projected Area}}$ The projected Area is LD where L is the length and D the diameter. The drag around long cylinders is more predictable than for short cylinders and the following applies to long cylinders. Much research has been carried out into the relationship between drag and Reynolds number. $Re = \dfrac{\rho u_o d}{\mu}$ and d is the diameter of the cylinder. At very small velocities, (Re <0.5) the fluid sticks to the cylinder all the way round and never separates from the cylinder. This produces a streamline pattern similar to that of an ideal fluid. The drag coefficient is at its highest and is mainly due to skin friction. The pressure distribution shows that the dynamic pressure is achieved at the front stagnation point and vacuum equal to three dynamic pressures exists at the top and bottom where the velocity is at its greatest.

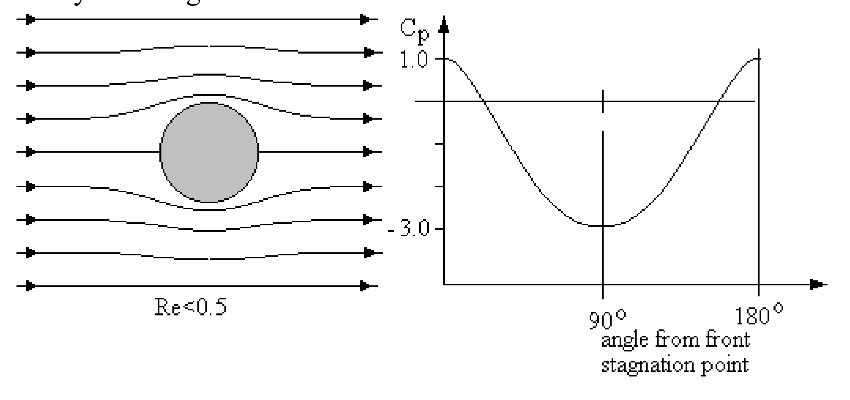


Fig.1.7

As the velocity increases the boundary layer breaks away and eddies are formed behind. The drag becomes increasingly due to the pressure build up at the front and pressure drop at the back.

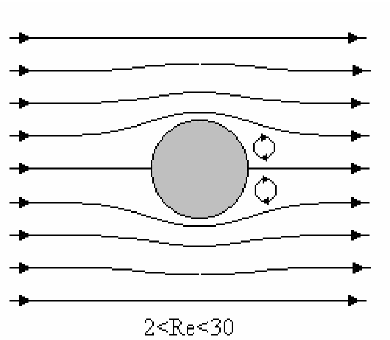


Fig.1.8

Further increases in the velocity cause the eddies to elongate and the drag coefficient becomes nearly constant. The pressure distribution shows that ambient pressure exists at the rear of the cylinder.

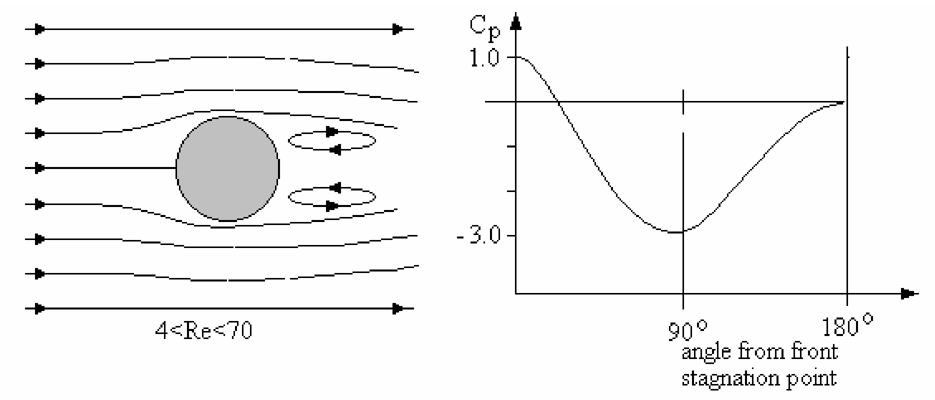


Fig.1.9

At a Reynolds number of around 90 the vortices break away alternatively from the top and bottom of the cylinder producing a vortex street in the wake. The pressure distribution shows a vacuum at the rear.

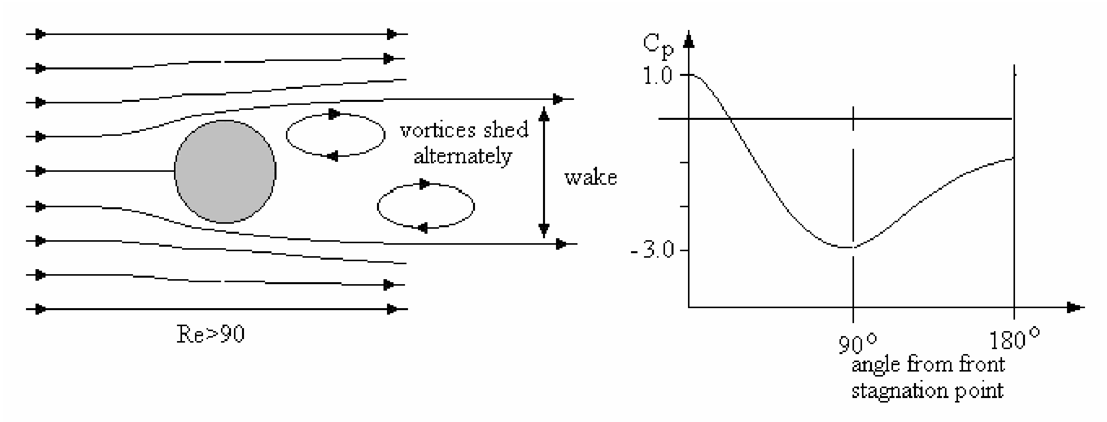


Fig.1.10

Up to a Reynolds number of about 2 x 10^5, the drag coefficient is constant with a value of approximately 1. The drag is now almost entirely due to pressure. Up to this velocity, the boundary layer has remained laminar but at higher velocities, flow within the boundary layer becomes turbulent. The point of separation moves back producing a narrow wake and a pronounced drop in the drag coefficient.

When the wake contains vortices shed alternately from the top and bottom, they produce alternating forces on the structure. If the structure resonates with the frequency of the vortex shedding, it may oscillate and produce catastrophic damage. This is a problem with tall chimneys and suspension bridges. The vortex shedding may produce audible sound.

Fig. 1.12 shows an approximate relationship between C_D and R_e for a cylinder and a sphere.

SELF ASSESSMENT EXERCISE No. 2

1. Calculate the drag force for a cylindrical chimney 0.9 m diameter and 50 m tall in a wind blowing at 30 m/s given that the drag coefficient is 0.8.
   The density of the air is 1.2 kg/m^3. (19.44 N)

2 Using the graph (fig.1.12) to find the drag coefficient, determine the drag force per metre length acting on an overhead power line 30 mm diameter when the wind blows at 8 m/s. The density of air may be taken as 1.25 kg/m^3 and the kinematic viscosity as 1.5 x 10^{-5} m^2/s. (1.8 N).

1.5 APPLICATION TO SPHERES

The relationship between drag and Reynolds number is roughly the same as for a cylinder but it is more predictable. The Reynolds number is $Re = \frac{\rho u_o d}{\mu}$ where d is the diameter of the sphere. The projected area of a sphere of diameter d is ¼ πd^2. In this case, the expression for the drag coefficient is as follows. $C_D = \frac{8R}{\rho u^2 \text{ x } \pi d^2}$.

At very small Reynolds numbers (less than 0.2) the flow stays attached to the sphere all the way around and this is called Stokes flow. The drag is at its highest in this region.

As the velocity increases, the boundary layer separates at the rear stagnation point and moves forward. A toroidal vortex is formed. For $0.2 < Re < 500$ the flow is called Allen flow.

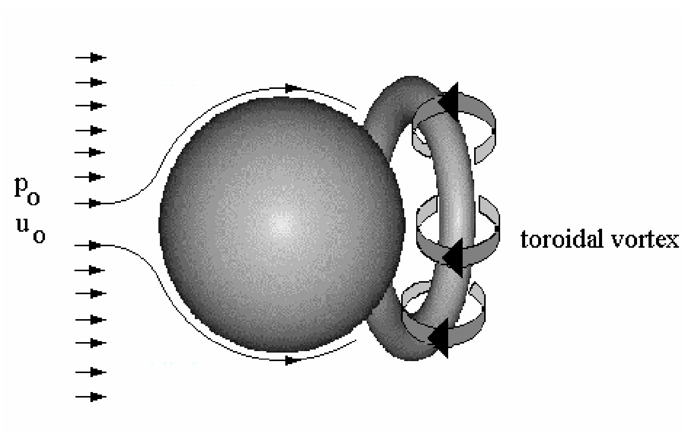


Fig.1.11

The breakaway or separation point reaches a stable position approximately 80^o from the front stagnation point and this happens when R_e is about 1000. For $500 < R_e$ the flow is called Newton flow. The drag coefficient remains constant at about 0.4. Depending on various factors, when R_e reaches 10^5 or larger, the boundary layer becomes totally turbulent and the separation point moves back again forming a smaller wake and a sudden drop in the drag coefficient to about 0.3. There are two empirical formulae in common use.

For $0.2 < R_e < 10^5$ $\quad C_D = \frac{24}{R_e} + \frac{6}{1 + \sqrt{R_e}} + 0.4$

For $R_e < 1000$ $$C_D = \frac{24}{R_e}\left[1 + 0.15\,\mathrm{Re}^{0.687}\right]$$

Fig. 1.12 shows this approximate relationship between C_D and Re.

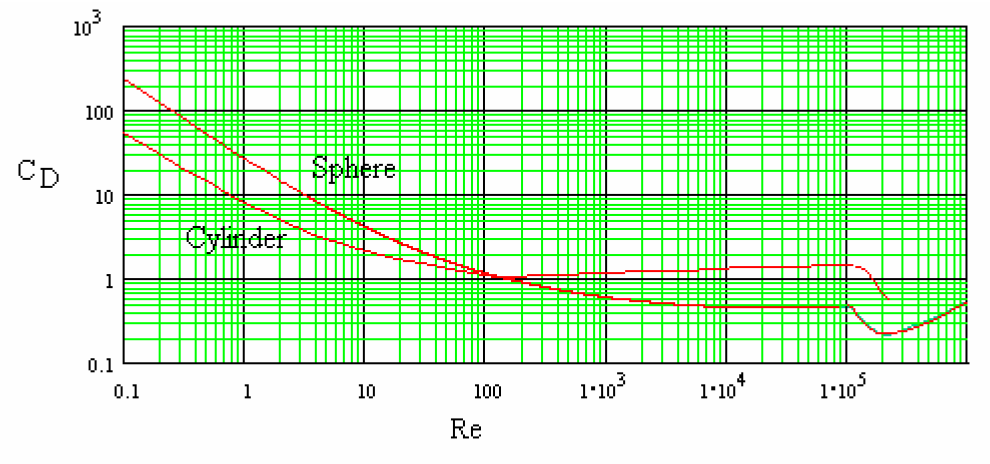


Fig.1.12

WORKED EXAMPLE 1.3

A sphere diameter 40 mm moves through a fluid of density 750 kg/m^3 and dynamic viscosity 50 cP with a velocity of 0.6 m/s. Note 1 cP = 0.001 Ns/m^2.

Calculate the drag on the sphere.

Use the graph to obtain the drag coefficient.

SOLUTION

$$\mathrm{Re} = \frac{\rho u d}{\mu} = \frac{750 \times 0.6 \times 0.04}{0.05} = 360$$

from the graph $C_D = 0.8$

$$C_D = \frac{2R}{\rho u^2 \text{ x projected Area}} \qquad \text{Projected area} = \pi\frac{d^2}{4} = \pi\frac{0.04^2}{4} = 1.2566 \times 10^{-3}\,\mathrm{m}^2$$

$$R = \frac{C_D \rho u^2 A}{2} = \frac{0.8 \times 750 \times 0.6^2 \times 1.2566 \times 10^{-3}}{2} = 0.136\ \mathrm{N}$$

1.6 TERMINAL VELOCITY

When a body falls under the action of gravity, a point is reached, where the drag force is equal and opposite to the force of gravity. When this condition is reached, the body stops accelerating and the terminal velocity reached. Small particles settling in a liquid are usually modelled as small spheres and the preceding work may be used to calculate the terminal velocity of small bodies settling in a liquid. A good application of this is the falling sphere viscometer described in earlier work.

For a body immersed in a liquid, the buoyant weight is W and this is equal to the viscous resistance R when the terminal velocity is reached.

R = W = volume x gravity x density difference $= \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

ρ_s = density of the sphere material
ρ_f = density of fluid
d = sphere diameter

STOKES' FLOW

For $R_e<0.2$ the flow is called Stokes flow and Stokes showed that $R = 3\pi d\mu u_t$
For a falling sphere viscometer, Stokes flow applies. Equating the drag force and the buoyant weight we get

$$3\pi d\mu u_t = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

$$\mu = \frac{d^2 g(\rho_s - \rho_f)}{18 u_t}$$

The terminal velocity for Stokes flow is $u_t = \dfrac{d^2 g(\rho_s - \rho_f)}{18\mu}$

This formula assumes a fluid of infinite width but in a falling sphere viscometer, the liquid is squeezed between the sphere and the tube walls and additional viscous resistance is produced. The Faxen correction factor F is used to correct the result.

WORKED EXAMPLE 1.4

The terminal velocity of a steel sphere falling in a liquid is 0.03 m/s. The sphere is 1 mm diameter and the density of the steel is 7830 kg/m^3. The density of the liquid is 800 kg/m^3. Calculate the dynamic and kinematic viscosity of the liquid.

SOLUTION

Assuming Stokes' flow the viscosity is found from the following equation.

$$\mu = \frac{d^2 g(\rho_s - \rho_f)}{18 u_t} = \frac{0.001^2 \text{ x } 9.81 \text{ x } (7830 - 800)}{18 \text{ x } 0.03} = 0.1277 \text{ Ns/m}^2 = 127.7 \text{ cP}$$

$$\nu = \frac{\mu}{\rho_s} = \frac{0.1277}{800} = 0.0001596 \text{ m}^2/\text{s} = 159.6 \text{ cSt}$$

Check the Reynolds number. $R_e = \dfrac{\rho_f u d}{\mu} = \dfrac{800 \text{ x } 0.03 \text{ x } 0.001}{0.0547} = 0.188$

As this is smaller than 0.2 the assumption of Stokes' flow is correct.

ALLEN FLOW

For $0.2 < R_e < 500$ the flow is called Allen flow and the following expression gives the empirical relationship between drag and Reynolds number. $\mathbf{C_D=18.5R_e^{-0.6}}$

Equating for C_D gives the following result. $C_D = \dfrac{8R}{\rho_f u_t^2 \ \pi d^2} = 18.5R_e^{-0.6}$

Substitute $R = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

$$C_D = \frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 18.5R_e^{-0.6} = 18.5\left(\frac{\rho_f u_t d}{\mu}\right)^{-0.6}$$

$$\frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 18.5\left(\frac{\rho_f u_t d}{\mu}\right)^{-0.6}$$

From this equation the velocity u_t may be found.

NEWTON FLOW

For $500 < R_e < 10^5$ C_D takes on a constant value of 0.44.

Equating for C_D gives the following. $C_D = \dfrac{8R}{\rho_f u_t^2 \ \pi d^2} = 0.44$

Substitute $R = \dfrac{\pi d^3 g(\rho_s - \rho_f)}{6}$

$$\frac{8dg(\rho_s - \rho_f)}{6\rho_f u_t^2} = 0.44$$

$$u_t = \sqrt{\frac{29.73dg(\rho_s - \rho_f)}{\rho_f}}$$

When solving the terminal velocity, you should always check the value of the Reynolds number to see if the criterion used is valid.

WORKED EXAMPLE 1.5

Small glass spheres are suspended in an up wards flow of water moving with a mean velocity of 1 m/s. Calculate the diameter of the spheres. The density of glass is 2630 kg/m^3. The density of water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.

SOLUTION

First, try the Newton flow equation. This is the easiest.

$$u_t = \sqrt{\frac{29.73d\,g(\rho_s - \rho_f)}{\rho_f}}$$

$$d = \frac{u_t^2 \rho_f}{29.73\,g(\rho_s - \rho_f)} = \frac{1^2 \text{ x } 1000}{29.73 \text{ x } 9.81 \text{ x } (2630 - 1000)} = 0.0021\,\text{m or } 2.1\,\text{mm}$$

Check the Reynolds number.

$$R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 1 \text{ x } 0.0021}{0.001} = 2103$$

The assumption of Newton flow was correct so the answer is valid.

WORKED EXAMPLE 1.6

Repeat the last question but this time with a velocity of 0.05 m/s. Determine the type of flow that exists.

SOLUTION

If no assumptions are made, we should use the general formula $C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$

$$R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 0.05 \text{ x } d}{0.001} = 50\,000d$$

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4$$

$$C_D = \frac{24}{50000d} + \frac{6}{1+\sqrt{50000d}} + 0.4$$

$$C_D = 0.00048d^{-1} + \frac{6}{1+223.6d^{0.5}} + 0.4$$

$$C_D = \frac{8dg(\rho_s - \rho_f)}{6\rho_f u^2} = \frac{8d \text{ x } 9.81 \text{ x } (2630 - 1000)}{6 \text{ x } 1000 \text{ x } 0.05^2} = 8528.16d$$

$$8528.16d = 0.00048d^{-1} + \frac{6}{1+223.6d^{0.5}} + 0.4$$

This should be solved by any method known to you such as plotting two functions and finding the point of interception.

$$f1(d) = 8528.16d$$

$$f2(d) = 0.00048d^{-1} + \frac{6}{1 + 223.6d^{0.5}} + 0.4$$

The graph below gives an answer of d = 0.35 mm.

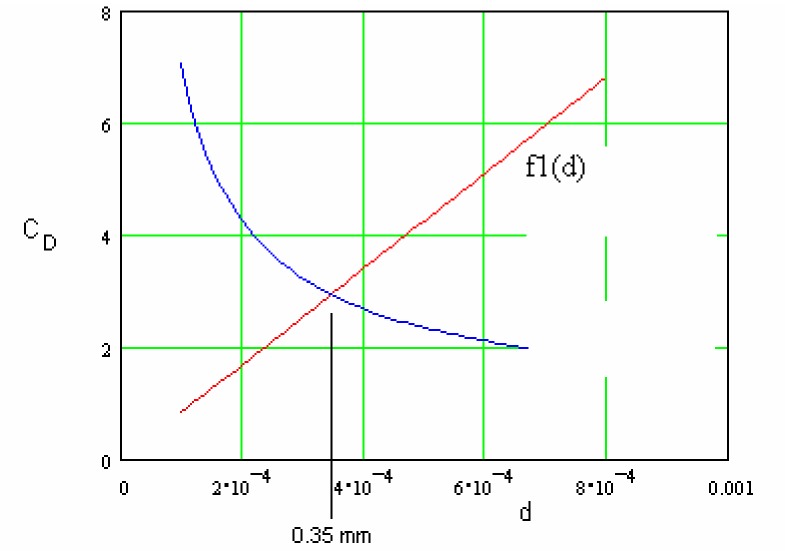


Fig. 1.13

Checking the Reynolds' number $R_e = \frac{\rho_f u_t d}{\mu} = \frac{1000 \text{ x } 0.05 \text{ x } 0.00035}{0.001} = 17.5$

This puts the flow in the Allen's flow section.

ANOTHER METHOD OF SOLUTION

It has been shown previously that the drag coefficient for a sphere is given by the formula $C_D = \dfrac{8R}{\pi d^2 \rho u^2}$. R is the drag force. One method of solving problems is to arrange the formula into the form $C_D R_e^2$ as follows.

$$C_D = \frac{8R}{\pi d^2 \rho_f u^2} x \frac{\rho_f \mu^2}{\rho_f \mu^2} = \frac{8R\rho_f}{\pi\mu^2} x \frac{\mu^2}{\rho_f^2 u^2 d^2} = \frac{8R\rho_f}{\pi\mu^2} x \frac{1}{R_e^2}$$

$$C_D R_e^2 = \frac{8R\rho_f}{\pi\mu^2}$$

If the sphere is falling and has reached its terminal velocity, R = buoyant weight.

$$R = \frac{\pi d^3 g(\rho_s - \rho_f)}{6}$$

$$C_D R_e^2 = \frac{\pi d^3 g(\rho_s - \rho_f)8\rho_f}{6\pi\mu^2}$$

$$C_D R_e^2 = \frac{4d^3 g\rho_f(\rho_s - \rho_f)}{3\mu^2} \ldots\ldots\ldots\ldots\ldots(A)$$

The drag coefficient for a sphere is related to the Reynolds number as described previously. There are two equations commonly used for this relationship as follows.

$$C_D = \frac{24}{R_e} + \frac{6}{1+\sqrt{R_e}} + 0.4 \ldots\ldots\ldots\ldots\ldots\ldots(B)$$

and

$$C_D = \frac{24}{R_e}\left[1 + 0.15R_e^{\ 0.687}\right] \ldots\ldots\ldots\ldots\ldots\ldots(C)$$

Either B or C may be used in the solution of problems. The general method is to solve $R_eC_D^2$ from equation A. Next compose a table of values of R_e, C_D, and $R_eC_D^2$. Plot $R_eC_D^2$ vertically and Re horizontally. Find the value of R_e that gives the required value of $R_eC_D^2$. From this the velocity may be deduced.

WORKED EXAMPLE 1.7

A sphere 1.5 mm diameter falls in water. The density of the sphere is 2500 kg/m^3. The density and dynamic viscosity of water is 997 kg/m^3 and 0.89 x 10^{-3} Ns/m^2 respectively. The drag coefficient is given by the formula $C_D = \frac{24}{R_e}\left[1 + 0.15R_e^{\ 0.687}\right]$. Determine the terminal velocity.

SOLUTION

$$C_D R_e^2 = \frac{4d^3 g\rho_f\left(\rho_s - \rho_f\right)}{3\mu^2} = \frac{4(0.0015)^3 x 9.81 x 997(2500 - 997)}{3(0.89x10^{-3})^2} = 83513$$

Next compile a table using the formula $C_D = \frac{24}{R_e}\left[1 + 0.15\,\mathrm{Re}^{0.687}\right]$.

R_e	0.1	1	10	100	1000
C_D	24.7	27.6	4.15	1.09	0.44
$C_D R_e^2$	2.47	27.6	415	109	438288

We are looking for a value of $C_D R_e^2$ = 83513 and it is apparent that this occurs when R_e is between 100 and 1000. By plotting or by narrowing down the figure by trial and error we find that the correct value of R_e is 356.

$R_e = 356 = \rho_f u d/\mu$

356 = 997 x u x 0.0015/0.89 x 10^{-3}

u = 0.212 m/s and this is the terminal velocity.

SELF ASSESSMENT EXERCISE No. 3

1. a. Explain the term Stokes flow and terminal velocity.

   b. Show that the terminal velocity of a spherical particle with Stokes flow is given by the formula $u = d^2g(\rho_s - \rho_f)/18\mu$

   Go on to show that $C_D=24/R_e$

2. Calculate the largest diameter sphere that can be lifted upwards by a vertical flow of water moving at 1 m/s. The sphere is made of glass with a density of 2630 kg/m^3. The water has a density of 998 kg/m^3 and a dynamic viscosity of 1 cP. (20.7 mm)

3. Using the same data for the sphere and water as in Q2, calculate the diameter of the largest sphere that can be lifted upwards by a vertical flow of water moving at 0.5 m/s. (5.95 mm).

4. Using the graph (fig. 1.12) to obtain the drag coefficient of a sphere, determine the drag on a totally immersed sphere 0.2 m diameter moving at 0.3 m/s in sea water. The density of the water is 1025 kg/m^3 and the dynamic viscosity is 1.05 x 10^{-3} Ns/m^2. (0.639 N).

5. A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. The density of the water is 1000 kg/m^3 and the dynamic viscosity is 1 cP.

   Calculate the terminal velocity assuming the drag coefficient is
   $C_D = 24\ R_e^{-1}\ (1+ 0.15R_e^{0.687})$ (Ans. 0.215 m/s

6. Similar to part of Q1 1990

   A glass sphere of density 2 690 kg/m^3 falls freely through water. Find the terminal velocity for a 4 mm diameter sphere and a 0.4 mm diameter sphere.

   The drag coefficient is $C_D = 8F/\{\pi d^2\rho u^2\}$
   This coefficient is related to the Reynolds number as shown.

R_e	15	20	25	30	35
C_D	3.14	2.61	2.33	2.04	1.87

   The density and viscosity of the water is 997 kg/m^3 and 0.89 x 10^{-3} N s/m^2.

   Answer 0.45 m/s and 0.06625 m/s.

7. Similar to part of Q4 1988.

   A glass sphere of diameter 1.5 mm and density 2 500 kg/m^3 is allowed to fall through water under the action of gravity. Find the terminal velocity assuming the drag coefficient is $C_D = 24\ R_e^{-1}(1+ 0.15R_e^{0.687})$
   (Ans. 0.215 m/s)

8. Similar to Q1 1986

The force F on a sphere of diameter d moving at velocity u_m in a fluid is given by
$F = C_D \{\pi d^2 \rho u_m^2\}/8$

For Reynolds numbers less than 1 000, C_D is given by $C_D = 24\ R_e^{-1}(1+ 0.15R_e^{0.687})$

Estimate the terminal velocity of a glass sphere 1 mm diameter and density 2 650 kg/m^3 in water of density 997 kg/m^3 and viscosity 0.89 x 10^{-3} N s/m^2.

Answer 0.15 m/s

2. BOUNDARY LAYERS

In this tutorial we will look at the shape of various types of boundary layers. We will look at the mathematical equations for the shape of the boundary layer and use them to solve problems.

You may recall that the definition of a BL is the thickness of that layer next to a surface in which the velocity grows from zero to a maximum value (or so close to a maximum as to be of no practical difference). This thickness is usually given the symbol δ (small delta).

The boundary layer, once established may have a constant thickness but, for example, when a flow meets the leading edge of a surface, the boundary layer will grow as shown (fig.2.1).

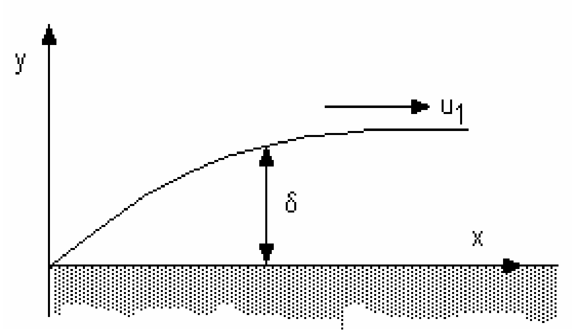


Fig.2.1

When the flow enters a pipe the BL builds up from all around the entrance and a cross section shows the layer meets at the centre (fig.2.2)

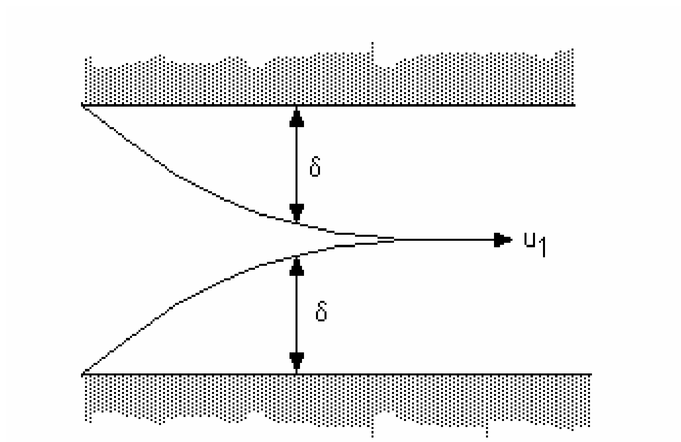


Fig.2.2

The symbol u_1 is used to designate the maximum velocity in the fully developed layer. The fully developed layer may be laminar or turbulent depending on the Reynolds' Number.

The velocity profile for a typical case is shown on fig.2.3.

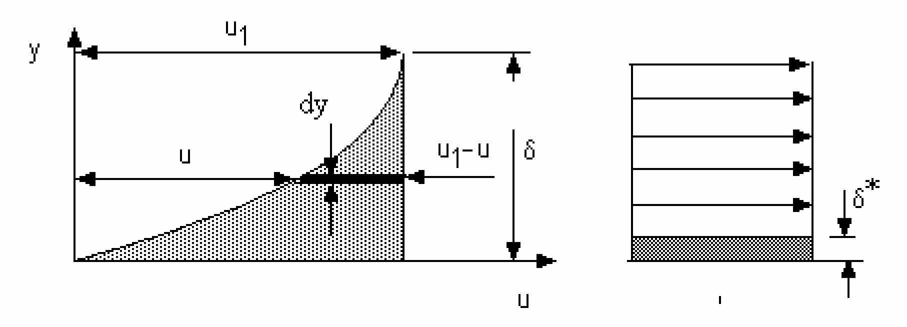


Fig.2.3

The shear stress between any two horizontal layers is τ. For a Newtonian Fluid the relationship between shear stress, dynamic viscosity (μ) and rate of shear strain (du/dy) is

$$\tau = \mu \, du/dy$$

At the wall the shear stress is called the WALL SHEAR STRESS, τ_o and occurs at y = 0. Note that the gradient du/dy is the rate of shear strain and it is steeper for turbulent flow than for laminar flow giving a greater shear resistance.

The solution of problems is simplified by the concepts of DISPLACEMENT THICKNESS AND MOMENTUM THICKNESS which we will now examine.

2.1. DISPLACEMENT THICKNESS δ^*

The flow rate within a boundary layer is less than that for a uniform flow of velocity u_1. The reduction in flow is equal to the area under the curve in fig.2.3. If we had a uniform flow equal to that in the boundary layer, the surface would have to be displaced a distance δ^* in order to produce the reduction. This distance is called the displacement thickness and it is given by :

$$\text{flow redution} = \int_0^\delta [\mathrm{u}_1 - u]dy = u_1\delta^*$$

If this is equivalent to a flow of velocity u_1 in a layer δ^* thick then :

$$\delta^* = \int_0^\delta \left[1 - \frac{u}{u_1}\right] dy$$

2.2. MOMENTUM THICKNESS θ

The momentum in a flow with a BL present is less than that in a uniform flow of the same thickness. The momentum in a uniform layer at velocity u_1 and height h is $\rho h u_1^2$. When a BL exists this is reduced by $\rho u_1^2\theta$. Where θ is the thickness of the uniform layer that contains the equivalent to the reduction. Using the same reasoning as before we get :

$$\theta = \int_0^\delta \left[\frac{u}{u_1}\right]\left[1 - \frac{u}{u_1}\right] dy$$

2.3. BOUNDARY LAYER LAWS

The velocity at any distance y above a surface is a function of the wall shear stress, the dynamic viscosity and the density.

$$u = \phi(y, \tau_o, \rho, \mu)$$

If you are familiar with the method of dimensional analysis you may wish to show for yourself that :

$$u(\rho/\tau_o)^{1/2} = \phi\{y(\tau_o)^{1/2}\}$$

Generally the law governing the growth of a BL is of the form $u = \phi(y)$ and the limits must be that u = 0 at the wall and $u = u_1$ in the fully developed flow. There are many ways in which this is expressed according to the Reynolds' Number for the flow. The important boundary conditions that are used in the formulation of boundary layer laws are:

1. The velocity is zero at the wall (u = 0 @ y = 0).

2. The velocity is a maximum at the top of the layer ($u = u_1$ @ $y = \delta$).

3. The gradient of the b.l. is zero at the top of the layer (du/dy = 0 @ $y = \delta$).

4. The gradient is constant at the wall (du/dy = C @ y = 0).

5. Following from (4) $d^2u/dy^2 = 0$ @ y = 0).

Let us start by considering LAMINAR BOUNDARY LAYERS.

2.3.1 LAMINAR BOUNDARY LAYERS

One of the laws which seem to work for laminar flow is $u = u_1 \sin(\pi y/2\delta)$

WORKED EXAMPLE No.2.1

Find the displacement thickness δ^* for a Laminar BL modelled by the equation $u = u_1 \sin(\pi y/2\delta)$

$$\delta^* = \int_0^\delta \left[1 - \frac{u}{u_1}\right] dy = \int_0^\delta \left[1 - \sin\left\{\frac{\pi y}{2\delta}\right\}\right]$$

$$\delta^* = \left[y + \frac{2\delta}{\pi}\cos\left\{\frac{\pi y}{2\delta}\right\}\right]_0^\delta = \{\delta + 0\} - \left\{0 + \frac{2\delta}{\pi}\right\} = 0.364\delta$$

Another way of expressing the shape of the laminar BL is with a power law. The next example is typical of that used in the examination.

WORKED EXAMPLE No.2.2

The velocity distribution inside a laminar BL over a flat plate is described by the cubic law :

$$u/u_1 = a_0 + a_1 y + a_2 y^2 + a_3 y^3$$

Show that the momentum thickness is $39\delta/280$

SOLUTION

At $y = 0$, $u = 0$ so it follows that $a_0 = 0$

$d^2u/dy^2 = 0$ @ $y = 0$ so $a_2 = 0$. Show for yourself that this is so.

The law is reduced to $u/u_1 = a_1 y + a_3 y^3$

at $y = \delta$, $u = u_1$ so $1 = a_1\delta + 3a_3\delta^2$

hence $a_1 = (1 - a_3\delta^3)/\delta$

Now differentiate and $du/dy = u_1(a_1 + 3a_3 y^2)$

at $y = \delta$, du/dy is zero so $0 = a_1 + 3a_3\delta^2$ so $a_1 = -3a_3\delta^2$

Hence by equating $a_1 = 3/2\delta$ and $a_3 = -1/2\delta^3$

Now we can write the velocity distribution as $u/u_1 = 3y/2\delta - (y/\delta)^3/2$

and $du/dy = u_1\{3/2\delta + 3y^2/2\delta^3\}$

If we give the term y/δ the symbol η we may rewrite the equation as:

$$u/u_1 = 3\eta/2 - \eta^3/2$$

The momentum thickness θ is given by :

$$\theta = \int_0^\delta \left[\frac{u}{u_1}\right]\left[1 - \frac{u}{u_1}\right] dy \quad \text{but dy} = \delta\, \mathrm{d}\eta$$

$$\theta = \int_0^1 \left\{\frac{3\eta}{2} - \frac{\eta^3}{2}\right\}\left\{1 - \frac{3\eta}{2} - \frac{\eta^3}{2}\right\} d\eta$$

Integrating gives :

$$\theta = \delta\left[\frac{3\eta^2}{4} - \frac{\eta^4}{8} - \frac{9\eta^3}{12} - \frac{\eta^7}{28} + \frac{3\eta^5}{10}\right]$$

between the limits $\eta=0$ and $\eta=1$ this evaluates to

$$\theta = 39\delta/280$$

WORKED EXAMPLE No.2.3

Show that $\delta/x = 4.64 R_e^{-0.5}$ for the same case as before.

SOLUTION

We must first go back to the basic relationship. From the previous page

$$du/dy = u_1\{3/2\delta + 3y^2/2\delta^3\}$$

At the wall where y=0 the shear stress is

$$\tau_o = \mu du/dy = \mu u_1\{3/2\delta + 3y^2/2\delta^3\} = (\mu u_1/\delta)\ \delta\ [(3/2\delta) + 3y^2/2\delta^3]$$

Putting $y/\delta = \eta$ we get :

$$\tau_o = (\mu u_1/\delta)\ \delta\ [(3/2\delta) + 3\delta^2/2\delta]$$

$$\tau_o = (\mu u_1/\delta)\ [(3/2) + 3\delta^2/2]$$

at the wall $\eta=0$ $\quad \tau_o = (\mu u_1/\delta)\ (3/2)$……….........(2.1)

The friction coefficient C_f is always defined as

$$C_f = \tau_o/(\rho\ u_1^2/2)\text{.............}(2.2)$$

It has been shown elsewhere that $C_f = 2d\theta/dx$. The student should search out this information from test books.

Putting $\theta = 39\delta/280$ (from the last example) then

$$C_f = 2d\theta/dx = (2\text{x}39/280)\ d\delta/dx \quad \text{..............}(2.3)$$

equating (2.2) and (2.3) gives

$$\tau_o = (\rho u_1^2)(39/280) d\delta/dx \text{ ....}(2.4)$$

equating (2.1) and (2.4) gives

$$(\rho u_1^2)(39/280) d\delta/dx = (\mu u/\delta)(3/2)$$

hence $\quad (3 \times 280)/(2 \times 39)(\mu dx)/\rho u) = \delta d\delta$

Integrating $\quad 10.77(\mu x/\ \rho u_1) = \delta^2/2 + C$

Since $\delta = 0$ at $x = 0$ (the leading edge of the plate) then $C=0$

hence $\quad \delta = \{21.54\mu x/\rho u_1\}^{½}$

dividing both sides by x gives $\quad \delta/x = 4.64(\mu/\rho u_1 x)^{-½} = 4.64R_e^{-½}$

NB $\quad R_{ex} = \rho u_1 x/\mu$ and is based on length from the leading edge.

SELF ASSESSMENT EXERCISE No. 4

1. The BL over a plate is described by $u/u_1=\sin(\pi y/2\delta)$. Show that the momentum thickness is 0.137δ.

2. The velocity profile in a laminar boundary layer on a flat plate is to be modelled by the cubic expression $u/u_1=a_0+a_1y + a_2y^2+a^3y^3$
where u is the velocity a distance y from the wall and u_1 is the main stream velocity.
Explain why a_0 and a_2 are zero and evaluate the constants a_1 and a_3 in terms of the boundary layer thickness δ.

Define the momentum thickness θ and show that it equals $39\delta/280$
Hence evaluate the constant A in the expression
$\delta/x = A\ (R_{e_x})^{-0.5}$
where x is the distance from the leading edge of the plate. It may be assumed without proof that the friction factor $C_f = 2\ d\theta/dx$

3. (a) The velocity profile in a laminar boundary layer is sometimes expressed in the form
$u/u_1=a_0+a_1(y/\delta)+a_2(y/\delta)^2+a_3\ (y/\delta)^3+a_4(y/\delta)^4$
where u_1 is the velocity outside the boundary layer and δ is the boundary layer thickness. Evaluate the coefficients a_0 to a_4 for the case when the pressure gradient along the surface is zero.

(b) Assuming a velocity profile $u/u_1=2(y/\delta) - (y/\delta)^2$ obtain an expression for the mass and momentum fluxes within the boundary layer and hence determine the displacement and momentum thickness.

4. When a fluid flows over a flat surface and the flow is laminar, the boundary layer profile may be represented by the equation

$$u/u_1= 2(\eta) - (\eta)^2 \quad \text{where } \eta = y/\delta$$

y is the height within the layer and δ is the thickness of the layer. u is the velocity within the layer and u_1 is the velocity of the main stream.

Show that this distribution satisfies the boundary conditions for the layer.

Show that the thickness of the layer varies with distance (x) from the leading edge by the equation $\delta=5.48x(R_{e_x})^{-0.5}$

It may be assumed that $\tau_o = \rho u_1^2\ d\theta/dx$

5. Define the terms *displacement thickness* δ^* and *momentum thickness* θ.
Find the ratio of these quantities to the boundary layer thickness δ if the velocity profile within the boundary layer is given by

$$u/u_1=\sin(\pi y/2\delta)$$

Show, by means of a momentum balance, that the variation of the boundary layer thickness δ with distance (x) from the leading edge is given by $\delta = 4.8(R_{e_x})^{-0.5}$

It may be assumed that $\tau_o = \rho u_1^2\, d\theta/dx$

Estimate the boundary layer thickness at the trailing edge of a plane surface of length 0.1 m when air at STP is flowing parallel to it with a free stream velocity u_1 of 0.8 m/s. It may be assumed without proof that the friction factor C_f is given by
$C_f = 2\, d\theta/dx$

N.B. standard data $\mu = 1.71 \times 10^{-5}$ N s/m^2. $\rho = 1.29$ kg/m^3.

6. In a laminar flow of a fluid over a flat plate with zero pressure gradient an approximation to the velocity profile is

$$u/u_1=(3/2)(\eta) - (1/2)(\eta)^3$$

$\eta = y/\delta$ and u is the velocity at a distance y from the plate and u_1 is the mainstream velocity. δ is the boundary layer thickness.

Discuss whether this profile satisfies appropriate boundary conditions.

Show that the local skin-friction coefficient C_f is related to the Reynolds' number (Re_x) based on distance x from the leading edge by the expression
$C_f = A\,(R_{e_x})^{-0.5}$
and evaluate the constant A.

It may be assumed without proof that $C_f = 2\, d\theta/dx$

and that θ is the integral of $(u/u_1)(1 - u/u_1)dy$ between the limits 0 and δ

2.3.2 TURBULENT BOUNDARY LAYERS

When a fluid flows at high velocities, the boundary layer becomes turbulent and the gradient at the wall becomes smaller so the wall shear stress is larger and the drag created on the surface increases.

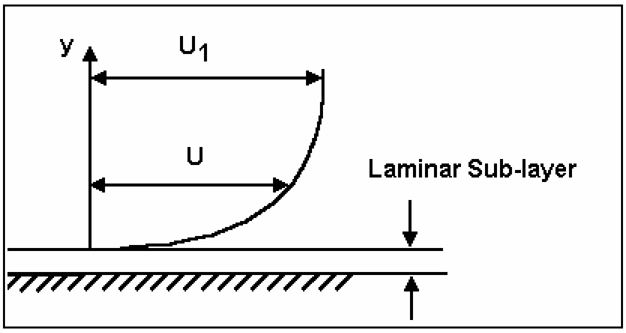


Fig. 2.4

Prandtl found that a law which fits the turbulent case well for Reynolds' numbers below 10^7 is:

$$u = u_1(y/\delta)^{1/7}$$

This is called the $1/7^{th}$ law.

The gradient of the B.L. is $du/dy = u_1\delta^{1/7}y^{-6/7}/7$

This indicates that at the wall where y=0, the gradient is infinite (horizontal). This is obviously incorrect and is explained by the existence of a laminar sub-layer next to the wall. In this layer the velocity grows very quickly from zero and merges with the turbulent layer. The gradient is the same for both at the interface of laminar and turbulent flow. The drag on the surface is due to the wall shear stress in the laminar sub-layer.

WORKED EXAMPLE No.2.4

Show that the mean velocity in a pipe with fully developed turbulent flow is 49/60 of the maximum velocity. Assume the 1/7th law.

For a pipe, the B.L. extends to the centre so δ = radius = R. Consider an elementary ring of flow.

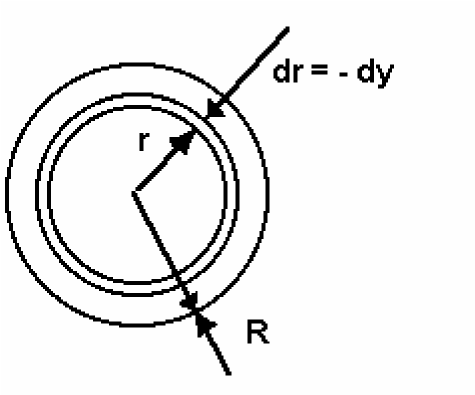


Fig.2.5

The velocity through the ring is u.

The volume flow rate through the ring is $2\pi rudr$

The volume flow rate in the pipe is $Q = 2\pi\int rudr$

Since δ = R then $u = u_1(y/R)^{1/7}$

also $r = R-y$

$$Q = 2\pi\int (R-y)udr = 2\pi\int u_1 R^{-1/7} (R-y)y^{1/7}dy$$

$$Q = 2\pi u_1 R^{-1/7}[Ry^{1/7} - y^{8/7}]$$

$$Q = 2\pi u_1 R^{-1/7}[(7/8)Ry^{8/7} - (7/15)y^{15/7}]$$

$$Q = (49/60)\pi R^2 u_1.$$

The mean velocity is defined by $u_m = Q/\pi R^2$

hence $u_m = (49/60)u_1$

2.4 FRICTION COEFFICIENT AND BOUNDARY LAYERS

Earlier it was explained that the friction coefficient C_f is the ratio of the wall shear stress to the dynamic pressure so :

$$C_f = 2\tau_o/(\rho u_m^2) \text{ ....(2.4.1)}$$

For smooth walled pipes, Blazius determined that $C_f = 0.079R_e^{-0.25}$ ....(2.4.2)

Equating (2.4.1) and (2.4.2) gives : $2\tau_o/(\rho u_m^2) = 0.079R_e^{-0.25}$

Note that u_m is the mean velocity and u_1 is the maximum velocity.

Research shows that $u_m=0.8u_1$

Also Note that $R_e=\rho u_1 D/\mu$ and $D = 2\delta$.

Hence $\tau_o=0.02125\rho u_1^2(\mu/\rho\delta u_1)^{0.25}$.............(2.4.3)

2.5 FORCE BALANCE IN THE BOUNDARY LAYER

The student should refer to textbooks for finer details of the following work.

Consider again the growth of the B.L. as the fluid comes onto a flat surface. A stream line for the flow is not parallel to the B.L. Now consider a control volume A, B, C, D.

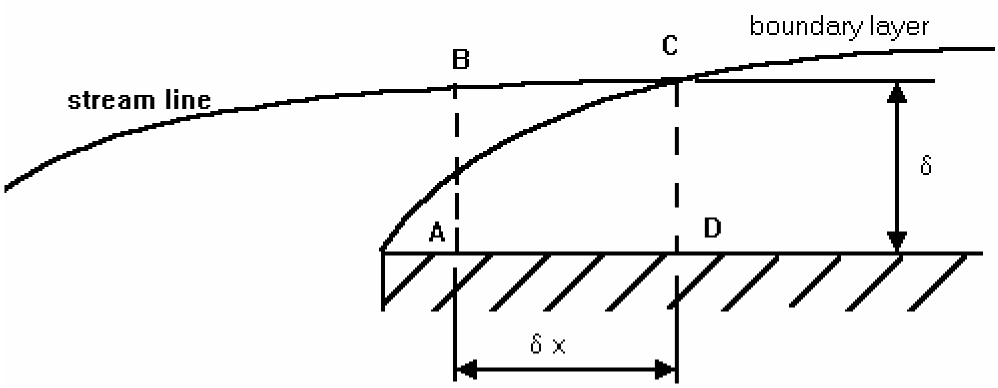


Fig.2.6

Balancing pressure force and shear force at the surface with momentum changes gives :

$$\tau_o = \rho\left(\frac{\delta}{\delta x}\right)\int_0^{\delta}[u-u_1]u\,dy + \rho\left(\frac{\delta u}{\delta x}\right)\int_0^{\delta}[u-u_1]u\,dy \text{...........(2.4.4)}$$

Using equations (2.4.2), (2.4.3) and (2.4.4) gives $(4/5)\delta^{5/4}= 0.231\{\mu/\rho u\}^{1/5}x = R_e^{-1/5}$

The shear force on the surface is $F_S= \tau_o$ x surface area

The surface skin friction coefficient is $C_f=2F_S/(\rho u_1^2) = 0.072R_e^{-1/5}$

Experiments have shown that a more accurate figure is : $C_f= 0.074R_e^{-1/5}$

SELF ASSESSMENT EXERCISE No. 5

1. Under what circumstances is the velocity profile in a pipe adequately represented by the 1/7 th power law $u/u_1=(y/R)^{1/7}$ where u is the velocity at distance y from the wall, R is the pipe radius and u_1 is the centre-line velocity ?

   The table shows the measured velocity profile in a pipe radius 30 mm. Show that these data satisfy the 1/7 th power law and hence evaluate

   (i) the centre-line velocity

   (ii) the mean velocity u_m

   (iii) the distance from the wall at which the velocity equals u_m.

1.0	2.0	5.0	10.0	15.0	20.0	y (mm)
1.54 1.70	1.94	2.14	2.26	2.36	u (m/s)	

2. (a) Discuss the limitations of the 1/7th power law $u/u_1=(y/R)^{1/7}$ for the velocity profile in a circular pipe of radius R, indicating the range of Reynolds numbers for which this law is applicable.

   (b) Show that the mean velocity is given by $49u_1/60$.

   (c) Water flows at a volumetric flow rate of 1.1 x 10^{-3} m^3/s in a tube of diameter 25 mm. Calculate the centre-line velocity and the distance from the wall at which the velocity is equal to the mean velocity.

   (d) Assuming that $C_f=0.079(Re)^{-0.25}$ evaluate the wall shear stress and hence estimate the laminar sub-layer thickness.

   μ = 0.89 x 10^{-3} N s/m^2. ρ=998 kg/m^3.

FLUID MECHANICS

TUTORIAL No.4

FLOW THROUGH POROUS PASSAGES

In this tutorial you will continue the work on laminar flow and develop Poiseuille's equation to the form known as the Carman - Kozeny equation. This equation is used to predict the flow rate through porous passages such as filter, filter beds and fluidised beds in combustion chambers.

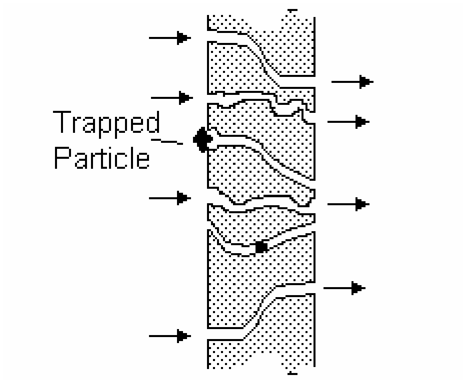
On completion of this tutorial you should be able to do the following.

- Derive the Carman - Kozeny equation.
- Solve problems involving the flow of fluids through a porous material.

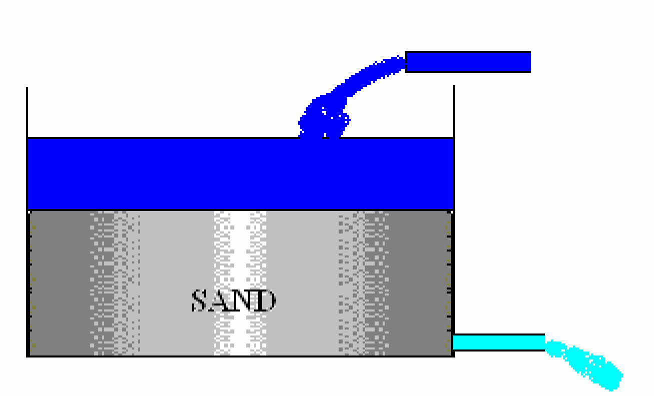
In order to do this tutorial you must be familiar with Poiseuille's equation for laminar flow and this is covered in tutorial 1.

FLOW THROUGH POROUS PASSAGES

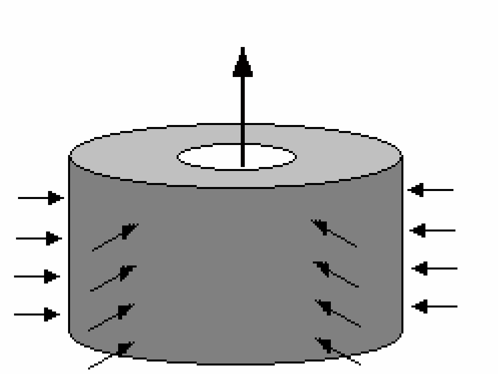
The following are examples where porous flow occurs.



A filter element made of thick sintered particles. This might be a cylinder with radial flow.



A sand bed filter for cleaning water. The water percolates down through the filter though long tortuous passages. The depth of water on top of the filter governs the rate at which the water is forced through.



A layer of rock through which water, gas or oil might seep. This is similar to a radial flow filter but on a much larger scale.

When a fluid passes through a porous material, it flows through long thin tortuous passages of varying cross section. The problem is how to calculate the flow rate based on nominal thickness of the layer. This was tackled by Kozeny and later by Carman. The result is a formula, which gives a mean velocity of flow in the direction at right angles to the layer plane in terms of its thickness and other parameters.

The passage between the particles is so small that the velocity in them is small and the flow is well and truly laminar.

Poiseuille's Equation for laminar flow states $\Delta p/l = -32\ \mu u'/D^2$.

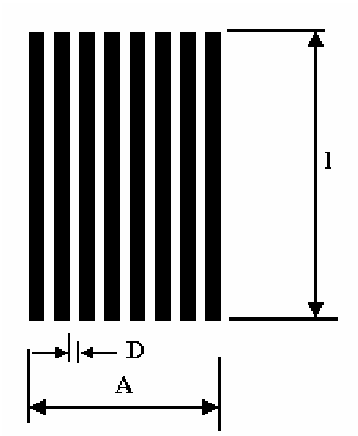


Figure 1

Kozeny modelled the layer as many small capillary tubes of diameter D making up a layer of cross sectional area A. The actual cross sectional area for the flow path is A'. The difference is the area of the solid material.

The ratio is $\varepsilon = A'/A$ and this is known as the ***porosity of the material***. The volume flow rate through the layer is Q.

Kozeny used the notion that Q = Au where u is the mean velocity at right angles to the layer.

The volume flow rate is also Q= A'u' where u' is the mean velocity in the tube.

Equating u' = uA/A'

ε = void fraction = A'/A. u'= u/ε.

Carman modified this formula when he realised that the actual velocity inside the tubes must be proportionally larger because the actual length is greater than the layer thickness. It follows that $u' = \frac{ul'}{\varepsilon l}$

where l is the layer thickness and l' the mean length of the passages. Substituting this in Poiseuille's Equation gives :

$$\frac{\Delta p}{l} = -\frac{32\mu u l'}{\varepsilon l D^2}$$

rearranging $$\frac{\Delta p}{l} = -\frac{32\mu u l'}{l\varepsilon D^2}$$

This is usually expressed as a pressure gradient in the direction of the mean flow (say x) and it becomes :

$$\frac{dp}{dx} = -\frac{32\mu u l'}{l\varepsilon D^2} \text{.......................(1)}$$

A_S= csa of the solid
A'= csa of the tubes
A = csa of the layer = A' + A_S.

$$\varepsilon = \frac{A'}{A} = \frac{A'}{A' + A_s}$$

Multiply top and bottom by thye length l and the areas become volumes so

$$\varepsilon = \frac{Q'}{Q' + Q_s}$$

where Q' is the volume of the tubes and Q_S is the volume of the solid.

$$Q' = \varepsilon(Q' + Q_s)$$

$$Q - Q' = Q_s = Q - Q\frac{Q'}{Q} = Q - \varepsilon Q = Q(1-\varepsilon)$$

$$Q_s = Q(1-\varepsilon)$$

$$Q = \frac{Q_s}{(1-\varepsilon)} = Q' + Q_s$$

$$Q' = \frac{\varepsilon Q_s}{(1-\varepsilon)}$$

S = Surface Area of the tubes . Divide both sides by S $\quad \frac{Q'}{S} = \frac{\varepsilon Q_s}{S(1-\varepsilon)}$

Q' is made up of tubes diameter D and length l' so $\quad Q' = \frac{n\pi D^2 l'}{4}$

and S = nπDl' where n is the number of tubes which cancels when these are substituted into the formula. This results in : $\quad D = \frac{4\varepsilon Q_s}{S(1-\varepsilon)}$

Next we consider the solid as made up of spherical particles of mean diameter d_S.

S = surface area of tubes but also the surface area of the solid particles.

Hence $Q_s = \frac{\pi d_s^3}{6}$ and $S = \pi d_s^2$ and $\frac{Q_s}{S} = \frac{d_s}{6}$

It follows that $\quad D = \frac{2\varepsilon d_s}{3(1-\varepsilon)}$

Substitute this into equation (1) and : $\quad \frac{dp}{dx} = -\frac{72\mu u l'(1-\varepsilon)^2}{l d_s^2 \varepsilon^3}$

Research has shown that (l'/l) is about 2.5 hence : $\quad \frac{dp}{dx} = -\frac{180\mu u(1-\varepsilon)^2}{d_s^2 \varepsilon^3}$

This is the Carman- Kozeny.

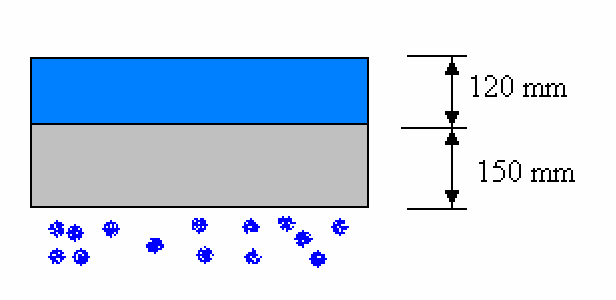
WORKED EXAMPLE No.1

Water is filtered through a sand bed 150 mm thick. The depth of water on top of the bed is 120 mm. The porosity ε is 0.4 and the mean particle diameter is 0.25 mm. The dynamic viscosity is 0.89 cP and the density is 998 kg/m^3.

Calculate the flow rate per square metre of area.

SOLUTION

The pressure difference across the sand bed is assumed to be the head of water since atmospheric pressure acts on top of the water and at the bottom of the bed.



First convert the head into pressure difference.

$\Delta p = \rho g(h_2 - h_1) = \{0 - (998 \text{ x } 9.81 \text{ x } 0.12)\} = -1174.8$ Pa.

The length of the bed (L) is 120 mm.

Assume that $dp/dx = \Delta p/L$ The dynamic viscosity is $0.89 \text{ x } 10^{-3}$ Ns/m^2.

Using the Carman- Kozeny equation where the pressure gradient is assumed to be linear.

$$\frac{dp}{dx} = -\frac{180\mu u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$$

$$\frac{dp}{dx} = -\frac{1174.8}{0.12} = -\frac{180 \text{ x } 0.89 \text{ x } 10^{-3} \text{ u } (1 - 0.4)^2}{0.25\text{x}10^{-3} \text{ x } 0.4^3}$$

$$7832.3 = 14418000u$$

$$u = 0.543\text{x}10^{-3}\ m/s$$

$$Q = 0.543\text{x}10^{-3}\ m^3/s \text{ per square meter of area.}$$

WORKED EXAMPLE No.2

Calculate the flow rate through a filter 70 mm outside diameter and 40 mm inside diameter and 100 mm long given that the pressure on the outside is 20 kPa greater than on the inside. The mean particle diameter d is 0.04 mm and the void fraction is 0.3. The dynamic viscosity is 0.06 N s/m^2.

SOLUTION

The flow is radial so -dp/dx = dp/dr since radius increases in the opposite sense to x in the derivation. The equation may be written as :

$$\frac{dp}{dx} = -\frac{180\mu u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$$

r is the radius. Putting in values:

$$\frac{dp}{dr} = \frac{180x0.06u(1-0.3)^2}{0.00004^2x0.3^3} = 122.5x10^9u$$

Consider an elementary cylinder through which the oil flows. The velocity normal to the surface is

$$u = \frac{Q}{2\pi Lr} = \frac{Q}{2\pi x0.1xr} = 1.591Qr^{-1}$$

hence:

$$dp = 122.5 \text{ x } 10^9 \text{ x } 1.591Q\, r^{-1}\, dr = 194.8975\, Q\, r^{-1}\, dr$$

Integrating between the outside and inside radius yields:

$$p = 194.89Q\ln\left(\frac{R_o}{R_i}\right)$$

$$p = 20000 = 194.89Q\ln\left(\frac{35}{20}\right)$$

$$Q = 183.3x10^9 m^3/s = 183.3mm^3/s$$

SELF ASSESSMENT EXCERCISE No.1

Q.1

Outline briefly the derivation of the Carman-Kozeny equation.

$$\frac{dp}{dl} = -\frac{180\mu u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$$

dp/dl is the pressure gradient, μ is the fluid viscosity, u is the superficial velocity, d_s is the particle diameter and ε is the void fraction.

A cartridge filter consists of an annular piece of material of length 150 mm and internal diameter and external diameters 10 mm and 20 mm. Water at 25°C flows radially inwards under the influence of a pressure difference of 0.1 bar. Determine the volumetric flow rate. (21.53 cm^3/s)

For the filter material take d = 0.05 mm and ε = 0.35.

μ=0.89 x 10^{-3} N s/m^2 and ρ= 997 kg/m^3.

Q.2

(a) Discuss the assumptions leading to the equation of horizontal viscous flow through a packed bed $\frac{dp}{dL} = -\frac{180\mu u(1-\varepsilon)^2}{d_s^2\varepsilon^3}$

where Δp is the pressure drop across a bed of depth L, void fraction ε and effective particle diameter d. u is the approach velocity and μ is the viscosity of the fluid.

(b) Water percolates downwards through a sand filter of thickness 15 mm, consisting of sand grains of effective diameter 0.3 mm and void fraction 0.45. The depth of the effectively stagnant clear water above the filter is 20 mm and the pressure at the base of the filter is atmospheric. Calculate the volumetric flow rate per m^2 of filter. (2.2 dm^3/s)

(Note the density and viscosity of water are given in the instructions on all exams papers)

μ=0.89 x 10^{-3} N s/m^2 and ρ= 997 kg/m^3

Q3.

Oil is extracted from a horizontal oil-bearing stratum of thickness 15 m into a vertical bore hole of radius 0.18 m. Find the rate of extraction of the oil if the pressure in the bore-hole is 250 bar and the pressure 300 m from the bore hole is 350 bar.

Take d= 0.05 mm, ε= 0.30 and $\mu = 5.0 \times 10^{-3}$ N s/m^2.

FLUID MECHANICS

TUTORIAL No.5

POTENTIAL FLOW

In this tutorial you will study the flow of ideal fluids. On completion you should be able to do the following.

- Define the stream function.
- Define the velocity potential.
- Understand the flow of an ideal fluid around a long cylinder.
- Understand the main points concerning vortices.

An ideal fluid has no viscosity (inviscid) and is incompressible. No such fluid exists but these assumptions make it possible to produce models for the flow of fluids in and around solid boundaries such as long cylinders. In particular, the concepts of POTENTIAL FLOW and STREAM FUNCTION give us useful mathematical models to study these phenomena.

1 STREAM FUNCTION

Consider the streamlines representing a 2 dimensional flow of a perfect fluid.

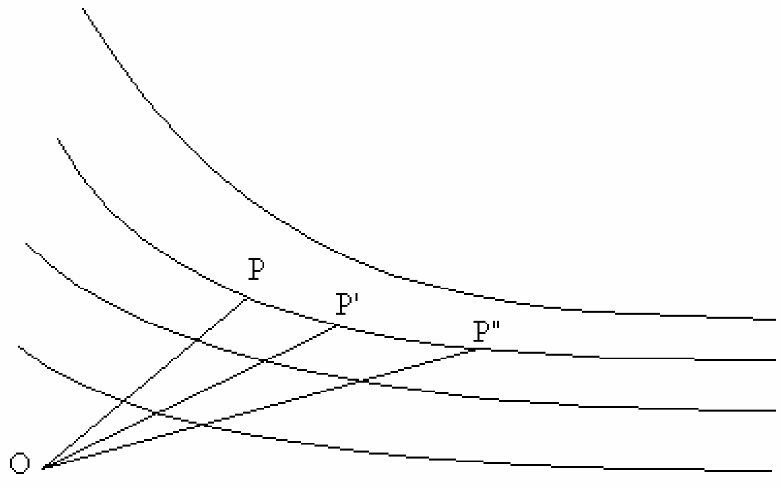


Figure 1

Flux is defined as the volume flow rate per metre depth normal to the page. The stream function is defined as the flux across the line O -P. The symbol used is ψ (psi). Since there is no flow rate normal to a stream line, then it follows that the stream function is the same between O and any point P, P' or P" on the same stream line. In other words, the stream line represents a constant value of the stream function.

It is easier to understand ψ in terms of small changes. Consider a short line of length ds perpendicular to a stream line. Let the velocity across this line have a mean value of v'. The flux crossing this line is hence v'ds and this is the small change in the stream function $d\psi$. It follows that $d\psi = v'ds$

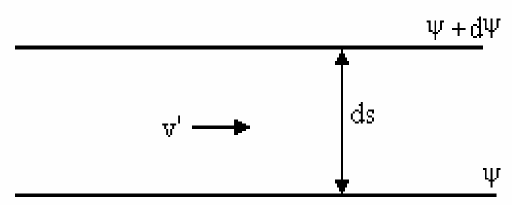


Figure 2

In this analysis, the stream function is positive when it crosses the line in an anti-clockwise direction (right to left on the diagram). This is quite arbitrary with some publications using clockwise as positive, others using anti-clockwise.

The stream function may be expressed with Cartesian or polar co-ordinates. The convention for velocity is that we use v for velocity in the y direction and u for velocity in the x direction. Consider a small flux entering a triangular area as shown. The fluid is incompressible so the volume per unit depth entering the area must be equal to that leaving. It follows that for a flux in the direction shown

$$d\psi + u\,dy = v\,dx \quad \text{and } d\psi = v'ds$$

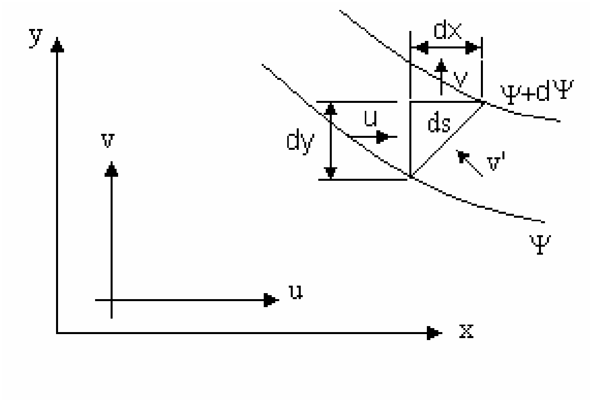


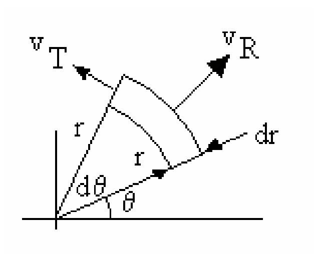
Figure 3

If the stream line is horizontal v' is velocity u and ds is dy hence $\qquad u = -d\psi/dy$

If the stream is vertical then v' is v and ds is dx hence $\qquad v = d\psi/dx$

When polar co-ordinates are used the flow directions are radial and tangential.

If the flow is radial and θ = 0, then v' becomes v_R and ds is r dθ.



$$v_R = -\frac{dy}{rdq}$$

If the flow is tangential and θ = 90° then v' becomes v_T and ds is dr hence

$$v_T = \frac{dy}{dr}$$

Figure 4

The sign convention agrees with the stream function being positive in a direction from right to left.

2 VELOCITY POTENTIAL

The velocity potential has a symbol ϕ. It is best explained as follows.

Consider a line along which the velocity v' varies. Over a short length ds the velocity potential varies by $d\phi$. Hence $d\phi = v'ds$ or $v' = d\phi/ds$. The velocity potential may be thought of as the product of velocity and length in the same direction. It follows that

$$\phi = \int v'\,ds$$

Some text books use a sign convention opposite to this and again this is arbitrary.

If the line is horizontal v' is velocity u and ds is dx hence $u = \frac{df}{dx}$

If the line is vertical then v' is v and ds is dy hence $v = \frac{df}{dy}$

If the flow is radial then v' is v_R and ds is dr hence $v_R = \frac{df}{dr}$

If the flow is tangential then v' is v_T and ds is r $d\theta$ hence $v_T = \frac{df}{rdq}$

The sign convention is positive for increasing radius and positive for anti- clockwise rotation.

Since v' is zero perpendicular to a stream line it follows that lines of constant ϕ run perpendicular to the stream lines. If these lines are superimposed on a flow we have a flow net.

Consider a flow with lines of constant ψ and ϕ as shown.

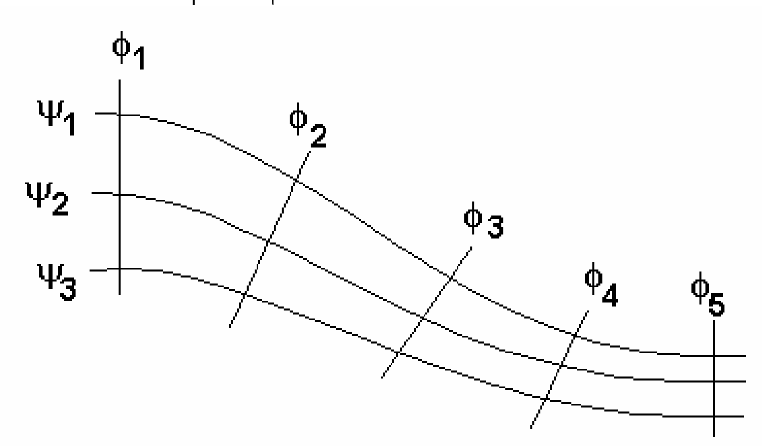


Figure 5

When we compare the velocity equations in terms of the stream function and the velocity potential we find :

$$u = -\frac{d\Psi}{dy} = \frac{df}{dx}\text{.............}(1)$$

$$v = \frac{d\Psi}{dx} = \frac{df}{dy}\text{.............}(2)$$

$$v_R = -\frac{d\Psi}{rdq} = \frac{df}{dr}\text{.........}(3)$$

$$v_T = \frac{d\Psi}{dr} = \frac{df}{rdq}\text{..........}(4)$$

3 UNIFORM FLOW

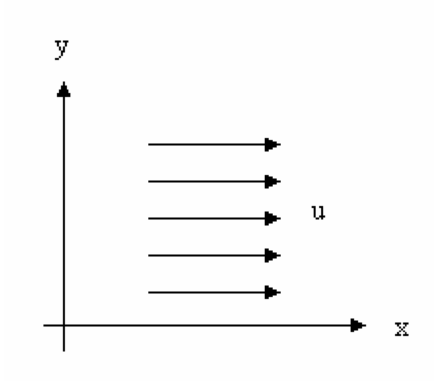


Figure 6

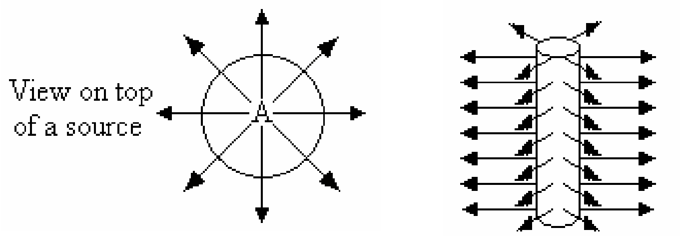
If the flow has a constant velocity u in the x direction and a uniform depth of 1 m then, the stream function is obtained from equation 1 and is $\psi = -uy$

From equation 3,

$$\frac{df}{dr} = -\frac{d(-uy)}{rdq} = -\frac{d(-ur\sin q)}{rdq} = u\cos q$$

4. SOURCE AND SINK

A line source is a single point 1m deep from which fluid appears and flows away radially. A line sink is a single point 1m deep at which flow disappears. The flow rate through any circle centred on the source or sink must be the same at all radii. All radii are stream lines.



STREAM FUNCTION

Consider a source at point A with a flow emerging 1 m deep at a rate of Q m^3/s. At radius r the radial velocity is Q/area = $Q/2\pi r = v_R$. Flux outwards is taken as positive. Some texts use the opposite sign convention.

At radius r the stream function is defined as $d\psi = v_R ds$

ds is a tiny arc. $d\psi = (Q/2\pi r)\, ds$

Note that text books and examiners often use m for the strength of the source and this has the same meaning as Q. A sink is the exact opposite of a source.

$d\psi = -(Q/2\pi r)\, ds$ for a sink.

The arc subtends an angle $d\theta$ and $ds = r\, d\theta$

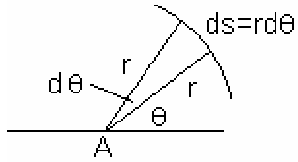


Figure 8

$d\Psi = \dfrac{Q}{2\pi r} r d\theta = \dfrac{Q}{2\pi} d\theta$ for a source.

$d\Psi = -\dfrac{Q}{2\pi r} r d\theta = -\dfrac{Q}{2\pi} d\theta$ for a sink

For a finite angle θ these become

$\Psi = \dfrac{Q}{2\pi}\theta$ for a source.

$\Psi = -\dfrac{Q}{2\pi}\theta$ for a sink.

VELOCITY POTENTIAL

Now consider a length in the radial direction.

$$ds = dr$$

At radius r the velocity potential is defined as

$$d\phi = v_R\, dr$$

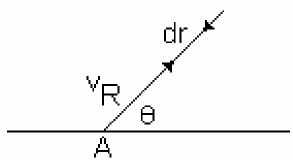


Figure 9

This becomes

$d\phi = \dfrac{Q}{2\pi r} dr$ for a source

$d\phi = -\dfrac{Q}{2\pi r} dr$ for a sink.

To find the expression for a length of one radius, we integrate with respect to r.

$\phi = \dfrac{Q}{2\pi} \ln r$ for a source

$\phi = -\dfrac{Q}{2\pi} \ln r$ for a sink.

From the preceding it may be deduced that the streamline are radial lines and the lines of constant ϕ are concentric circles.

5 DOUBLET

A doublet is formed when an equal source and a sink are brought close together. Consider a source and sink of equal strength placed at A and B respectively. The stream function for point P relative to A and B are respectively

$$\Psi_B = \frac{Q}{2p} q_2 \text{ for the source}$$

$$\Psi_A = -\frac{Q}{2p} q_1 \text{ for the sink}$$

$$\Psi_P = \Psi_B + \Psi_A = \frac{Q}{2p}(q_2 - q_1)$$

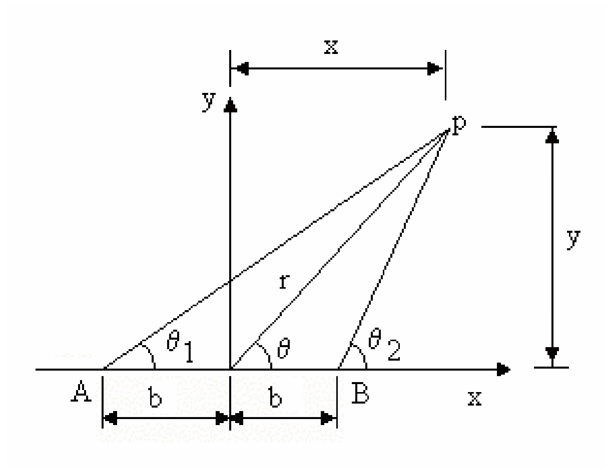


Figure 10

Referring to the diagram

$$\tan\theta_1 = \frac{y}{x+b}, \tan\theta_2 = \frac{y}{x-b}$$

$$\tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

$$\tan(\theta_2 - \theta_1) = \frac{y(x-b) - y(x+b)}{1 + \frac{y^2}{x^2 - b^2}}$$

$$\tan(\theta_2 - \theta_1) = \frac{2by}{x^2 - b^2 + y^2}$$

As $b \to 0$, $b^2 \to 0$ and the tan of the angle becomes the same as he angle itself in radians.

$$(\theta_2 - \theta_1) = \frac{2by}{x^2 + y^2}$$

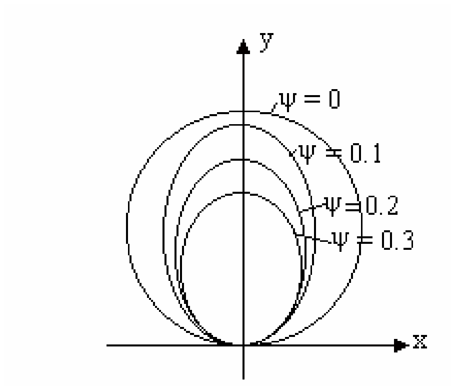
$$\Psi_p = \frac{Q}{2\pi}\left[\frac{2by}{x^2 + y^2}\right]$$

When the source and sink are brought close together we have ***DOUBLET*** but b remains finite.

Let $B = (Qb/\pi)$ $\qquad \Psi = \frac{By}{x^2 + y^2}$

Since $y = r\sin\theta$ and $x^2 + y^2 = r^2$ then

$$\Psi = \frac{Br\sin q}{r^2} = \frac{B\sin q}{r}$$



$\psi = 0$ is the streamline across which there is no flux and this is a circle so it can be used to represent a cylinder.

e 11

6. COMBINATION OF UNIFORM FLOW AND SOURCE OR SINK

For this development, consider the case for the source at the origin of the x – y co-ordinates with a uniform flow of velocity u from left to right. The development for a sink in a uniform flow follows the same principles. The uniform flow encounters the flux from the source producing a pattern as shown. At large values of x the flow has become uniform again with velocity u. The flux from the source is Q. this divides equally to the top and bottom. At point s there is a stagnation point where the radial velocity from the source is equal and opposite of the uniform velocity u.

The radial velocity is $Q/2\pi r$. Equating to u we have **r = Q/2pu** and this is the distance from the origin to the stagnation point.

For uniform flow $\Psi_1 = -uy$ For the source $\Psi_2 = Q\theta/2\pi$. The combined value is Y **= -uy + Qq/2p**

The flux between the origin and the stagnation point S is half the flow from the source. Hence, the flux is Q/2 and the angle θ is π radian (180^o). The dividing streamline emanating from S is the zero streamline $\Psi = 0$. Since no flux crosses this streamline, the dividing streamline could be a solid boundary. When the flow is uniform, we have:

$\Psi = 0 = -uy + Q\theta/2\pi = -uy + Q\pi/2\pi$ $\qquad$ $y = -uy + Q/2$ $\qquad$ $y = Q/2u$

y is the distance from the x axis to the zero stream line where the flow is uniform (at large values of x). The thickness of the uniform stream emerging from the source is $t = 2y$.

Hence $\qquad$ **t =Q/u**.

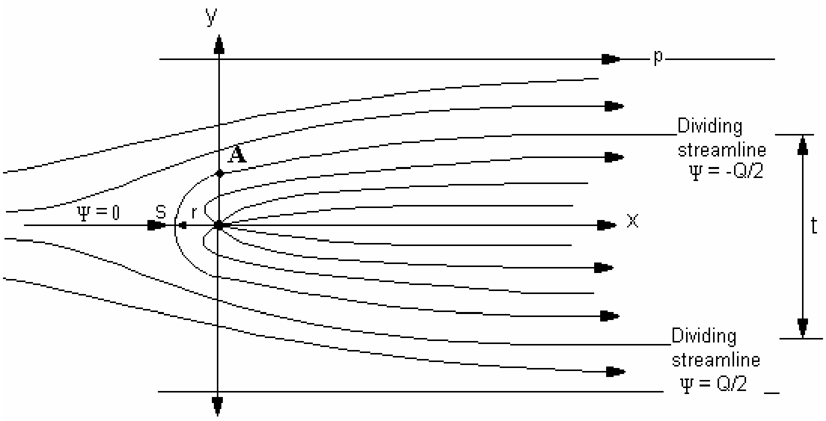


Figure 12

PRESSURE

Consider points S and A. At S there is a pressure p_s and no velocity. At point A there is a velocity v_A and pressure p_A. Applying Bernoulli between these points, we have:

$p_s = p_A + \rho\, v_A^2/2$

$p_s - p_A = \rho\, v_A^2/2$

To solve the pressure difference we need to know the velocity. At point A we can solve this as follows. The velocity is the resultant velocity of the uniform flow u and the radial velocity from the source v_R.

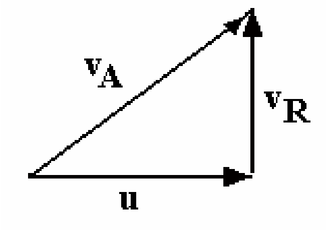


Figure 13

$v_A^2 = u^2 + v_R^2$

The stream line at point A is $\Psi = 0$ hence $0 = -uy + Q\theta/2\pi$ hence $y = Q\theta/2\pi u$

At this point $\theta = \pi/2$ (90^o) so $y = Q/4u$

This is the distance to point A along the y axis.

$v_R = Q/2\pi r$. The radius at point A is Q/4u hence $v_R = 2u/\pi$

$v_A^2 = u^2 + (2u/\pi)^2 = u^2 \{1 + 4/\pi^2\}$

$p_s - p_A = \rho\, v_A^2/2 = (\rho u^2/2)\{1 + 4/\pi^2\}$

WORKED EXAMPLE No.1

A uniform flow of fluid with a density of 800 kg/m^3 is from left to right with a velocity u = 2 m/s. It is combined with a source of strength Q = 8 m^2/s at the origin. Calculate:

1. The distance to the stagnation point.
2. The width of the flow stream emanating from the source when it has reached a uniform state.
3. The pressure difference between the stagnation point and the point where the zero streamline crosses the y axis.

SOLUTION

From the preceding work

Distance to stagnation point = $Q/2\pi u = 8/(2\pi \times 2) = 2/\pi$ metres

$t = Q/u = 8/2 = 4$ m

$p_s - p_A = (\rho u^2/2)\{1 + 4/\pi^2\}$

$p_s - p_A = (800 \times 2^2/2)\{1 + 4/\pi^2\}$

$p_s - p_A = 1600(1.405) = 2248$ N/m^2

If a sink is placed at the origin, the flow pattern is like this.

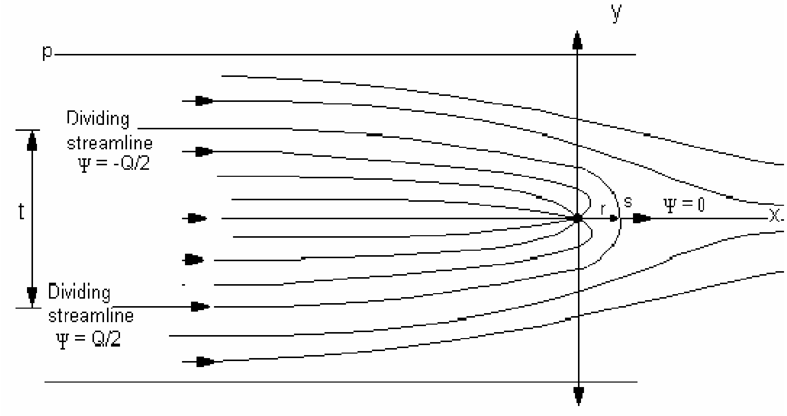


Figure 14

The analysis is similar and yields the same result.

7. FLOW AROUND A LONG CYLINDER

When an ideal fluid flows around a long cylinder, the stream lines and velocity potentials form the same pattern as a doublet placed in a constant uniform flow. It follows that we may use a doublet to study the flow pattern around a cylinder. The result of combining a doublet with a uniform flow at velocity u is shown below.

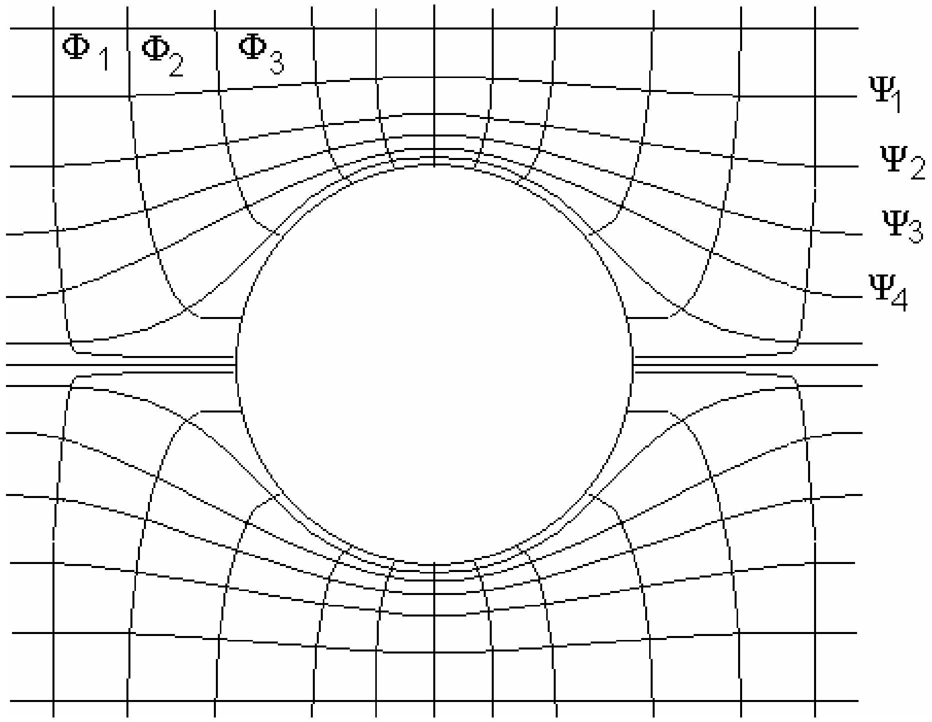


Figure 15

Consider a doublet at the origin with a uniform flow from left to right. The stream function for point p is obtained by adding the functions for a doublet and a uniform flow.

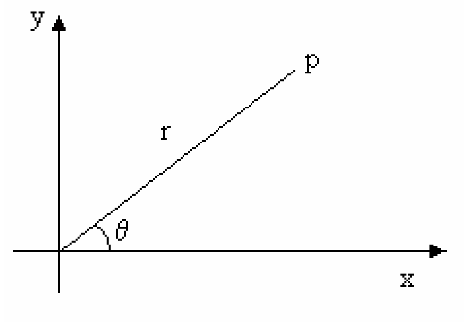


Figure 16

For a doublet is $\psi = B\sin\theta/r$

For a uniform flow $\psi = -uy$.

For the combined flow is $\psi = B\sin\theta/r - uy$

Where B = (Qb/π) From the diagram we have $y = r\sin\theta$ and substituting this into the stream function gives

$$\Psi = \frac{B\sin q}{r} - ur\sin q = \left(\frac{B}{r} - ur\right)\sin q$$

$$\frac{d\Psi}{dr} = \left(\frac{-B}{r^2} - u\right)\sin q \qquad (5)$$

$$\Psi = \left(\frac{B}{r} - ur\right)\cos q \qquad (6)$$

The equation is usually given in the form $\Psi = \left(\frac{B}{r} - Ar\right)\cos\theta$ where A = u

The stream functions may be converted into velocity potentials by use of equations 3 and 5 or 4 and 6 as follows.

Equation 4

$$v_T = \frac{d\Psi}{dr} = \frac{df}{rdq}$$

$$\frac{d\Psi}{dr} = \frac{df}{rdq}$$

$$r\frac{d\Psi}{dr} = \frac{df}{dq}$$

$$r\left\{-\frac{B}{r^2} - A\right\}\sin q = \frac{df}{dq}$$

$$\left\{-\frac{B}{r} - Ar\right\}\sin q = \frac{df}{dq}$$

$$\left\{\frac{B}{r} + Ar\right\}\cos q = f$$

Equation 3

$$v_R = -\frac{d\Psi}{rdq} = \frac{df}{dr}$$

$$-\frac{d\Psi}{dq} = r\frac{df}{dr}$$

$$-\left(\frac{B}{r} - Ar\right)dr\cos q = r\frac{df}{dr}$$

$$-\left(\frac{B}{r^2} + A\right)dr\cos q = df$$

$$\left(\frac{B}{r} + Ar\right)\cos q = f$$

At any given point in the flow with co-ordinates r,θ the velocity has a radial and tangential component. The true velocity v_θ is the vector sum of both which, being at a right angle to each other, is found from Pythagoras as

$$v_\theta = \sqrt{v_R^2 + v_T^2}$$

From equation 3 and 4 we can show that

$$v_R = -u\left\{\frac{R^2}{r^2} - 1\right\}\cos\theta \ldots\ldots\ldots\ldots(7)$$

$$v_T = -u\left\{\frac{R^2}{r^2} + 1\right\}\sin\theta \ldots\ldots\ldots\ldots(8)$$

R is the radius of the cylinder. From these equations we may find the true velocity at any point in the flow.

WORKED EXAMPLE No. 2

The velocity potential for an ideal fluid flowing around a long cylinder is given by

$$\left\{\frac{B}{r} + Ar\right\}\cos\theta = \phi$$

The cylinder has a radius R and is placed in a uniform flow of velocity u, which affects the velocity near to the cylinder. Determine the constants A and B and determine where the maximum velocity occurs.

SOLUTION

The values of the constants depend upon the quadrant selected to solve the boundary conditions. This is because the sign of the tangential velocity and radial velocity are different in each quadrant. Which ever one is used, the final result is the same. Let us select the quadrant from 90° to 180°.

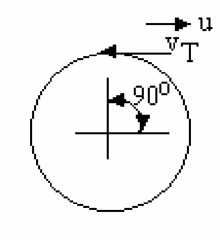
At a large distance from the cylinder and at the 90° position the velocity is from left to right so at this point v_T = -u. From equation 4 we have

$$v_T = \frac{d\phi}{r d\theta}$$

$$\phi = \left\{\frac{B}{r} + Ar\right\}\cos\theta$$

$$v_T = -\frac{1}{r}\left\{\frac{B}{r} + Ar\right\}\sin\theta$$

$$v_T = -\left\{\frac{B}{r^2} + A\right\}\sin\theta$$



Putting r = infinity and θ = 90° and remembering that $+v_T$ is anticlockwise +u is left to right, we have

$$v_T = -u = -\left\{\frac{B}{r^2} + A\right\}\sin\theta = -\{0 + A\}x1$$

Hence v_T = -A = -u so A = u as expected from earlier work.

At angle 180° with r = R, the velocity is only radial in directions and is zero because it is arrested.

From equation 3 we have $$v_R = \frac{d\phi}{dr} = \left(-\frac{B}{r^2} + A\right)\cos\theta$$

Putting r = R and v_R = 0 and θ = 180 we have

$$0 = \left(-\frac{B}{R^2} + A\right)(-1)$$

$$0 = \left(\frac{B}{R^2} - A\right)$$

Put A = u $$0 = \frac{B}{R^2} - u \qquad B = uR^2$$

Substituting for $B = uR^2$ and A = u we have

$$\phi = \left\{\frac{B}{r} + Ar\right\}\cos\theta = \left\{\frac{uR^2}{r} + ur\right\}\cos\theta$$

At the surface of the cylinder r = R the velocity potential is

$$\phi = \{uR + uR\}\cos\theta = 2uR\cos\theta$$

The tangential velocity on the surface of the cylinder is then

$$v_T = \frac{d\phi}{rd\theta} = -\left\{\frac{B}{r^2} + A\right\}\sin\theta$$

$$v_T = -\left\{\frac{uR^2}{r^2} + u\right\}\sin\theta$$

$$v_T = -2u\sin\theta$$

This is a maximum at θ = ± 90° where the streamlines are closest together so the maximum velocity is 2u on the top and bottom of the cylinder.

WORKED EXAMPLE No.3

The potential for flow around a cylinder of radius a is given by

$$f = ux\left[1 + \frac{a^2}{x^2 + y^2}\right]$$

where x and y are the Cartesian co-ordinates with the origin at the middle. Derive an expression for the stream function ψ.

SOLUTION

First convert from Cartesian to polar co-ordinates.

$x^2 + y^2 = r^2$ $\quad\quad$ $x = r\cos\theta$

$$f = ur\cos q\left[1 + \frac{a^2}{r^2}\right]$$

$$v_T = \frac{d\Psi}{dr} = \frac{df}{rdq}$$

$$r\frac{d\Psi}{dr} = \frac{df}{dq} = -u\left[r + \frac{a^2}{r}\right]\sin q$$

$$\frac{d\Psi}{dr} = -u\left[1 + \frac{a^2}{r^2}\right]\sin q$$

$$\Psi = -u\left[r - \frac{a^2}{r}\right]\sin q$$

$$\Psi = ur\left[\frac{a^2}{r^2} - 1\right]\sin q$$

Now change back to Cartesian co-ordinates

$$\Psi = ur\left[\frac{a^2}{x^2 + y^2} - 1\right]\sin q \;= uy\left[\frac{a^2}{x^2 + y^2} - 1\right]$$

8. PRESSURE DISTRIBUTION AROUND

The velocity of the main stream flow is u and the pressure is p'. When it flows over the surface of the cylinder the pressure is p because of the change in velocity. The pressure change is p - p'.

The dynamic pressure for a stream is defined as $\rho u^2/2$

The pressure distribution is usually shown in the dimensionless form

$$2(p - p')/(\rho u^2)$$

For an infinitely long cylinder placed in a stream of mean velocity u we apply Bernoulli's equation between a point well away from the stream and a point on the surface. At the surface the velocity is entirely tangential so :

$$p' + \rho u^2/2 = p + \rho v_T^2/2$$

From the work previous this becomes

$$p' + \rho u^2/2 = p + \rho(2u\sin\theta)^2/2$$

$$p - p' = \rho u^2/2 - (\rho/2)(4u^2\sin^2\theta) = (\rho u^2/2)(1 - 4\sin^2\theta)$$

$$(p - p')/(\rho u^2/2) = 1 - 4\sin^2\theta$$

If this function is plotted against angle we find that the distribution has a maximum value of 1.0 at the front and back, and a minimum value of -3 at the sides.

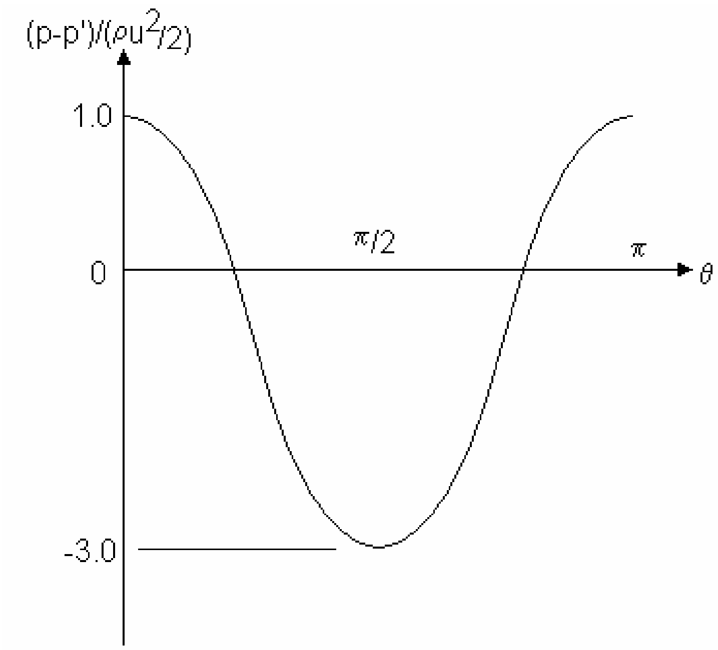


Figure 17

9. THE FLOW OF REAL INCOMPRESSIBLE FLUIDS AROUND A CYLINDER

This is covered in detail in tutorial 3. When the fluid is real, it has viscosity and where it flows over a surface a boundary layer is formed. Remember a boundary layer is the thickness of the layer in which the velocity grows from zero at the surface to a maximum in the main stream.

When the fluid flows around a cylinder, the tangential velocity reaches a theoretical maximum on the top edge. This means the velocity increases around the leading edge. The flow may be laminar or turbulent depending on conditions. If it remains laminar, then the boundary layer gets thinner as shown below. A point may be reached where the layer thickness is reduced to zero and then it actually becomes reversed with eddies forming as shown. At this point the boundary layer separates from the surface and a wake is formed.

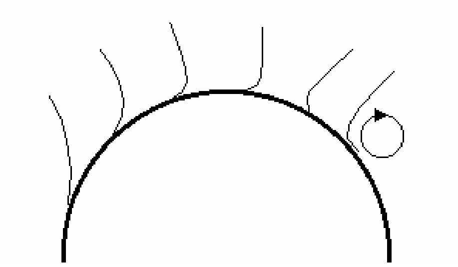


Figure 18

Research shows that the drag coefficient reduces with increased stream velocity and then remains constant when the boundary layer achieves separation. If the mainstream velocity is further increased, turbulent flow sets in around the cylinder and this produces a marked drop in the drag. This is shown below on the graph of C_D against Reynolds's number. The point where the sudden drop occurs is at a critical value of Reynolds's number of 5×10^5.

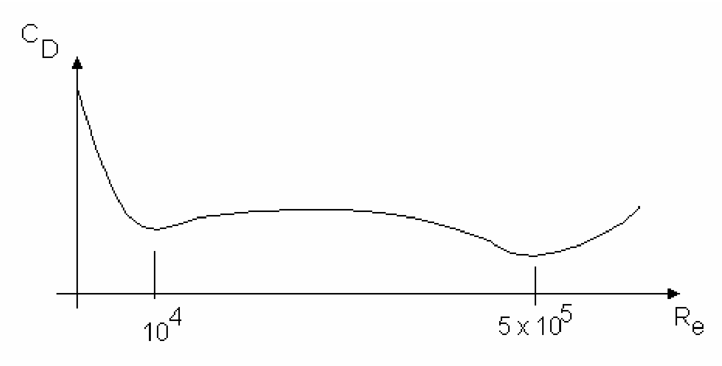


Figure 19

The drag coefficient is defined as : C_D = Drag Force/$\rho A(u^2/2)$

where A is the area normal to the flow in cases such as this.

The student should read up details of boundary layer formation, wakes and separation as this work is only a brief description of what occurs.

WORKED EXAMPLE No.4

Water flows around a cylinder 80 mm radius. At large distances from the cylinder, the velocity is 7.5 m/s in the x direction and the pressure is 1 bar. Find the velocity and pressure at the point x = -90 mm and y = 20 mm. The velocity and stream functions are as given in the last example.

SOLUTION

$v_R = u[R^2/r^2 - 1]\cos\theta$
$v_T = u[1 + R^2/r^2]\sin\theta$
changing the co-ordinates into angle we have
$\theta = \tan^{-1}(y/x) = 167.5^o$

R = 0.08 m u = 7.5 m/s
$r = (x^2 + y^2)^{1/2} = 92.19$ mm

$v_R = 7.5[0.08^2/0.092^2 - 1]\cos 167.5^o$
$v_R = 1.785$ m/s
$v_T = 7.5[1 + 0.08^2/0.092^2]\sin 167.5^o$
$v_T = 2.85$ m/s

The true velocity is the vector sum of these two so
$v = (1.785^2 + 2.85^2)^{1/2} = 3.363$ m/s

Applying Bernoulli between the mainstream flow and this point we have

$1 \times 10^5/\rho g + 7.5^2/2g = p/\rho g + 3.363^2/2g$

p = 122.47 kPa

SELF ASSESSMENT EXCERCISE No.1

1.a. Show that the potential function $\phi = A(r + B/r)\cos\theta$ represents the flow of an ideal fluid around a long cylinder. Evaluate the constants A and B if the cylinder is 40 mm radius and the velocity of the main flow is 3 m/s. (A = 3 m/s and B = 0.0016)

b. Obtain expressions for the tangential and radial velocities and hence the stream function ψ.

c. Evaluate the largest velocity in the directions parallel and perpendicular to the flow direction. (6 m/s for tangential velocity)

d. A small neutrally buoyant particle is released into the stream at r = 100 mm and $\theta = 150^o$. Determine the distance at the closest approach to the cylinder. (66.18 mm)

2.a. Show that the potential function $\phi = (Ar + B/r)\cos\theta$ gives the flow of an ideal fluid around a cylinder. Determine the constants A and B if the velocity of the main stream is u and the cylinder is radius R.

b. Find the pressure distribution around the cylinder expressed in the form $(p - p')/(\rho u^2/2)$ as a function of angle.

c. Sketch the relationship derived above and compare it with the actual pressure profiles that occur up to a Reynolds number of 5×10^5.

3. Show that in the region y>0 the potential function $\phi = a \ln[x^2 + (y-c)^2] + a \ln[x^2 + (y+c)^2]$ gives the 2 dimensional flow pattern associated with a source distance c above a solid flat plane at y=0.

b. Obtain expressions for the velocity adjacent to the plane at y = 0. Find the pressure distribution along this plane.

c. Derive an expression for the stream function ϕ.

4. A uniform flow has a sink placed in it at the origin of the Cartesian co-ordinates. The stream function of the uniform flow and sink are $\psi_1 = Uy$ and $\psi_2 = B\theta$
Write out the combined stream function in Cartesian co-ordinates.

Given U=0.001 m/s and B= -0.04 $m^3/3$ per m thickness, derive the velocity potential.

Determine the width of the flux into the sink at a large distance upstream.
(Ans. 80π m)

10. VORTICES

10.1 CIRCULATION

Consider a stream line that forms a closed loop. The velocity of the streamline at any point is tangential to the radius of curvature R. the radius is rotating at angular velocity ω. Now consider a small length of that streamline ds.

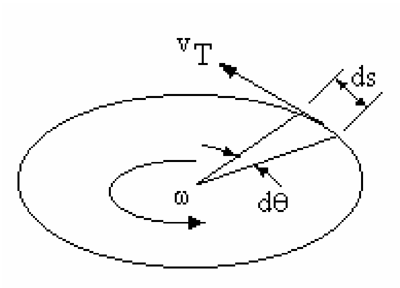


Figure 20

The circulation is defined as $K = \int v_T \, ds$ and the integration is around the entire loop.

Substituting $v_T = \omega R$ $\quad ds = R\, d\theta$

$K = \int \omega R^2 \, d\theta$ The limits are 0 and 2π

$K = 2\pi \omega R^2$

In terms of v_T $\quad K = 2\pi v_T R$

10.2 VORTICITY

Vorticity is defined as $G = \int v_T \, ds/A$ where A is the area of the rotating element.

The area of the element shown in the diagram is a small sector of arc ds and angle dθ.

$$dA = \frac{dq}{2p} pR^2 = R^2 \frac{dq}{2}$$

$$A = \int R^2 \frac{dq}{2}$$

$$G = \frac{\int wR^2 dq}{\int R^2 \frac{dq}{2}} = 2w \quad \text{at any point.}$$

It should be remembered in this simplistic approach that ω may vary with angle.

10.3 VORTICES

Consider a cylindrical mass rotating about a vertical axis. The streamlines form concentric circles. The angular velocity of the streamlines are the same at all radii for a forced vortex , but varies with radius for a free vortex.

Consider a small annular element between two streamlines. The streamlines are so close that the circumference of each is the same and length $2\pi r$. Let the depth be dh, a small part of the actual depth.

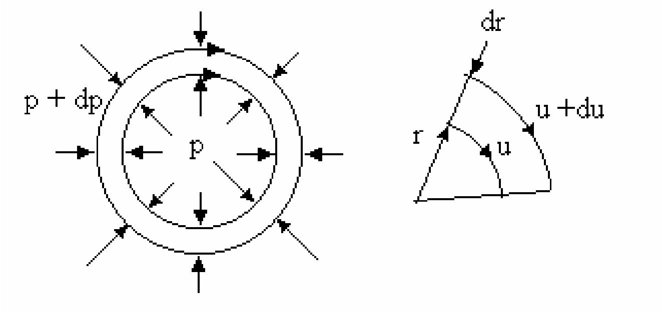


Figure 21

The velocity of the outer streamline is u + du and the inner streamline is u. The pressure at the inner streamline is p and at the outer streamline is p + dp.

The mass of the element is $\rho 2\pi r\, dh\, dr$

The centrifugal force acting on the mass is $\rho 2\pi r\, dh\, dr\, u^2/r$

It must be the centrifugal force acting on the element that gives rise to the change in pressure dp. It follows that

$dp\, 2\pi r\, dh = \rho 2\pi r\, dh\, dr\, u^2/r$ and $dp/\rho = u^2\, dr/r$

Changing pressure into head $dp = \rho g\, dh$ so $dh/dr = u^2/gr$

The kinetic head at the inner streamline is $u^2/2g$

Differentiating w.r.t. radius we get $u\, du/(g\, dr)$

The total energy may be represented as a Head H where H = Total Energy/mg

The rate of change of energy head with radius is dH/dr. It follows that this must be the sum of the rate of change of pressure and kinetic heads so

$dH/dr = u^2/gr + u\, du/(g\, dr)$

10.3.1 FREE VORTEX

A free vortex is one with no energy added nor removed so dH/dr = 0. It is also irrotational which means that although the streamlines are circle and the individual molecules orbit the axis of the vortex, they do not spin. This may be demonstrated practically with a vorticity meter that is a float with a cross on it. The cross can be seen to orbit the axis but not spin as shown.

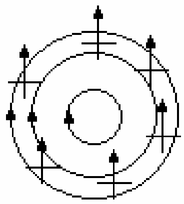


Figure 22

Since the total head H is the same at all radii it follows the dH/dr = 0. The equation reduces to

$$u/r + du/dr = 0$$

$$dr/r + du/v = 0$$

Integrating $\ln u + \ln r = \text{Constant}$

$$\ln (ur) = \text{constant}$$

$$\mathbf{ur = C}$$

Note that a vortex is positive for anti-clockwise rotation. C is the strength of the free vortex with units of m^2/s

10.3.2 STREAM FUNCTION FOR A FREE VORTEX

The tangential velocity was shown to be linked to the stream function by

$$d\Psi = v_T\, dr$$

Substituting $v_T = C/r$

$$d\Psi = C\, dr/r$$

Suppose the vortex has an inner radius of a and an outer radius of R.

$$\Psi = C \int dr/r = C \ln (R/a)$$

10.3.3 VELOCITY POTENTIAL FOR A FREE VORTEX

The velocity potential was defined in the equation $d\phi = v_T\, r\, d\theta$

Substituting $v_T = C/r$ and integrating.

$$\phi = \int (C/r) r\, d\theta$$

Over the limits 0 to θ we have

$$\phi = C\theta$$

10.3.4 SURFACE PROFILE OF A FREE VORTEX

It was shown earlier that $dh/dr = u^2/gr$ where h is the depth. Substituting u=C/r we get

$$dh/dr = C^2/gr^3$$
$$dh = C^2gr^{-3}\ dr/g$$

Integrating between a small radius r and large radius R we get

$$\mathbf{h_2 - h_1 = (C^2/2g)(1/R^2 - 1/r^2)}$$

Plotting h against r produces a shape like this.

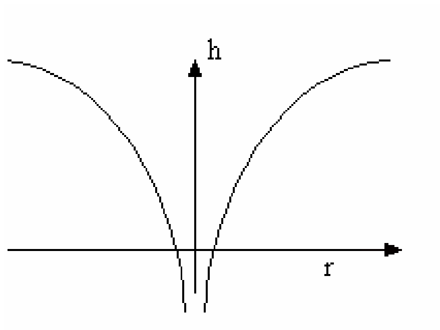


Figure 23

10.3.5 FORCED VORTEX

A forced vortex is one in which the whole cylindrical mass rotates at one angular velocity ω. It was shown earlier that $dH/dr = u^2/gr + u\ du/(g\ dr)$ where h is the depth. Substituting $u = \omega r$ and noting $du/dr = \omega$ we have

$$dH/dr = (\omega r)^2/gr + \omega^2 r/g$$
$$dH/dr = 2\omega^2 r/g$$

Integrating without limits yields

$$H = \omega^2 r^2/g + A$$

H was also given by

$$H = h + u^2/2g = h + \omega^2 r^2/2g$$

Equating we have

$$h = \omega^2 r^2/2g + A$$

At radius r $h_1 = \omega^2 r^2/2g + A$

At radius R $h_2 = \omega^2 R^2/2g + A$

$$\mathbf{h_2 - h_1 = (w^2/2g)(R^2 - r^2)}$$

This produces a parabolic surface profile like this.

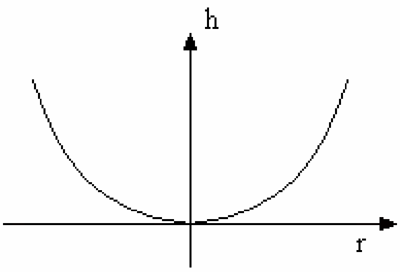


Figure 24

WORKED EXAMPLE No.5

A free vortex of strength C is placed in a uniform flow of velocity u. Derive the stream function and velocity potential for the combined flow.

SOLUTION

The derivation of the stream function and velocity potential for a free vortex is given previously as

$\Psi = C \ln (r/a)$ and $\phi = C\theta$

The corresponding functions for a uniform flow are

$$\Psi = -uy = -ur\sin\theta \text{ and } \phi = ur\cos\theta$$

Combining the functions we get

$$\Psi = C \ln (r/a) - ur\sin\theta$$
$$\phi = C\theta + ur\cos\theta$$

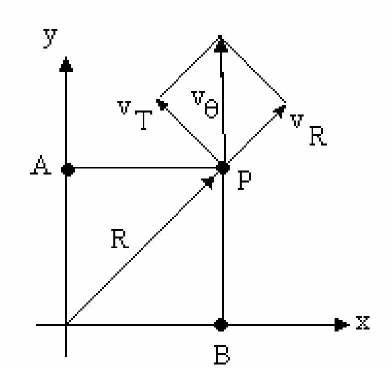
WORKED EXAMPLE No.6

The strength of a free vortex is 2 m^2/s and it is placed in a uniform flow of 3 m/s in the x direction. Calculate the pressure difference between the main stream and a point at x= 0.5 and y=0.5. The density of the fluid is 997 kg/m^3.

SOLUTION

The velocity of the combined flow at this point is v_θ. This the vector sum of the radial and tangential velocities so

$$v_\theta = \{v_T^2 + v_R^2\}^{1/2}$$



C = 2 u = 3

$$v_R = d\phi /dr = u \cos\theta$$
$$v_T = d\Psi /dr = C/r - u \sin\theta$$

At point A $\theta = 90^o$ R = 0.5 $v_R = 0$ hence v_T = 7 m/s and $v_{\theta A}$ = 7m/s

At point B $\theta = 0^o$ R = 0.5 v_R = 3 m/s hence v_T = 4 m/s and $v_{\theta B}$ = 5 m/s
The mainstream pressure is p and the velocity is u.

Apply Bernoulli between the main stream and point A .

$$p + \rho u^2/2 = p_A + \rho v_{\theta A}^2/2$$

Apply Bernoulli between the main stream and point B .

$$p + \rho u^2/2 = p_B + \rho v_{\theta B}^2/2$$

The pressure difference is then

$$p_A - p_B = (\rho/2)\{v_{\theta B}^2 - v_{\theta A}^2\} = -11\ 964 \text{ Pascal}$$

WORKED EXAMPLE 7

A rectangular channel 1 m deep carries 2 m^3/s of water around a 90^o bend with an inner radius of 2 m and outer radius of 4 m. Treating the around the bend as part of a free vortex, determine the difference in levels between the inner and outer edge.

SOLUTION

Free vortex ur = C m^2/s
ψ = C ln (R/r)

ψ = Flux = Flow/depth = 2 m^2/s and this must be the same across any radial line on the bend.

Putting r = 2 m and R = 4 m

C ln (4/2) = 2 hence C = 2.885

ψ = 2.885 ln (R/r)

The surface profile of a free vortex is $h_2 - h_1 = (C^2/2g)(1/r_1^2 - 1/r_2^2)$

Let the inside level of the bend be 0 so h_2 is the change in level over the bend.

$h_2 = \{(2.885)^2/2g)\}(1/2^2 - 1/4^2) = 0.08$ m

SELF ASSESSMENT EXERCISE No.2

1. Define the following terms.

   Stream function.
   Velocity potential function.
   Streamline
   Stream tube
   Circulation
   Vorticity.

2. A free vortex of with circulation $K = 2\pi v_T R$ is placed in a uniform flow of velocity u.

   Derive the stream function and velocity potential for the combined flow.

   The circulation is 7 m^2/s and it is placed in a uniform flow of 3 m/s in the x direction. Calculate the pressure difference between a point at x= 0.5 and y=0.5.

   The density of the fluid is 1000 kg/m^3.

   (Ans. 6695 Pascal)

APPLIED FLUID MECHANICS

TUTORIAL No.6

DIMENSIONAL ANALYSIS

When you have completed this tutorial you should be able to do the following.

- Explain the basic system of dimensions.
- Find the relationship between variables affecting a phenomenon.
- Define and use dimensionless numbers.
- Solve problems by the use of model tests.
- Solve typical exam questions.

1. BASIC DIMENSIONS

All quantities used in engineering can be reduced to six basic dimensions. These are the dimensions of

Mass	M
Length	L
Time	T
Temperature	θ
Electric Current	I
Luminous Intensity	J

The last two are not used in fluid mechanics and temperature is only used sometimes.

All engineering quantities can be defined in terms of the four basic dimensions M,L,T and θ. We could use the S.I. units of kilogrammes, metres, seconds and Kelvins, or any other system of units, but if we stick to M,L,T and θ we free ourselves of any constraints to a particular system of measurements.

Let's now explain the above with an example. Consider the quantity *density*. The S.I. units are kg/m^3 and the imperial units are lb/in^3. In our system the units would be $Mass/Length^3$ or M/L^3. It will be easier in the work ahead if we revert to the inverse indice notation and write it as ML^{-3}.

Other engineering quantities need a little more thought when writing out the basic $MLT\theta$ dimensions. The most important of these is the unit of force or the Newton in the S.I. system. Engineers have opted to define force as that which is needed to accelerate a mass such that 1 N is needed to accelerate 1 kg at 1 m/s^2. From this we find that the Newton is a derived unit equal to 1 kg m/s^2. In our system the dimensions of force become MLT^{-2}. This must be considered when writing down the dimensions of anything containing force.

Another unit that produces problems is that of angle. Angle is a ratio of two sides of a triangle and so has no units nor dimensions at all. This also applies to revolutions which are angular measurements. Strain is also a ratio and has no units nor dimensions. Angle and strain are in fact examples of dimensionless quantities that will be considered in detail later.

WORKED EXAMPLE No. 1

Write down the basic dimensions of pressure p.

SOLUTION

Pressure is defined as p = Force/Area

The S.I. unit of pressure is the Pascal which is the name for $1N/m^2$.
Since force is MLT^{-2} and area is L^2 then the basic dimensions of pressure are

$$ML^{-1}T^{-2}$$

When solving problems it is useful to use a notation to indicate the MLT dimensions of a quantity and in this case we would write

$$[p] = ML^{-1}T^{-2}$$

WORKED EXAMPLE No.2

Deduce the basic dimensions of dynamic viscosity.

SOLUTION

Dynamic viscosity was defined in an earlier tutorial from the formula $\tau = \mu du/dy$

τ is the shear stress, du/dy is the velocity gradient and μ is the dynamic viscosity. From this we have $\mu = \tau\, dy/du$

Shear stress is force/area.
The basic dimensions of force are MLT^{-2}
The basic dimensions of area are L^2.
The basic dimensions of shear stress are $ML^{-1}T^{-2}$.
The basic dimensions of distance y are L.
The basic dimensions of velocity v are LT^{-1}.
It follows that the basic dimension of dy/du (a differential coefficient) is T.
The basic dimensions of dynamic viscosity are hence $(ML^{-1}T^{-2})(T) = ML^{-1}T^{-1}$.

$[\mu] = ML^{-1}T^{-1}$.

2. LIST OF QUANTITIES AND DIMENSIONS FOR REFERENCE.

AREA	$(LENGTH)^2$	L^2
VOLUME	$(LENGTH)^3$	L^3
VELOCITY	LENGTH/TIME	LT^{-1}
ACCELERATION	$LENGTH/(TIME^2)$	LT^{-2}
ROTATIONAL SPEED	REVOLUTIONS/TIME	T^{-1}
FREQUENCY	CYCLES/TIME	T^{-1}
ANGULAR VELOCITY	ANGLE/TIME	T^{-1}
ANGULAR ACCELERATION	$ANGLE/(TIME)^2$	T^{-2}
FORCE	MASS X ACCELERATION	MLT^{-2}
ENERGY	FORCE X DISTANCE	ML^2T^{-2}
POWER	ENERGY/TIME	ML^2T^{-3}
DENSITY	MASS/VOLUME	ML^{-3}
DYNAMIC VISCOSITY	STRESS/VELOCITY GRADIENT	$ML^{-1}T^{-1}$
KINEMATIC VISCOSITY	DYN. VISCOSITY/DENSITY	L^2T^{-1}
PRESSURE	FORCE/AREA	$ML^{-1}T^{-2}$
SPECIFIC HEAT CAPACITY	ENERGY/(MASS X TEMP)	$L^2T^{-2}\theta^{-1}$
TORQUE	FORCE X LENGTH	ML^2T^{-2}
BULK MODULUS	PRESSURE/STRAIN	$ML^{-1}T^{-2}$

3. HOMOGENEOUS EQUATIONS

All equations must be homogeneous. Consider the equation $F = 3 + T/R$
F is force, T is torque and R is radius. Rearranging we have $3 = F - T/R$

Examine the units. F is Newton. T is Newton metre and R is metre.

hence $3 = F\ (N) - T/R\ (N\ m)/m)$
$3 = F(N) - T/R\ (N)$

It follows that the number 3 must represent 3 Newton. It also follows that the unit of F and T/R must both be Newton. If this was not so, the equation would be nonsense. In other words all the components of an equation that add together must have the same units. You cannot add dissimilar quantities. For example you cannot say that 5 apples + 6 pears = 11 plums. This is clearly nonsense. When all parts of an equation that add together have the same dimensions, then the equation is homogeneous.

WORKED EXAMPLE No.3

Show that the equation Power = Force x velocity is homogeneous in both S.I. units and basic dimensions.

SOLUTION

The equation to be checked is $P = F v$

The S.I. Unit of power (P) is the Watt. The Watt is a Joule per second. A Joule is a Newton metre of energy. Hence a Watt is 1 N m/s.

The S.I. unit of force (F) is the Newton and of velocity (v) is the metre/second.

The units of F v are hence N m/s.

It follows that both sides of the equation have S.I. units of N m/s so the equation is homogeneous.

Writing out the MLT dimensions of each term we have

$$[P] = ML^2T^{-3}$$
$$[v] = LT^{-1}$$
$$[F] = MLT^{-2}$$

Substituting into the equation we have $ML^2T^{-3} = MLT^{-2}\ LT^{-1} = ML^2T^{-3}$

Hence the equation is homogeneous.

4. INDECIAL EQUATIONS

When a phenomenon occurs, such as a swinging pendulum as shown in figure 3.44we observe the variables that effect each other. In this case we observe that the frequency, (f) of the pendulum is affected by the length (l) and the value of gravity (g). We may say that frequency is a function of l and g. In equation form this is as follows.

$$f = \phi(l,g)$$ where ϕ is the function sign.

When we remove the function sign we must put in a constant because there is an unknown number and we must allocate unknown indices to l and g because we do know not what if any they are. The equation is written as follows.

$$f = C\, l^a g^b$$

C is a constant and has no units. a and b are unknown indices.

This form of relating variables is called an indecial equation. The important point here is that because we know the units or dimensions of all the variables, we can solve the unknown indices.

WORKED EXAMPLE No.4

Solve the relationship between f, l and g for the simple pendulum.

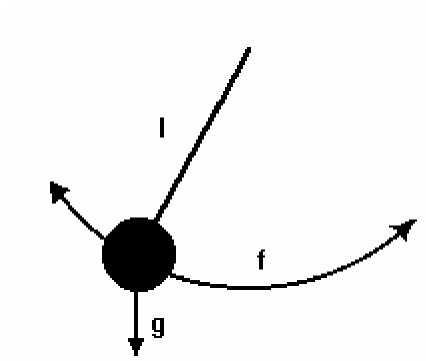


Fig.1

SOLUTION

First write down the indecial form of the equation (covered overleaf).

$f = C\, l^a\, g^b$

Next write down the basic dimensions of all the variables.

$[f] = T^{-1}$

$[l] = L^1$

$[g] = LT^{-2}$

Next substitute the dimensions in place of the variables.

$$T^{-1} = (L^1)^a\,(LT^{-2})^b$$

Next tidy up the equation. $T^{-1} = L^{1a}\, L^b T^{-2b}$

Since the equation must be homogeneous then the power of each dimension must be the same on the left and right side of the equation. If a dimension does not appear at all then it is implied that it exists to the power of zero. We may write them in until we get use to it. The equation is written as follows.

$$M^0 L^0 T^{-1} = L^{1a}\, LT^{-2b} M^0$$

Next we equate powers of each dimension. First equate powers of Time.

$T^{-1} = T^{-2b}$

$-1 = -2b$

$b = 1/2$

Next equate powers of Length.

$L^0 = L^{1a}\, L^b$

$0 = 1a + b$ hence $a = -b = -1/2$

$M^0 = M^0$ yields nothing in this case.

Now substitute the values of a and b back into the original equation and we have the following.

$f = C\, l^{-½}\, g^{½}$

$f = C\, (g/l)^{½}$

The frequency of a pendulum may be derived from basic mechanics and shown to be

$f = (1/2\pi)(g/l)^{½}$

If we did not know how to find C = (½ π) from basic mechanics, then we know that if we conducted an experiment and measured the values f for various values of l and g, we could find C by plotting a graph of f against $(g/l)^{½}$. This is the importance of dimensional analysis to fluid mechanics. We are able to determine the basic relationships and then conduct experiments and determine the remaining unknown constants. We are able to plot graphs because we know what to plot against what.

SELF ASSESSMENT EXERCISE No.1

1. It is observed that the velocity 'v' of a liquid leaving a nozzle depends upon the pressure drop 'p' and the density 'ρ'. Show that the relationship between them is of the form

$$v = C\left(\frac{p}{\rho}\right)^{\frac{1}{2}}$$

2. It is observed that the speed of a sound in 'a' in a liquid depends upon the density 'ρ' and the bulk modulus 'K'. Show that the relationship between them is

$$a = C\left(\frac{K}{\rho}\right)^{\frac{1}{2}}$$

3. It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'. Show that the relationship between them is

$$f = C\left(\frac{F}{ml}\right)^{\frac{1}{2}}$$

5. DIMENSIONLESS NUMBERS

We will now consider cases where the number of unknown indices to be solved, exceed the number of equations to solve them. This leads into the use of dimensionless numbers.

Consider that typically a problem uses only the three dimensions M, L and T. This will yield 3 simultaneous equations in the solution. If the number of variables in the equation gives 4 indices say a, b, c and d, then one of them cannot be resolved and the others may only be found in terms of it.

In general there are n unknown indices and m variables. There will be m-n unknown indices. This is best shown through a worked example.

WORKED EXAMPLE No. 5

The pressure drop per unit length 'p' due to friction in a pipe depends upon the diameter 'D' , the mean velocity 'v' , the density 'ρ' and the dynamic viscosity 'μ'. Find the relationship between these variables.

SOLUTION

$p = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

p is pressure per metre

$[p] = ML^{-2}T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

$$ML^{-2}T^{-2} = L^a\,(LT^{-1})^b\,(ML^{-3})^c(ML^{-1}T^{-1})^d$$

$$ML^{-2}T^{-2} = L^{a+b-3c-d}\,M^{c+d}\,T^{-b-d}$$

The problem is now deciding which index not to solve. The best way is to use experience gained from doing problems. Viscosity is the quantity that causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a, b and c in terms of d.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve

MASS $1 = c + d$ hence $c = 1 - d$

LENGTH $-2 = a + b - 3c - d$

$-2 = a + (2 - d) - 3(1 - d) - d$ hence $a = -1 - d$

Next put these back into the original formula.

$$p = K\ D^{-1-d}\ v^{2-d}\ \rho^{1-d}\ \mu^{d}$$

Next group the quantities with same power together as follows :

$$p = K\{\rho v^2 D^{-1}\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

Remember that p was pressure drop per unit length so the pressure loss over a length L is

$$P = K\ L\{\rho\ v^2 D^{-1}\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

We have two unknown constants K and d. The usefulness of dimensional analysis is that it tells us the form of the equation so we can deduce how to present experimental data. With suitable experiments we could now find K and d.

Note that this equation matches up with Poiseuille's equation which gives the relationship as :

$$p = 32\ \mu\ L\ v\ D^{-2}$$

It may be deduced that K = 32 and d = 1 (laminar flow only)

The term $\{\rho v D \mu^{-1}\}$ has no units. If you check it out all the units will cancel. This is a ***DIMENSIONLESS NUMBER***, and it is named after Reynolds.

Reynolds Number is denoted R_e.The whole equation can be put into a dimensionless form as follows.

$$\{p\ \rho^{-1}\ L^{-1}\ v^{-2} D^{1}\} = K\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

$$\{p\ \rho^{-1}\ L^{-1}\ v^{-2} D^{1}\} = \text{function}\ (R_e)$$

This is a dimensionless equation. The term $\{p\rho^{-1}\ L^{-1} v^{-2} D^{1}\}$ is also a dimensionless number.

Let us now examine another similar problem.

WORKED EXAMPLE No.6

Consider a sphere moving through an viscous fluid completely submerged. The resistance to motion R depends upon the diameter D, the velocity v, the density ρ and the dynamic viscosity μ. Find the equation that relates the variables.

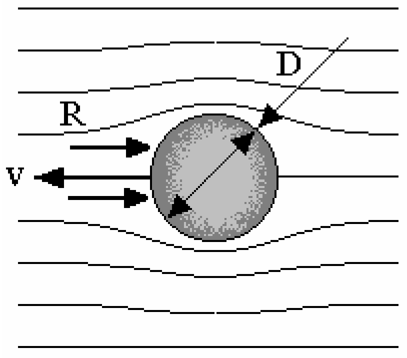


Fig.2

SOLUTION

$R = \text{function}(D\ v\ \rho\ \mu) = K\ D^a\ v^b\ \rho^c\ \mu^d$

First write out the MLT dimensions.

$[R] = ML^1T^{-2}$

$[D] = L$ $\quad ML^1T^{-2} = L^a\,(LT^{-1})^b\,(ML^{-3})^c(ML^{-1}T^{-1})^d$

$[v] = LT^{-1}$ $\quad ML^1T^{-2} = L^{a+b-3c-d}\ M^{c+d}\ T^{-b-d}$

$[\rho] = ML^{-3}$

$[\mu] = ML^{-1}T^{-1}$

Viscosity is the quantity which causes viscous friction so the index associated with it (d) is the one to identify. We will resolve a,b and c in terms of d as before.

TIME $-2 = -b - d$ hence $b = 2 - d$ is as far as we can resolve b

MASS $1 = c + d$ hence $c = 1 - d$

LENGTH $1 = a + b - 3c - d$

$1 = a + (2 - d) - 3(1-d) - d$ hence $a = 2-d$

Next put these back into the original formula.

$$R = K\ D^{2-d}\ v^{2-d}\ \rho^{1-d}\ \mu^d$$

Next group the quantities with same power together as follows :

$$R= K\{\rho v^2 D^2\}\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$
$$R\{\rho v^2 D^2\}^{-1} = K\ \{\mu\ \rho^{-1} v^{-1} D^{-1}\}^d$$

The term $\{\rho v D \mu^{-1}\}$ is the Reynolds Number R_e and the term $R\{\rho v^2 D^2\}^{-1}$ is called the Newton Number N_e. Hence the relationship between the variables may be written as follows.

$$R\{\rho v^2 D^2\}^{-1} = \text{function } \{\rho v D\ \mu^{-1}\}$$
$$N_e = \text{function } (R_e)$$

Once the basic relationship between the variables has been determined, experiments can be conducted to find the parameters in the equation. For the case of the sphere in an incompressible fluid we have shown that

$$N_e = \text{function } (R_e)$$

Or put another way $\quad N_e = K\ (R_e)^n$

where K is a constant of proportionality and n is an unknown index (equivalent to -d in the earlier lines). In logarithmic form the equation is

$$\log(N_e) = \log(K) + n \log(R_e)$$

This is a straight line graph from which log K and n are taken. Without dimensional analysis we would not have known how to present the information and plot it. The procedure now would be to conduct an experiment and plot log(Ne) against log(Re). From the graph we would then determine K and n.

6. BUCKINGHAM'S Π (Pi) THEORY

Many people prefer to find the dimensionless numbers by intuitive methods. Buckingham's theory is based on the knowledge that if there are m basic dimensions and n variables, then there are m - n dimensionless numbers. Consider worked example No.6 again. We had the basic equation

$$R = \text{function}(D\ v\ \rho\ \mu)$$

There are 5 quantities and there will be 3 basic dimensions ML and T. This means that there will be 2 dimensionless numbers Π_1 and Π_2. These numbers are found by choosing two prime quantities (R and μ).

Π_1 is the group formed between μ and D v ρ

Π_2 is the group formed between R and D v ρ

First taking μ. Experience tells us that this will be the Reynolds number but suppose we don't know this.

The dimensions of μ are $ML^{-1}T^{-1}$

The dimensions of D v ρ must be arranged to be the same.

$$\mu = \Pi_1\ D^a v^b\ \rho^c$$

$$M^1L^{-1}T^{-1} = \Pi_1\ (L)^a\ (LT^{-1})^b\ (ML^{-3})^c$$

Time	**-1 = -b**	**b = 1**
Mass		**c = 1**
Length	-1 = a + b -3c	
	-1 = a + 1 – 3	a=1

$$\mu = \Pi_1\ D^1\ v^1\ \rho^1 \qquad \Pi_1 = \frac{\mu}{Dv\rho}$$

The second number must be formed by combining R with ρ,v and D

$$R = \Pi_2\ D^a\ v^b\ \rho^c$$

$$MLT^{-2} = \Pi_2\ (L)^a\ (LT^{-1})^b (ML^{-3})^c$$

Time	**-2 = -b**	**b = 2**
Mass		**c = 1**
Length	**1 = a + b -3c**	
	1 = a + 2 – 3	**a = 2**

$$R = \Pi_2\ D^2\ v^2\ \rho^1 \qquad \Pi_2 = \frac{R}{\rho v^2 D^2}$$

The dimensionless equation is $\mathbf{\Pi_2 = f(\Pi_1)}$

WORKED EXAMPLE No.7

The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' through a compressible fluid is dependant upon the density 'ρ' and the bulk modulus 'K'. The resistance is primarily due to the compression of the fluid in front of the sphere. Show that the dimensionless relationship between these quantities is N_e = function (M_a)

SOLUTION

$R = \text{function}(D\ v\ \rho\ K) = C\ D^a\ v^b\ \rho^c\ K^d$

There are 3 dimensions and 5 quantities so there will be 5 –3 = 2 dimensionless numbers. Identify that the one dimensionless group will be formed with R and the other with K.

Π_1 is the group formed between K and D v ρ

Π_2 is the group formed between R and D v ρ

$K = \Pi_2 D^a\ v^b \rho^c$

$[K] = ML^{-1}\ T^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$ML^{-1}T^{-2} = L^a\ (LT^{-1})^b\ \ (ML^{-3})^c$

$ML^{-1}T^{-2} = L^{a+b-3c}\ M^c\ \ T^{-b}$

Time -2 = -b **b = 2**
Mass **c = 1**
Length -1 = a + b -3c
-1 = a + 2 – 3 **a = 0**

$K = \Pi_2\ \ D^o\ v^2\ \ \rho^{\ 1}$

$$\Pi_2 = \frac{K}{\rho v^2}$$

$R = \Pi_1 D^a\ v^b \rho^c$

$[R] = MLT^{-2}$

$[D] = L$

$[v] = LT^{-1}$

$[\rho] = ML^{-3}$

$MLT^{-2} = L^a\ (LT^{-1})^b\ \ (ML^{-3})^c$

$ML^1T^{-2} = L^{a+b-3c}\ M^c\ \ T^{-b}$

Time -2 = -b **b = 2**
Mass **c = 1**
Length **1 = a + b -3c**
1 = a + 2 – 3 **a = 2**

$R = \Pi_1\ \ D^2\ v^2\ \ \rho^{\ 1}$

$$\Pi_1 = \frac{R}{\rho v^2 D^2}$$

It was shown earlier that the speed of sound in an elastic medium is given by the following formula. $a = (k/\rho)^{½}$

It follows that $(k/\rho) = a^2$ and so $\Pi_2 = (a/v)^2$

The ratio v/a is called the Mach number (Ma) so $(Ma)^{-2}$

Π_1 is the Newton Number Ne.

The equation may be written as $\Pi_1 = \phi\Pi_2\ N_e$ or $Ne = \phi(M_a)$

SELF ASSESSMENT EXERCISE No.2

1. The resistance to motion 'R' for a sphere of diameter 'D' moving at constant velocity 'v' on the surface of a liquid is due to the density 'ρ' and the surface waves produced by the acceleration of gravity 'g'. Show that the dimensionless equation linking these quantities is N_e = function(F_r)

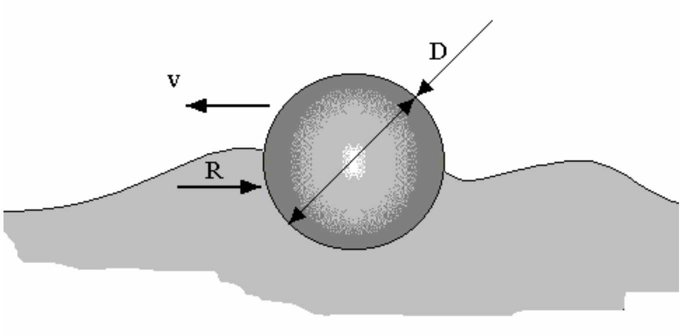


Fig. 3

F_r is the Froude number and is given by $F_r = \sqrt{\frac{v^2}{gD}}$

Here is a useful tip. It is the power of g that cannot be found.

2. The Torque 'T' required to rotate a disc in a viscous fluid depends upon the diameter 'D' , the speed of rotation 'N' the density 'ρ' and the dynamic viscosity 'μ'. Show that the dimensionless equation linking these quantities is :

$$\{T\, D^{-5}\, N^{-2}\ \rho^{-1}\} = \text{function}\ \{\rho\, N\, D^{2}\ \ \mu^{-1}\}$$

MORE DIFFICULT PROBLEMS

The problems so far seen have one unknown index in the solution. When there are two (or more) unknown indexes, the procedure is the same as before. A group of quantities must be formed for each unknown index left in the penultimate part of the solution.

SELF ASSESSMENT EXERCISE No.3

1. The resistance to motion 'R' of a sphere travelling through a fluid which is both viscous and compressible, depends upon the diameter 'D' , the velocity 'v' , the density 'ρ' , the dynamic viscosity 'μ' and the bulk modulus 'K'. Show that the complete relationship between these quantities is :

$$N_e = \text{function}\{R_e\}\ \{M_a\}$$

where $N_e = R\ \rho^{-1}\ v^{-2}\ D^{-2}$

$$R_e = \rho\, v\ D\, \mu^{-1}$$

$$M_a = v/a \quad \text{and} \quad a = (k/\rho)^{0.5}$$

7 MODEL TESTING

When we test a model in order to predict the performance of the real thing, the results are only valid when the forces acting on the model are in the same ratio to each other as they are on the real thing. When this occurs we have DYNAMIC SIMILARITY.

It will be shown that in order to have dynamic similarity, the model must also be a true scale model, in other words we must have GEOMETRIC SIMILARITY.

7.1 DYNAMIC SIMILARITY

We have already seen that certain dimensionless numbers occur in problems of fluid mechanics. Each of these is associated with a particular kind of force.

The Newton Number N_e is associated with total resistance.

The Reynolds Number R_e is associated with viscous resistance.

The Mach Number M_a is associated with compression wave resistance.

The Froude Number F_r is associated with surface wave resistance.

There are others and all dimensionless numbers can take various forms. In order to obtain dynamic similarity, these dimensionless numbers must have the same values on the model and the real thing. Consider for example the resistance to motion of a sphere due to viscosity and compressibility of the fluid. The dimensionless equation is:

$$N_e = \phi(R_e)(M_a)$$

To ensure that the viscous, compression and resistance forces are in the same ratio to each other on the model and on the object, then the three numbers must be the same on both. This is often difficult or impossible to obtain when there are more than three numbers for reasons which will become apparent.

7.2 GEOMETRIC SIMILARITY

In much of the forgoing work, the work has been about a sphere of diameter D so that only one actual length dimension was needed to define both the shape and size of the object. If we tried the same analysis for a submarine or an aeroplane, we should include all the linear dimensions necessary to define the shape and this would be enormous. Consider the following problem that needs two linear dimensions and it is the one we looked at previously in a slightly different way.

The pressure drop p in a pipe depends upon the diameter D, the length l, the density ρ and the viscosity μ. Dimensional analysis shows that :

$$\frac{p}{\rho v^2} = \phi\left(\frac{l}{D}\right)\left(\frac{\rho v D}{\mu}\right)$$

$p/(\rho v^2)$ is a form of the Newton number and $(\rho vD/\mu)$ is a form of the Reynolds number. It could have been arranged for Reynolds number to include l instead of D.

Because we needed two linear dimensions D and l, we now have another dimensionless number (l/D) that is the ratio of the two. In a model test this must be made the same as for the object and if the ratio is the same then geometric similarity exists.

If many such linear dimensions exist in a problem, then many dimensionless numbers will be created which are all the possible ratios of any one with all the others. To avoid all this work, we usually just assume a characteristic length. This is valid when geometric similarity exists as will become apparent.

We may express our equation as :

$$Ne = \phi\left(\frac{l}{d}\right)(\mathrm{Re})$$

Removing the function sign gives :

$$Ne = K\left(\frac{l}{d}\right)(\mathrm{Re})^n$$ where K is the constant of proportionality.

If we make the value of R_e the same on the model and the real object and if we have geometric similarity , then since the function is the same for both (K and n) then it follows that the Newton number must be the same also. In other words since

$$Ne_{object} = K\left(\frac{l}{d}\right)(\mathrm{Re})^n{}_{object} = Ne_{\mathrm{mod}\,el} = K\left(\frac{l}{d}\right)(\mathrm{Re})^n{}_{\mathrm{mod}\,el}$$

Then $\{\mathrm{Ne}\}_{\mathrm{object}} = \{\mathrm{Ne}\}_{\mathrm{model}}$

From this the resisting force may be predicted. Note that if we had many linear dimensions and many ratios like l/D, then they would also cancel so it is not necessary to include them, just a characteristic length. Let us finish this problem now as a worked example.

WORKED EXAMPLE No.8

The pipe in the previous analysis is 200 m long and 0.5 m diameter and must carry water with a mean velocity of 0.2 m/s. In order to predict the pressure drop, a model is made to a scale of 1/10. Calculate the velocity at which water must flow in the model in order to obtain dynamic similarity.

SOLUTION

For this section we must obtain dynamic similarity by equating the Reynolds numbers. Hence :

$$(\rho vD/\mu)_{model} = (\rho vD/\mu)_{object}$$

The density and viscosity will be the same in both since the same water is used so

$$(vD)_{model} = (vD)_{object}$$

$$v_{model} \times D/10 = 2 \times D \quad \text{hence } v_{model} = 2 \text{ m/s}$$

When the model is tested at the velocity, the pressure drop is found to be 100kPa. Predict the pressure drop in the real pipe.

Since R_e is now the same and l/D is the same for both cases then the Newton number is the same so

$$p/(\rho v^2)_{model} = p/(\rho v^2)_{pipe}$$

Again density and viscosity cancel so we have

$$100/2^2 = p/0.2^2$$

p = 1 kPa on the full size pipe.

WORKED EXAMPLE No.9a

The resistance to motion R of a hydrofoil depends upon the characteristic length l, the velocity v, the density ρ and the acceleration of gravity g.

It may be shown that $N_e = f(F_r)$ where $Ne = R/(\rho v^2 l^2)$ and $F_r = v/(gl)^{1/2}$

In order to predict the resistance of a hydrofoil, a model is made to a scale of 1/20. The actual hydrofoil must move at 0.8 m/s over water. Calculate the velocity of the model that gives dynamic similarity on the same water.

SOLUTION

For dynamic similarity the Froude numbers must be made the same.

$v/(gl)^{1/2}$ model $= v/(gl)^{1/2}$(hydrofoil)

$v/(l)^{1/2}$ model $= v/(l)^{1/2}$(hydrofoil)

$v_{model} \times (20/l)^{1/2} = 0.8 / l^{1/2}$

$v_{model} = 0.8 / 20^{1/2}$

$v_{model} = 0.179$ m/s

WORKED EXAMPLE No.9b

The model is tested at 0.179 m/s and the resistance to motion was found to be 2.2N. Predict the resistance of the hydrofoil at 0.8 m/s.

SOLUTION

Since the Froude number is the same and the function is the same then the Newton number must be the same for both.

$R/(\rho v^2 l^2)_{model} = R/(\rho v^2 l^2)_{hydrofoil}$

Since the density is the same then $\{2.2 \times 20^2/(0.179\ l)^2\} = \{R/(vl)^2$

R = 17 570 N

SELF ASSESSMENT EXERCISE No.4

1. (a) The viscous torque produced on a disc rotating in a liquid depends upon the characteristic dimension D, the speed of rotation N, the density ρ and the dynamic viscosity μ. Show that :

$$\{T/(\rho N^2 D^5)\} = f(\rho N D^2/\mu)$$

(b) In order to predict the torque on a disc 0.5 m diameter which rotates in oil at 200 rev/min, a model is made to a scale of 1/5. The model is rotated in water. Calculate the speed of rotation for the model which produces dynamic similarity.

For the oil the density is 750 kg/m^3 and the dynamic viscosity is 0.2 Ns/m^2.

For water the density is 1000 kg/m^3 and the dynamic viscosity is 0.001 Ns/m^2.

(The answer is 18.75 rev/min)

(c) When the model is tested at 18.75 rev/min the torque was 0.02 Nm. Predict the torque on the full size disc at 200 rev/min. (Ans 5 333 N)

2. The resistance to motion of a submarine due to viscous resistance is given by :

$\{R/(\rho v^2 D^2)\} = f(\rho v D/\mu)$ where D is the characteristic dimension.

The submarine moves at 8 m/s through sea water. In order to predict its resistance, a model is made to a scale of 1/100 and tested in fresh water. Determine the velocity at which the model should be tested. (690.7 m/s)

The density of sea water is 1036 kg/m^3

The density of fresh water is 1000 kg/m^3

The viscosity of sea water is 0.0012 N s/m^2.

The viscosity of fresh water is 0.001 N s/m^2.

When run at the calculated speed, the model resistance was 200 N. Predict the resistance of the submarine. (278 N).

3. The resistance of an aeroplane is due to, viscosity and compressibility of the fluid. Show that:

$$\{R/(\rho v^2 D^2)\} = f(M_a)(R_e)$$

An aeroplane is to fly at an altitude of 30 km at Mach 2.0. A model is to be made to a suitable scale and tested at a suitable velocity at ground level. Determine the velocity of the model that gives dynamic similarity for the Mach number and then using this velocity determine the scale which makes dynamic similarity in the Reynolds number. (680.6 m/s and 1/61.86)

The properties of air are

sea level	a= 340.3 m/s	μ= 1.7897 x 10^{-5}	ρ = 1.225 kg/m^3
30 km	a= 301.7 m/s	μ= 1.4745 x 10^{-5}	ρ = 0.0184 kg/m^3

When built and tested at the correct speed, the resistance of the model was 50 N. Predict the resistance of the aeroplane. (2 259 N).

4. The force on a body of length 3 m placed in an air stream at 1 bar and moving at 60 m/s is to be found by testing a scale model. The model is 0.3 m long and placed in high pressure air moving at 30 m/s. Assuming the same temperature and viscosity, determine the air pressure which produced dynamic similarity.

The force on the model is found to be 500 N. Predict the force on the actual body. (Ans. 20 bar and 10 kN).

5. Show by dimensional analysis that the velocity profile near the wall of a pipe containing turbulent flow is of the form $u^+ = f(y^+)$

where $u^+ = u(\rho/\tau_o)^{1/2}$ and $y^+ = y(\rho\tau_o)^{1/2}/\mu$

When water flows through a smooth walled pipe 60 mm bore diameter at 0.8 m/s, the velocity profile is $u^+ = 2.5\ln(y^+) + 5.5$

Find the velocity 10 mm from the wall.
The friction coefficient is $C_f = 0.079\, R_e^{-0.25}$.

Answer 0.85 m/s

FLUID MECHANICS

TUTORIAL No.7

FLUID FORCES

When you have completed this tutorial you should be able to

- Solve forces due to pressure difference.
- Solve problems due to momentum changes.
- Solve problems involving pressure and momentum changes.
- Solve forces on pipe bends.
- Solve problems on stationary vanes.
- Construct blade vector diagrams for moving vanes.
- Calculate the momentum changes over a moving vane.

Let's start by examining forces due to pressure changes.

1. PRESSURE FORCES

Consider a duct as shown in fig.1. First identify the control volume on which to conduct a force balance. The inner passage is filled with fluid with pressure p_1 at inlet and p_2 at outlet. There will be forces on the outer surface of the volume due to atmospheric pressure. If the pressures of the fluid are measured relative to atmosphere (i.e. use gauge pressures) then these forces need not be calculated and the resultant force on the volume is due to that of the fluid only. The approach to be used here is to find the forces in both the x and y directions and then combine them to find the resultant force.

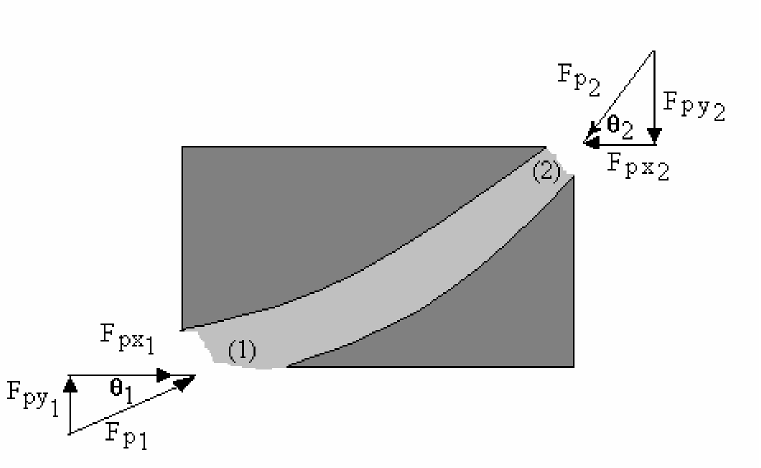


Fig. 1

The force normal to the plane of the bore is pA.

At the inlet (1) the force is $F_{p_1} = p_1 A_1$

At the outlet (2) the force is $F_{p_2} = p_2 A_2$

These forces must be resolved vertically and horizontally to give the following.

$F_{px1} = F_{p1} \cos \theta_1$ (to the right)

$F_{px2} = F_{p1} \cos \theta_2$ (to the left)

The total horizontal force $F_H = F_{px_1} - F_{px_2}$

$F_{py_1} = F_{p_1} \sin \theta_1$ (up)

$F_{py_2} = F_{p_2} \sin \theta_2$ (down)

The total vertical force $F_V = F_{py_1} - F_{py_2}$

WORKED EXAMPLE No. 1

A nozzle has an inlet area of 0.005 m^2 and it discharges into the atmosphere. The inlet gauge pressure is 3 bar. Calculate the resultant force on the nozzle.

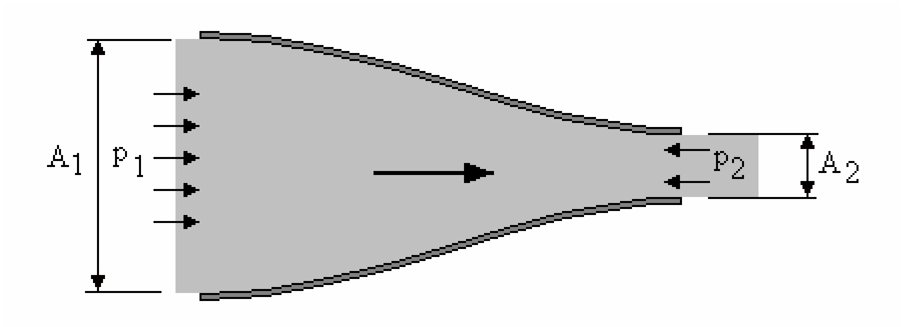


Fig.2

SOLUTION

Since the areas are only in the vertical plane, there is no vertical force. $F_V = 0$

Using gauge pressures, the pressure force at exit is zero. $F_{px2} = 0$

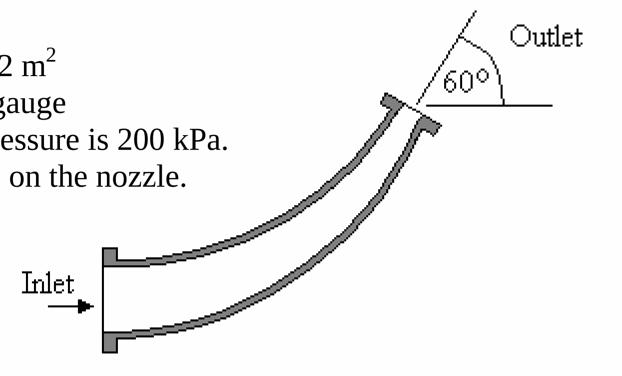
$F_{px1} = 3 \times 10^5 \times 0.005 = 1500$ N

$F_H = 1500 - 0 = 1500$ N to the right.

WORKED EXAMPLE No.2

The nozzle shown has an inlet area of 0.002 m^2 and an outlet area of 0.0005 m^2. The inlet gauge pressure is 300 kPa and the outlet gauge pressure is 200 kPa. Calculate the horizontal and vertical forces on the nozzle.

Fig.3



SOLUTION

$Fp_1 = 300 \times 10^3 \times 0.002 = 600$ N

$Fpx_1 = 600$ N

$Fpy_1 = 0$ N since the plane is vertical.

$Fp_2 = 200 \times 10^3 \times 0.0005 = 100$ N

$Fpx_2 = 100 \times \cos 60^o = 50$ N

$Fpy_2 = 100 \times \sin 60^o = 86.67$ N

Total Horizontal force $F_H = 600 - 50 = 550$ N

Total vertical force $F_V = 0 - 86.67$ N $= - 86.67$ N

2. MOMENTUM FORCES

When a fluid speeds up or slows down, inertial forces come into play. Such forces may be produced by either a change in the magnitude or the direction of the velocity since either change in this vector quantity produces acceleration.

For this section, we will ignore pressure forces and just study the forces due to velocity changes.

2.1 NEWTON'S 2nd LAW OF MOTION

This states that ***the change in momentum of a mass is equal to the impulse given to it.***

Impulse = Force x time
Momentum = mass x velocity
Change in momentum = Δmv

Newton's second law may be written as $\Delta mv = Ft$

Rearrange to make F the subject. $\Delta mv/t = F$

Since $\Delta v/t$ = acceleration 'a' we get the usual form of the law $F = ma$

The mass flow rate is m/t and at any given moment this is dm/dt or m' and for a constant flow rate, only the velocity changes.

In fluids we usually express the second law in the following form. $F = (m/t)\,\Delta v = m'\Delta v$
$m'\Delta v$ is the rate of change of momentum so the second law may be restated as

F = Rate of change of momentum

F is the impulsive force resulting from the change. Δv is a vector quantity.

2.2 APPLICATION TO PIPE BENDS

Consider a pipe bend as before and use the idea of a control volume.

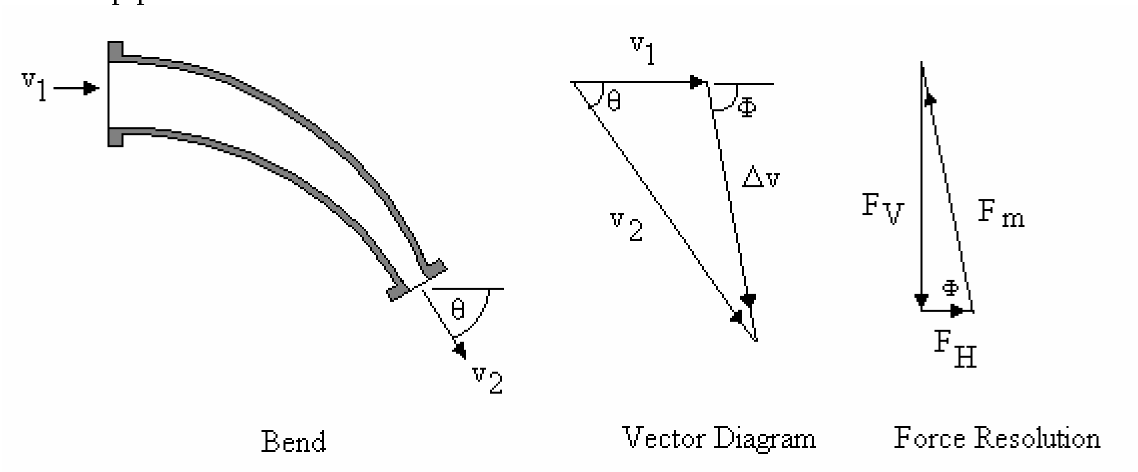


Fig.4

First find the vector change in velocity using trigonometry.

$$\tan\Phi = \frac{v_2\sin\theta}{v_2\cos\theta - v_1} \qquad \Delta v = \left\{(v_2\sin\theta)^2 + (v_2\cos\theta - v_1)^2\right\}^{\frac{1}{2}}$$

Alternatively Δv could be found by drawing the diagram to scale and measuring it.

If we had no change in magnitude then $v_1 = v_2 = v$ then **$\Delta v = v\,\{2(1 - \cos\theta)\}^{½}$**

The momentum force acting on the fluid is $F_m = m'\Delta v$

The force is a vector quantity which must be in the direction of Δv. Every force has an equal and opposite reaction so there must be a force on the bend equal and opposite to the force on the fluid. This force could be resolved vertically and horizontally such that

$F_H = F_m\cos\Phi$ and $F_V = F_m\sin\Phi$

This theory may be applied to turbines and pump blade theory as well as to pipe bends.

SELF ASSESSMENT EXERCISE No.1

1. A pipe bends through an angle of 90° in the vertical plane. At the inlet it has a cross sectional area of 0.003 m^2 and a gauge pressure of 500 kPa. At exit it has an area of 0.001 m^2 and a gauge pressure of 200 kPa.

   Calculate the vertical and horizontal forces due to the pressure only.
   (Answers 200 N and 1500 N).

2. A pipe bends through an angle of 45° in the vertical plane. At the inlet it has a cross sectional area of 0.002 m^2 and a gauge pressure of 800 kPa. At exit it has an area of 0.0008 m^2 and a gauge pressure of 300 kPa.

   Calculate the vertical and horizontal forces due to the pressure only.
   (Answers 169.7 N and 1430 N).

3. Calculate the momentum force acting on a bend of 130° that carries 2 kg/s of water at 16m/s velocity.

   Determine the vertical and horizontal components. (Answers 24.5 N and 52.6 N)

4. Calculate the momentum force on a 180° bend that carries 5 kg/s of water. The pipe is 50 mm bore diameter throughout. The density is 1000 kg/m^3.
   (Answer 25.46 N)

5. A horizontal pipe bend reduces from 300 mm bore diameter at inlet to 150 mm diameter at outlet. The bend is swept through 50° from its initial direction.
   The flow rate is 0.05 m^3/s and the density is 1000 kg/m^3. Calculate the momentum force on the bend and resolve it into two perpendicular directions relative to the initial direction.
   (Answers 108.1 N and 55.46 N).

3. COMBINED PRESSURE AND MOMENTUM FORCES

Now we will look at problems involving forces due to pressure changes and momentum changes at the same time. This is best done with a worked example since we have covered the theory already.

WORKED EXAMPLE No.3

A pipe bend has a cross sectional area of 0.01 m^2 at inlet and 0.0025 m^2 at outlet. It bends 90^o from its initial direction. The velocity is 4 m/s at inlet with a pressure of 100 kPa gauge. The density is 1000 kg/m^3. Calculate the forces acting parallel and perpendicular to the initial direction.

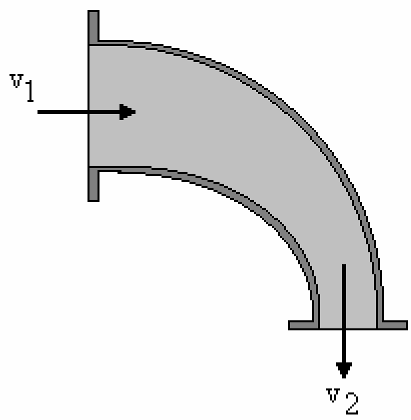


Fig. 5

SOLUTION

$v_1 = 4$m/s. Since $\rho A_1 v_1 = \rho A_2 v_2$ then $v_2 = 16$ m/s

We need the pressure at exit. This is done by applying Bernoulli between (1) and (2) as follows.

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$
$$100 \times 10^3 + \tfrac{1}{2}\, 1000 \times 4^2 = p_2 + 1000 \times \tfrac{1}{2}\, 16^2$$
$$p_2 = 0 \text{ kPa gauge}$$

Now find the pressure forces.

$$F_{px_1} = p_1 A_1 = 1200 \text{ N}$$

$F_{py_2} = p_2 A_2 = 0$ N Next solve the momentum forces.

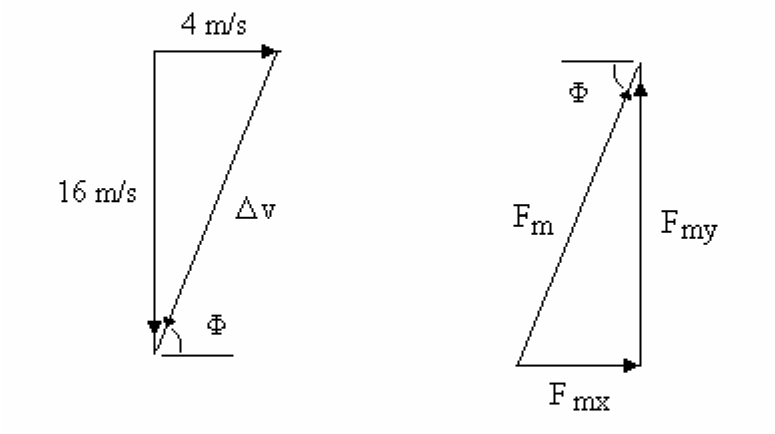


Fig.6

$m' = \rho A v = 40$ kg/s
$\Delta v = (4^2 + 16^2)^{½} = 16.49$ m/s
$F_m = m'\Delta v = 659.7$ N
$\phi = \tan^{-1}(16/4) = 75.96^o$

RESOLVE

$F_{my} = 659.7 \sin 75.96 = 640$ N
$F_{mx} = 659.7 \cos 75.96 = 160$ N

Total forces in x direction = 1200 + 160 = 1360 N
Total forces in y direction = 0 + 640 = 640 N

ALTERNATIVE SOLUTION

Many people prefer to solve the complete problems by solving pressure and momentum forces in the x or y directions as follows.

x direction $m'v_1 + p_1A_1 = F_X = 1200$ N

y direction $m'v_2 + p_2A_2 = F_Y = 640$ N

When the bend is other than 90^o this has to be used more carefully because there is an x component at exit also.

4. APPLICATIONS TO STATIONARY VANES

When a jet of fluid strikes a stationary vane, the vane decelerates the fluid in a given direction. Even if the speed of the fluid is unchanged, a change in direction produces changes in the velocity vectors and hence momentum forces are produced. The resulting force on the vane being struck by the fluid is an ***impulsive force.*** Since the fluid is at atmospheric pressure at all times after leaving the nozzle, there are no forces due to pressure change.

4.1 FLAT PLATE NORMAL TO JET

Consider first a jet of liquid from a nozzle striking a flat plate as shown in figure 7.

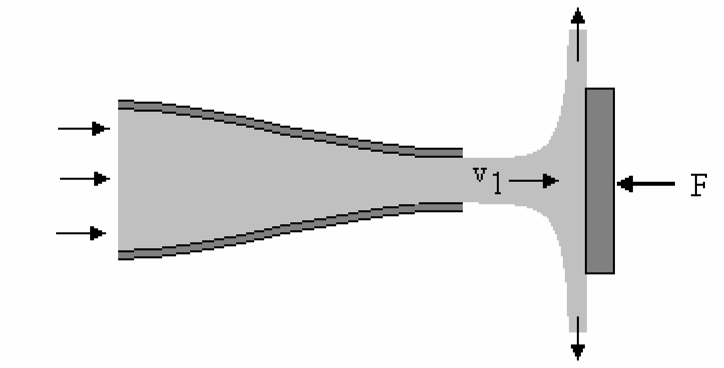


Fig.7

The velocity of the jet leaving the nozzle is v_1. The jet is decelerated to zero velocity in the original direction. Usually the liquid flows off sideways with equal velocity in all radial directions with no splashing occurring. The fluid is accelerated from zero in the radial directions but since the flow is equally divided no resultant force is produced in the radial directions. This means the only force on the plate is the one produced normal to the plate. This is found as follows.

m' = mass flow rate.
Initial velocity = v_1.
Final velocity in the original direction = $v_2 = 0$.
Change in velocity $= \Delta v = v_2 - v_1 = -v_1$
Force $= m'\Delta v = -mv_1$

This is the force required to produce the momentum changes in the fluid. The force on the plate must be equal and opposite so

$$\mathbf{F = m'v_1 = \rho A\, v_1}$$

WORKED EXAMPLE No.4

A nozzle has an exit diameter of 15 mm and discharges water into the atmosphere. The gauge pressure behind the nozzle is 400 kPa. The coefficient of velocity is 0.98 and there is no contraction of the jet. The jet hits a stationary flat plate normal to its direction. Determine the force on the plate. The density of the water is 1000 kg/m^3. Assume the velocity of approach into the nozzle is negligible.

SOLUTION

The velocity of the jet is $v_1 = C_v(2\Delta p/\rho)^{½}$

$v_1 = 0.98\ (2\text{x } 400\ 000/1000)^{½} = 27.72$ m/s

The nozzle exit area A = π x $0.015^2/4 = 176.7 \times 10^{-6}$ m^2.

The mass flow rate is $\rho A v_1 = 1000 \times 176.7 \times 10^{-6} \times 27.72 = 4.898$ kg/s.

The force on the vane = 4.898 x 27.72 = **135.8 N**

4.2 FLAT PLATE AT ANGLE TO JET

If the plate is at an angle as shown in fig. 8 then the fluid is not completely decelerated in the original direction but the radial flow is still equal in all radial directions. All the momentum normal to the plate is destroyed. It is easier to consider the momentum changes normal to the plate rather than normal to the jet.

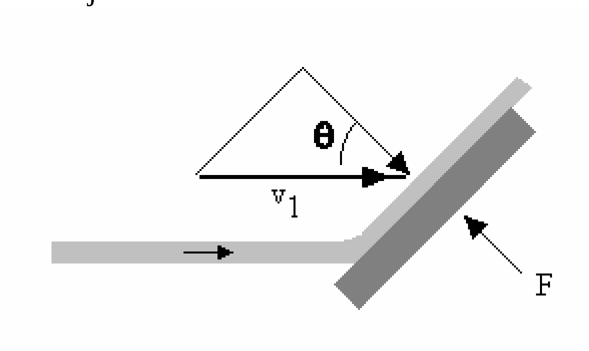


Fig.8

Initial velocity normal to plate = $v_1 \cos\theta$.
Final velocity normal to plate = 0.
Force normal to plate = $m'\Delta v = 0 - \rho A\, v_1 \cos\theta$.
This is the force acting on the fluid so the force on the plate is

$$\mathbf{m'\, v_1 \cos\theta \quad or \quad \rho A\, v_1^2 \cos\theta.}$$

If the horizontal and vertical components of this force are required then the force must be resolved.

WORKED EXAMPLE No. 5

A jet of water has a velocity of 20 m/s and flows at 2 kg/s. The jet strikes a stationary flat plate. The normal direction to the plate is inclined at 30^o to the jet. Determine the force on the plate in the direction of the jet.

SOLUTION

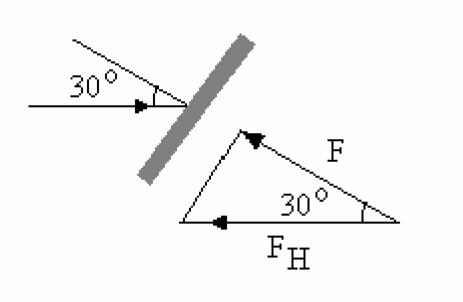


Fig.9

The force normal to the plate is $mv_1 \cos\theta = 2 \times 20\cos 30^o = 34.64$ N.

The force in the direction of the jet is found by resolving.

$$F_H = F/\cos 30^o = 34.64/\cos 30^o = 40 \text{ N}$$

4.3 CURVED VANES

When a jet hits a curved vane, it is usual to arrange for it to arrive on the vane at the same angle as the vane. The jet is then diverted from with no splashing by the curve of the vane. If there is no friction present, then only the direction of the jet is changed, not its speed.

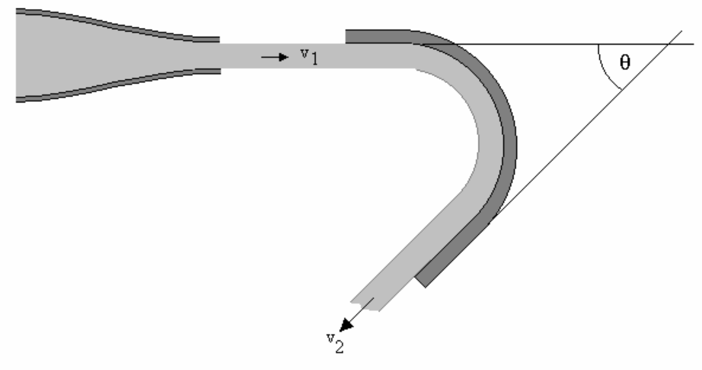


Fig.10

This is the same problem as a pipe bend with uniform size. v_1 is numerically equal to v_2.

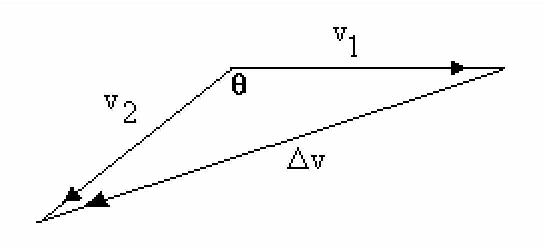


Fig.11

If the deflection angle is θ as shown in figs.10 and 11 then the impulsive force is

$$\mathbf{F = m'\Delta v = m'\, v_1\{2(1 - \cos\theta)\}^{1/2}}$$

The direction of the force on the fluid is in the direction of Δv and the direction of the force on the vane is opposite. The force may be resolved to find the forces acting horizontally and/or vertically.

It is often necessary to solve the horizontal force and this is done as follows.

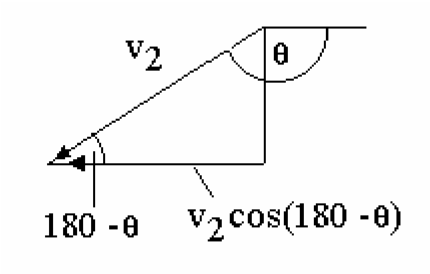


Fig.12

Initial horizontal velocity = $v_{H1} = v_1$

Final horizontal velocity = $v_{H2} = -v_2 \cos(180 - \theta) = v_2 \cos\theta$

Change in horizontal velocity = Δv_{H1}

Since $v_2 = v_1$ this becomes $\Delta v_h = \{v_2 \cos\theta - v_1\} = v_1\{\cos\theta - 1\}$

Horizontal force on fluid = $m'v_1\{\cos\theta - 1\}$

The horizontal force on the vane is opposite so

$$\textbf{Horizontal force} = \mathbf{m'\Delta v_H = m'v_1\{1 - \cos\theta\}}$$

WORKED EXAMPLE No.6

A jet of water travels horizontally at 16 m/s with a flow rate of 2 kg/s. It is deflected 130^o by a curved vane. Calculate resulting force on the vane in the horizontal direction.

SOLUTION

The resulting force on the vane is $F = m' v_1\{2(1 - \cos\theta)^{½}$

$$F = 2 \times 16 \{2(1 - \cos 130^o)\}^{½} = 58 \text{ N}$$

The horizontal force is

$$F_H = m' v_1\{\cos\theta - 1\}$$
$$F_H = 2 \times 16 \times (1 - \cos 130)$$
$$F_H = 52.6 \text{ N}$$

SELF ASSESSMENT EXERCISE No. 2

Assume the density of water is 1000 kg/m^3 throughout.

1. A pipe bends through 90^o from its initial direction as shown in fig.13. The pipe reduces in diameter such that the velocity at point (2) is 1.5 times the velocity at point (1). The pipe is 200 mm diameter at point (1) and the static pressure is 100 kPa. The volume flow rate is 0.2 m^3/s. Assume there is no friction. Calculate the following.

   a) The static pressure at (2).

   b) The velocity at (2).

   c) The horizontal and vertical forces on the bend F_H and F_V.

   d) The total resultant force on the bend.

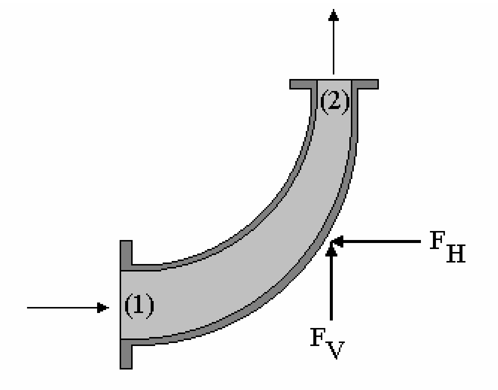


Fig.13

2. A nozzle produces a jet of water. The gauge pressure behind the nozzle is 2 MPa. The exit diameter is 100 mm. The coefficient of velocity is 0.97 and there is no contraction of the jet. The approach velocity is negligible. The jet of water is deflected 165^{o} from its initial direction by a stationary vane. Calculate the resultant force on the nozzle and on the vane due to momentum changes only.
(Answers 29.5 kN and 58.5 kN).

3. A stationary vane deflects 5 kg/s of water 50^{o} from its initial direction. The jet velocity is 13 m/s. Draw the vector diagram to scale showing the velocity change. Deduce by either scaling or calculation the change in velocity and go on to calculate the force on the vane in the original direction of the jet.
(Answer 49.8 N).

4. A jet of water travelling with a velocity of 25 m/s and flow rate 0.4 kg/s is deflected 150^{o} from its initial direction by a stationary vane. Calculate the force on the vane acting parallel to and perpendicular to the initial direction.
(Answers 18.66 N and 5 N)

5. A jet of water discharges from a nozzle 30 mm diameter with a flow rate of 15 dm^{3}/s into the atmosphere. The inlet to the nozzle is 100 mm diameter. There is no friction nor contraction of the jet. Calculate the following.

   i. The jet velocity.(21.22 m/s)
   ii. The gauge pressure at inlet. (223.2 kPa)
   iii. The force on the nozzle. (2039 N)

   The jet strikes a flat stationary plate normal to it. Determine the force on the plate.
   (312 N)

5. MOVING VANES

When a vane moves away from the jet as shown on fig.14, the mass flow arriving on the vane is reduced because some of the mass leaving the nozzle is producing a growing column of fluid between the jet and the nozzle. This is what happens in turbines where the vanes are part of a revolving wheel. We need only consider the simplest case of movement in a straight line in the direction of the jet.

5.1 MOVING FLAT PLATE

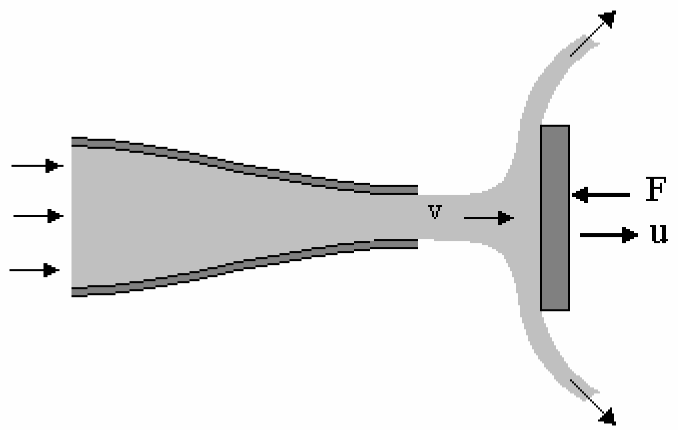


Fig.14

The velocity of the jet is v and the velocity of the vane is u. If you were on the plate, the velocity of the fluid arriving would be v - u. This is the relative velocity, that is, relative to the plate. The mass flow rate arriving on the plate is then

$$\mathbf{m' = \rho A(v-u)}$$

The initial direction of the fluid is the direction of the jet. However, due to movement of the plate, the velocity of the fluid as it leaves the edge is not at 90^o to the initial direction. In order to understand this we must consider the fluid as it flows off the plate. Just before it leaves the plate it is still travelling forward with the plate at velocity u. When it leaves the plate it will have a true velocity that is a combination of its radial velocity and u. The result is that it appears to come off the plate at a forward angle as shown.

We are only likely to be interested in the force in the direction of movement so we only require the change in velocity of the fluid in this direction.

The initial forward velocity of the fluid = v
The final forward velocity of the fluid = u
The change in forward velocity = v-u
The force on the plate = m'ρv = m' (v-u)
Since m' = ρA(v-u) then the force on the plate is

$$\mathbf{F = \rho A(v-u)^2}$$

5.2 MOVING CURVED VANE

Turbine vanes are normally curved and the fluid joins it at the same angle as the vane as shown in fig.15.

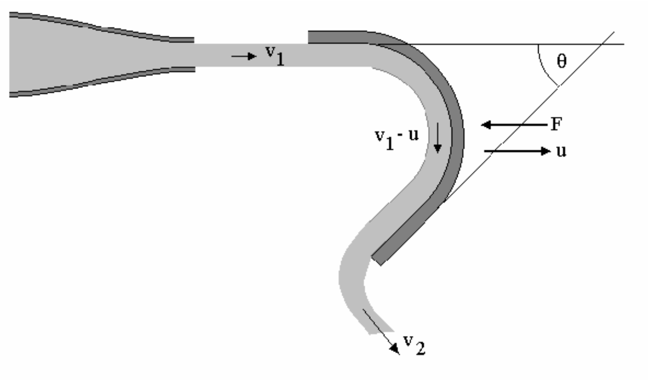


Fig.15

The velocity of the fluid leaving the nozzle is v_1. This is a true or absolute velocity as observed by anyone standing still on the ground. The fluid arrives on the vane with relative velocity v_1-u as before. This is a relative velocity as observed by someone moving with the vane. If there is no friction then the velocity of the fluid over the surface of the vane will be v_1-u at all points. At the tip where the fluid leaves the vane, it will have two velocities. The fluid will be flowing at v_1-u over the vane but also at velocity u in the forward direction. The true velocity v_2 at exit must be the vector sum of these two.

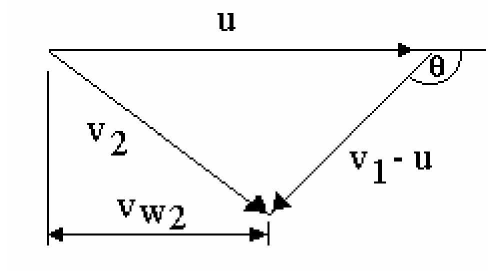


Fig.16

If we only require the force acting on the vane in the direction of movement then we must find the horizontal component of v_2. Because this direction is the direction in which the vane is whirling about the centre of the wheel, it is called the velocity of whirl v_{w2}. The velocity v_1 is also in the direction of whirling so it follows that $v_1 = v_{w1}$.

V_{w2} may be found by drawing the vector diagram (fig.16) to scale or by using trigonometry. In this case you may care to show for yourself that $v_{w2} = u + (v_1 - u)(\cos\theta)$

The horizontal force on the vane becomes $F_H = m' (v_{w1} - v_{w2}) = m' (v_1 - v_{w2})$

You may care to show for yourself that this simplifies down to $Fh = m'(v_1 - u)(1 - \cos\theta)$

This force moves at the same velocity as the vane. The power developed by a force is the product of force and velocity. This is called the Diagram Power (D.P.) and the diagram power developed by a simple turbine blade is D.P. = $m'u(v_1 - u)(1 - \cos\theta)$

This work involving the force on a moving vane is the basis of turbine problems and the geometry of the case considered is that of a simple water turbine known as a Pelton Wheel. You are not required to do this in the exam. It is unlikely that the examination will require you to calculate the force on the moving plate but the question in self assessment exercise 5 does require you to calculate the exit velocity v_2.

WORKED EXAMPLE No.7

A simple turbine vane as shown in fig.15 moves at 40 m/s and has a deflection angle of 150^o. The jet velocity from the nozzle is 70 m/s and flows at 1.7 kg/s.

Calculate the absolute velocity of the water leaving the vane and the diagram power.

SOLUTION

Drawing the vector diagram (fig.16) to scale, you may show that v_2 = 20.5 m/s. This may also be deduced by trigonometry. The angle at which the water leaves the vane may be measured from the diagram or deduced by trigonometry and is 46.9^o to the original jet direction.

D.P. = $m'u(v_1\text{-}u)(1+\cos\theta)$ = 1.7 x 40(70-40)(1 - cos 150) = 3807 Watts

SELF ASSESSMENT EXERCISE No.3

1. A vane moving at 30 m/s has a deflection angle of 90^o. The water jet moves at 50 m/s with a flow of 2.5 kg/s. Calculate the diagram power assuming that all the mass strikes the vane.
   (Answer 1.5 kW).

2. Figure 10 shows a jet of water 40 mm diameter flowing at 45 m/s onto a curved fixed vane. The deflection angle is 150^o. There is no friction. Determine the magnitude and direction of the resultant force on the vane. (4916 N)

   The vane is allowed to move away from the nozzle in the same direction as the jet at a velocity of 18 m/s. Draw the vector diagram for the velocity at exit from the vane and determine the magnitude and direction of the velocity at exit from the vane. (14.53 m/s)

FLUID MECHANICS

TUTORIAL No.8A

WATER TURBINES

When you have completed this tutorial you should be able to

- Explain the significance of specific speed to turbine selection.

- Explain the general principles of
  Pelton Wheels
  Kaplan Turbines
  Francis Turbine

- Construct blade vector diagrams for moving vanes for a Pelton Wheels and a Francis Turbine

- Deduce formulae for power and efficiency for turbines.

- Solve numerical problems for a Pelton Wheels and a Francis Turbine

1. INTRODUCTION

A water turbine is a device for converting water (fluid) power into shaft (mechanical) power. A pump is a device for converting shaft power into water power.

Two basic categories of machines are the rotary type and the reciprocating type. Reciprocating motors are quite common in power hydraulics but the rotary principle is universally used for large power devices such as on hydroelectric systems.

Large pumps are usually of the rotary type but reciprocating pumps are used for smaller applications.

1.1 THE SPECIFIC SPEED FOR VARIOUS TYPES OF TURBINES

The power 'P' of any rotary hydraulic machine (pump or motor) depends upon the density 'ρ' , the speed 'N', the characteristic diameter 'D', the head change 'ΔH', the volume flow rate 'Q' and the gravitational constant 'g'. The general equation is:

$$P = f(\rho, N, D, \Delta H, Q, g)$$

It is normal to consider $g\Delta H$ as one quantity. $P = f\{\rho, N, D, (g\Delta H), Q\}$

There are 6 quantities and 3 dimensions so there are three dimensionless groups Π_1, $\Pi_2$and Π_3. First form a group with P and ρND.

$$P = \varphi(\rho N D) = \Pi_1 \rho^a N^b D^c$$

$$M^1 L^2 T^{-3} = \left(ML^{-3}\right)^a \left(T^{-1}\right)^b \left(D^1\right)^c$$

Mass $1 = a$ Time $-3 = -b$ $b = 3$ Length $2 = -3a + c = -3 + c$ $c = 5$

$$P = \Pi_1 \rho^1 N^3 D^5 \qquad \Pi_1 = \frac{P}{\rho\ N^3 D^5} = \text{ Power Coefficient}$$

Next repeat the process between Q and ρND

$$Q = \varphi(\rho N D) = \Pi_2 \rho^a N^b D^c$$

$$M^3 T^{-1} = \left(ML^{-3}\right)^a \left(T^{-1}\right)^b \left(D^1\right)^c$$

Time $-1 = -b$ $b = 1$ Mass $0 = a$ Length $3 = -3a + c$ $c = 3$

$$Q = \Pi_2 \rho^0 N^1 D^3 \qquad \Pi_2 = \frac{Q}{ND^3} = \text{ Flow Coefficient}$$

Next repeat the process between gΔH and ρND

$$(g\,\Delta H) = \varphi(\rho N D) = \Pi_3 \rho^a N^b D^c$$

$$M^0 L^2 T^{-2} = \left(ML^{-3}\right)^a \left(T^{-1}\right)^b \left(D^1\right)^c$$

Mass $0 = a$ Time $-2 = -b$ $b = 2$ Length $2 = -3a + c$ $c = 2$

$$Q = \Pi_3 \rho^0 N^2 D^2 \qquad \Pi_3 = \frac{Q}{N^2 D^2} = \text{ Head Coefficient}$$

Finally the complete equation is $\dfrac{P}{\rho N^3 D^5} = \varphi\left(\dfrac{Q}{ND^2}\right)\left(\dfrac{g\,\Delta H}{N^2 D^2}\right)$

SPECIFIC SPEED Ns

The specific speed is a parameter used for pumps and turbines to determine the best design to match a given pumped system. The formula may be derived from consideration of the pump geometry or by dimensional analysis. The latter will be used here.

$$\frac{P}{\rho N^3 D^5} = \varphi\left(\frac{Q}{ND^2}\right)\left(\frac{g\,\Delta H}{N^2 D^2}\right)$$

The three dimensionless numbers represent the Power coefficient, the flow coefficient and the Head coefficient respectively. Now consider a family of geometrically similar machines operating at dynamically similar conditions. For this to be the case the coefficients must have the same values for each size. Let the 3 coefficients be Π_1, Π_2 and Π_3 such that

$$\Pi_1 = \frac{P}{\rho N^3 D^5} \qquad \Pi_2 = \frac{Q}{ND^3} \quad D = \left(\frac{Q}{N\Pi_2}\right)^{\frac{1}{3}} \qquad \Pi_3 = \frac{g\,\Delta H}{N^2 D^2} \quad D = \left(\frac{g\,\Delta H}{N^2 \Pi_3}\right)^{\frac{1}{2}}$$

$$\text{Equating} \quad \left(\frac{Q}{N\Pi_2}\right)^{\frac{1}{3}} = \left(\frac{g\,\Delta H}{N^2 \Pi_3}\right)^{\frac{1}{2}} \qquad \frac{1}{N}\left(\frac{g\,\Delta H}{\Pi_3}\right)^{\frac{1}{2}} = \frac{Q^{\frac{1}{3}}}{\Pi_2^{\frac{1}{3}} N^{\frac{1}{3}}}$$

$$\frac{(\Delta H)^{\frac{1}{2}}}{Q^{\frac{1}{3}} N^{\frac{2}{3}}} = \frac{\Pi_3^{\frac{1}{2}}}{\Pi_2^{\frac{1}{3}} g^{\frac{1}{2}}} = \text{constant} \qquad \frac{(\Delta H)^{\frac{1}{2}}}{KQ^{\frac{1}{3}}} = N^{\frac{2}{3}}$$

$$\left[\frac{(\Delta H)^{\frac{1}{2}}}{KQ^{\frac{1}{3}}}\right]^{\frac{3}{2}} = N = \frac{(\Delta H)^{\frac{3}{4}}}{K^{\frac{1}{2}} Q^{\frac{1}{2}}} \qquad \frac{NQ^{\frac{1}{2}}}{(\Delta H)^{\frac{3}{4}}} = K^{-\frac{1}{2}} = \text{constant} \qquad N_s = \frac{NQ^{\frac{1}{2}}}{(\Delta H)^{\frac{3}{4}}}$$

Ns is a dimensionless parameter that and the units used are normally rev/min for speed, m^3/s for flow rate and metres for head. Other units are often used and care should be taken when quoting Ns values.

It follows that for a given speed, the specific speed is large for large flows and low heads and small for small flows and large heads. The important value is the one that corresponds to the conditions that produce the greatest efficiency. The diagram illustrates how the design affects the specific speed.

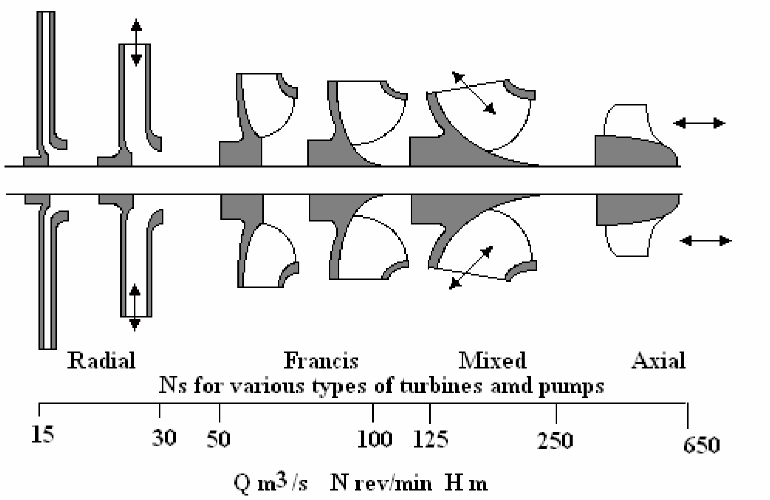


Figure 1

2. GENERAL PRINCIPLES OF TURBINES.

WATER POWER

This is the fluid power supplied to the machine in the form of pressure and volume.

Expressed in terms of pressure head the formula is $\quad W.P. = mg\Delta H$

M is the mass flow rate in kg/s and ΔH is the pressure head difference over the turbine in metres. Remember that $\Delta p = \rho g \Delta H$

Expressed in terms of pressure the formula is $\quad W.P. = Q\Delta p$

Q is the volume flow rate in m^3/s. Δp is the pressure drop over the turbine in N/m^2 or Pascals.

SHAFT POWER

This is the mechanical, power output of the turbine shaft. The well known formula is

$$S.P. = 2\pi NT$$

Where T is the torque in Nm and N is the speed of rotation in rev/s

DIAGRAM POWER

This is the power produced by the force of the water acting on the rotor. It is reduced by losses before appearing as shaft power. The formula for D.P. depends upon the design of the turbine and involves analysis of the velocity vector diagrams.

HYDRAULIC EFFICIENCY

This is the efficiency with which water power is converted into diagram power and is given by

$$\eta_{hyd} = D.P./W.P.$$

MECHANICAL EFFICIENCY

This is the efficiency with which the diagram power is converted into shaft power. The difference is the mechanical power loss.

$$\eta_{mech} = S.P./D.P.$$

OVERALL EFFICIENCY

This is the efficiency relating fluid power input to shaft power output.

$$\eta_{o/a} = S.P./W.P.$$

It is worth noting at this point that when we come to examine pumps, all the above expressions are inverted because the energy flow is reversed in direction.

The water power is converted into shaft power by the force produced when the vanes deflect the direction of the water. There are two basic principles in the process, ***IMPULSE and REACTION.***

IMPULSE occurs when the direction of the fluid is changed with no pressure change. It follows that the magnitude of the velocity remains unchanged.

REACTION occurs when the water is accelerated or decelerated over the vanes. A force is needed to do this and the reaction to this force acts on the vanes.

Impulsive and reaction forces are determined by examining the changes in velocity (magnitude and direction) when the water flows over the vane. The following is a typical analysis.

The vane is part of a rotor and rotates about some centre point. Depending on the geometrical layout, the inlet and outlet may or may not be moving at the same velocity and on the same circle. In order to do a general study, consider the case where the inlet and outlet rotate on two different diameters and hence have different velocities.

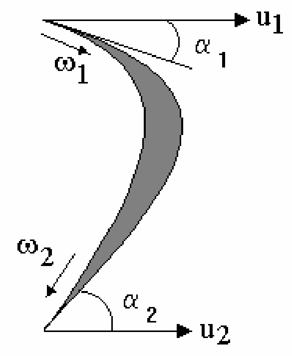


Fig.2

u_1 is the velocity of the blade at inlet and u_2 is the velocity of the blade at outlet. Both have tangential directions.ω_1 is the relative velocity at inlet and ω_2 is the relative velocity at outlet.

The water on the blade has two velocity components. It is moving tangentially at velocity u and over the surface at velocity ω. The absolute velocity of the water is the vector sum of these two and is denoted v. At any point on the vane $v = \omega + u$

At inlet, this rule does not apply unless the direction of v_1 is made such that the vector addition is true. At any other angle, the velocities will not add up and the result is chaos with energy being lost as the water finds its way onto the vane surface. The perfect entry is called ***"SHOCKLESS ENTRY"*** and the entry angle β_1 must be correct. This angle is only correct for a given value of v_1.

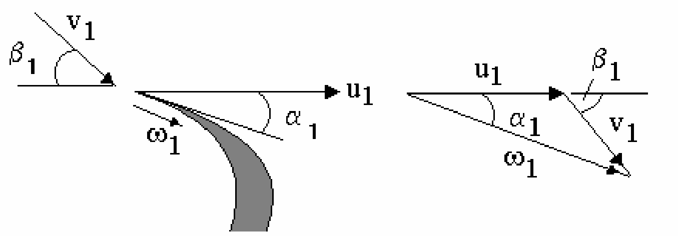


Fig.3

INLET DIAGRAM

For a given or fixed value of u_1 and v_1, shockless entry will occur only if the vane angle α_1 is correct or the delivery angle β_1 is correct. In order to solve momentum forces on the vane and deduce the flow rates, we are interested in two components of v_1. These are the components in the direction of the vane movement denoted v_w (meaning velocity of whirl) and the direction at right angles to it v_R (meaning radial velocity but it is not always radial in direction depending on the wheel design). The suffix (1) indicates the entry point. A typical vector triangle is shown.

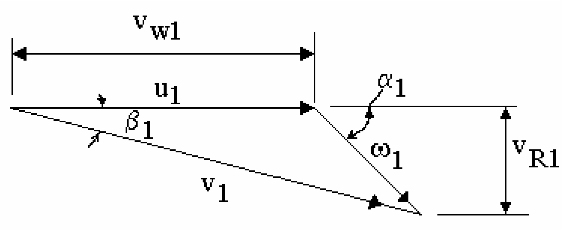


Fig.4

OUTLET DIAGRAM

At outlet, the absolute velocity of the water has to be the vector resultant of u and ω and the direction is unconstrained so it must come off the wheel at the angle resulting. Suffix (2) refers to the outlet point. A typical vector triangle is shown.

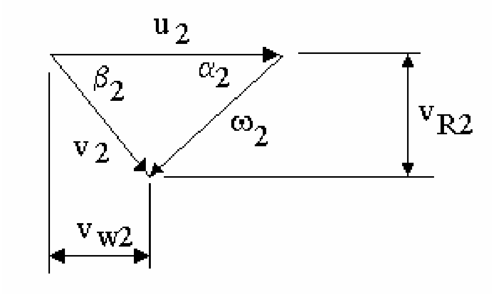


Fig. 5

DIAGRAM POWER

Diagram power is the theoretical power of the wheel based on momentum changes in the fluid. The force on the vane due to the change in velocity of the fluid is F = mΔv and these forces are vector quantities. m is the mass flow rate. The force that propels the wheel is the force developed in the direction of movement (whirl direction). In order to deduce this force, we should only consider the velocity changes in the whirl direction (direction of rotation) Δv_w. The power of the force is always the product of force and velocity. The velocity of the force is the velocity of the vane (u). If this velocity is different at inlet and outlet it can be shown that the resulting power is given by

$$\text{D.P.} = m\,\Delta v_w = m\,(u_1 v_{w1} - u_2\, v_{w2})$$

3. PELTON WHEEL

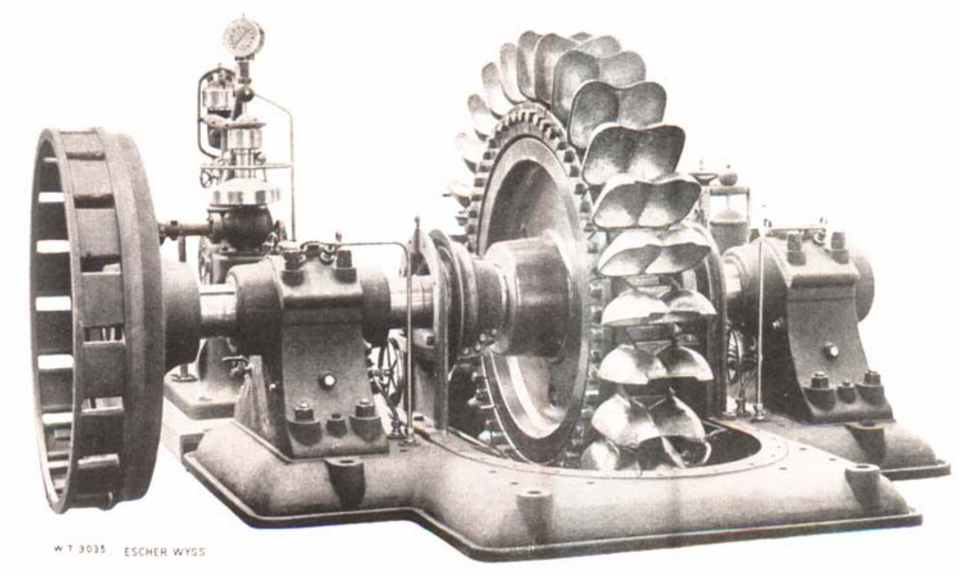


Fig. 6 Pelton Wheel With Case Removed

Pelton wheels are mainly used with high pressure heads such as in mountain hydroelectric schemes. The diagram shows a layout for a Pelton wheel with two nozzles.

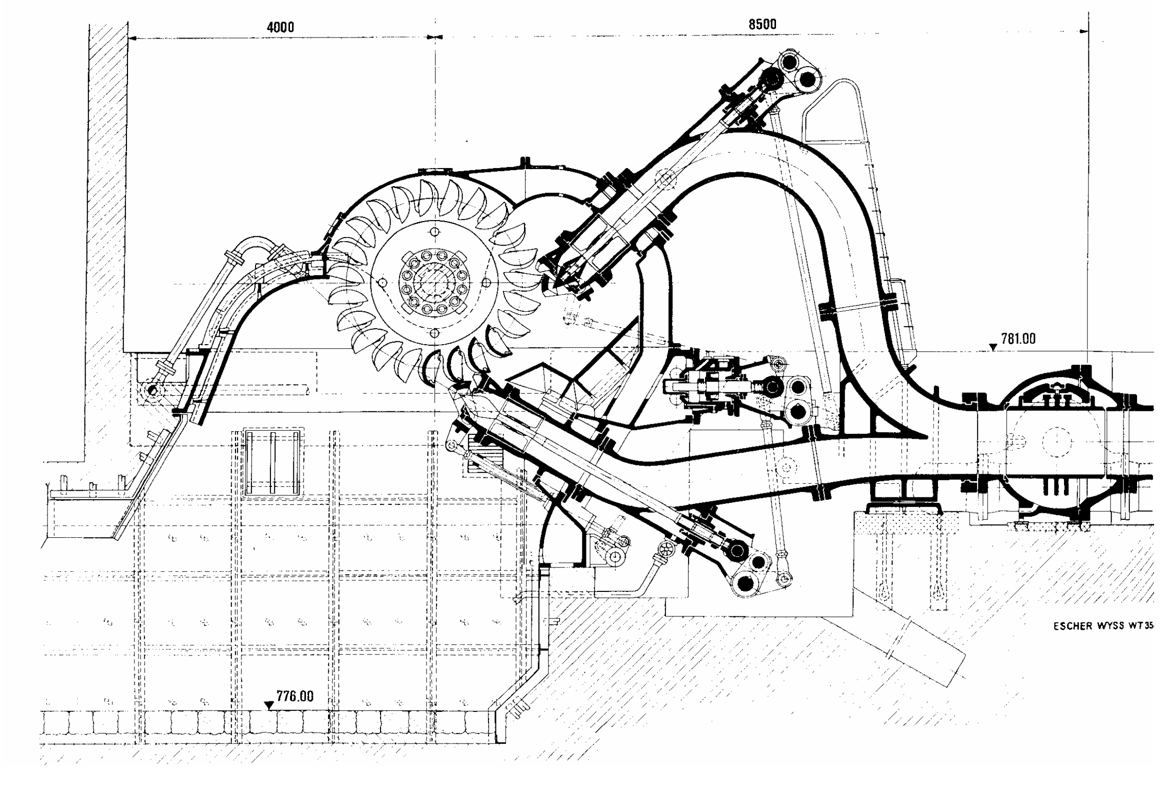


Fig.7 Schematic Diagram Of Pelton Wheel With Two Nozzles

3.1 GENERAL THEORY

The Pelton Wheel is an impulse turbine. The fluid power is converted into kinetic energy in the nozzles. The total pressure drop occurs in the nozzle. The resulting jet of water is directed tangentially at buckets on the wheel producing impulsive force on them. The buckets are small compared to the wheel and so they have a single velocity $u = \pi ND$
D is the mean diameter of rotation for the buckets.

The theoretical velocity issuing from the nozzle is given by

$$v_1 = (2gH)^{1/2} \text{ or } v_1 = (2p/\rho)^{1/2}$$

Allowing for friction in the nozzle this becomes

$$v_1 = C_v(2gH)^{1/2} \text{ or } v_1 = C_v(2p/\rho)^{1/2}$$

H is the gauge pressure head behind the nozzle, p the gauge pressure and c_v the coefficient of velocity and this is usually close to unity.

The mass flow rate from the nozzle is

$$m = C_c \rho A v_1 = C_c \rho A C_v (2gH)^{1/2} = C_d \rho A (2gH)^{1/2}$$

C_c is the coefficient of contraction (normally unity because the nozzles are designed not to have a contraction).

C_d is the coefficient of discharge and $C_d = C_c C_v$

In order to produce no axial force on the wheel, the flow is divided equally by the shape of the bucket. This produces a zero net change in momentum in the axial direction.

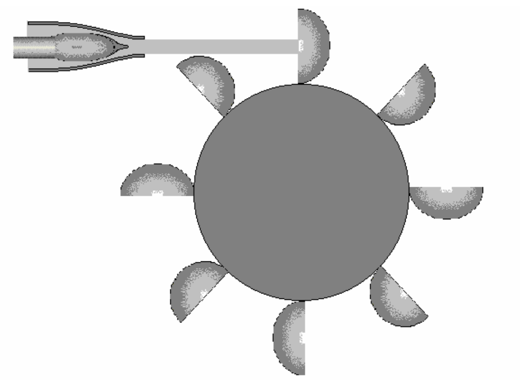


Fig.8
Layout of Pelton Wheel with One Nozzle

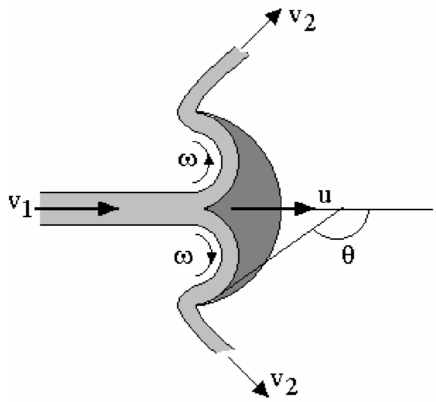


Fig. 9
Cross Section Through Bucket

The water is deflected over each half of the bucket by an angle of θ degrees. Since the change in momentum is the same for both halves of the flow, we need only consider the vector diagram for one half. The initial velocity is v_1 and the bucket velocity u_1 is in the same direction. The relative velocity of the water at inlet (in the middle) is ω_1 and is also in the same direction so the vector diagram is a straight line.

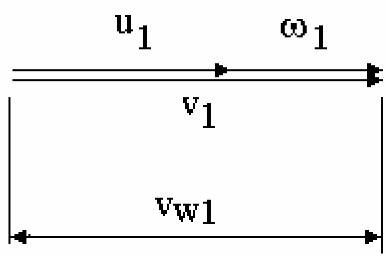


Fig. 10

If the water is not slowed down as it passes over the bucket surface, the relative velocity ω_2 will be the same as ω_1. In reality friction slows it down slightly and we define a blade friction coefficient as $k = \omega_2/\omega_1$

The exact angle at which the water leaves the sides of the bucket depends upon the other velocities but as always the vectors must add up so that

Note that $u_2 = u_1 = u$ since the bucket has a uniform velocity everywhere.

The vector diagram at exit is as shown.

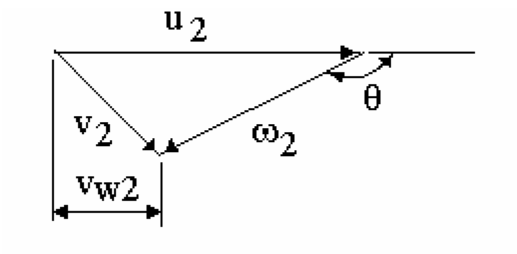


Fig. 11

It is normal to use ω_1 and u as common to both diagrams and combine them as shown.

Since $u_2 = u_1 = u$ the diagram power becomes I

Examining the combined vector diagram shows that $\Delta v_W = \omega_1 - \omega_2\cos\theta$

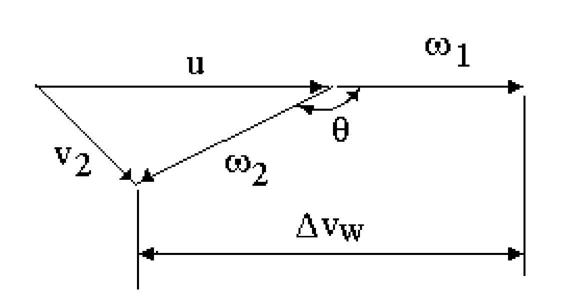


Fig. 12

Hence $\quad$ D.P.= $mu(\omega_1 - \omega_2\cos\theta)$ but $\omega_2 = k\omega_1$

D.P.= $mu\omega_1(1 - k\cos\theta)$ but $\omega_1 = v_1 - u$

D.P.= $mu(v_1-u)(1- k\cos\theta)$

WORKED EXAMPLE No. 1

A Pelton wheel is supplied with 1.2 kg/s of water at 20 m/s. The buckets rotate on a mean diameter of 250 mm at 800 rev/min. The deflection angle is 165^o and friction is negligible. Determine the diagram power. Draw the vector diagram to scale and determine Δv_W.

SOLUTION

$u = \pi ND/60 = \pi \times 800 \times 0.25/60 = 10.47$ m/s

D.P = $mu(v_1-u)(1- k\cos\theta)$

D.P = $1.2 \times 10.47 \times (20 - 10.47)(1 - \cos 165) = 235$ Watts

You should now draw the vector diagram to scale and show that $\Delta v_W = 18.5$ m/s

3.2 CONDITION FOR MAXIMUM POWER

If the equation for diagram power is used to plot D.P against u, the graph is as shown below.

Clearly the power is zero when the buckets are stationary and zero when the buckets move so fast that the water cannot catch up with them and strike them. In between is a velocity which gives maximum power. This may be found from max and min theory.

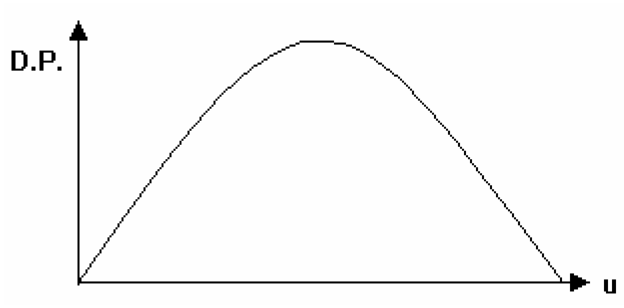


Fig. 13

$$\frac{d(D.P.)}{du} = \frac{d\{mu(v_1 - u)(1 - k\cos\theta)\}}{du} \qquad \frac{d(D.P.)}{du} = \frac{d\{m(uv_1 - u^2)(1 - k\cos\theta)\}}{du}$$

$$\frac{d(D.P.)}{du} = m(v_1 - 2u)(1 - k\cos\theta)$$

For a maximum value $m(v_1 - 2u)(1 - k\cos\theta) = 0$ Hence for maximum power $v_1 = 2u$

3.3 SPECIFIC SPEED N_S FOR PELTON WHEELS

You may have already covered the theory for specific speed in dimensional analysis but for those who have not, here is a brief review.

Specific speed is a parameter which enables a designer to select the best pump or turbine for a given system. It enables the most efficient matching of the machine to the head and flow rate available. One definition of specific speed for a turbine is : $N_S = NQ^{1/2}(H)^{-3/4}$

N is the speed in rev/min, Q is the volume flow rate in m^3/s and H is the available head in metres. The equation may be developed for a Pelton Wheel as follows.

$u = \pi ND/60 = K_1ND$ D = mean wheel diameter $N = u/(K_1D)$

u = bucket velocity $v_j = K_2H^{1/2}$ H = head behind the nozzle

v_j = nozzle velocity Now for a fixed speed wheel, $u = K_3v_j$ Hence

$$N = \frac{K_3v_j}{K_1D} = \frac{K_3K_2H^{\frac{1}{2}}}{K_1D} = \frac{K_4H^{\frac{1}{2}}}{D}$$

$$Q = A_jv_j = \frac{\pi d^2}{4}v_j \quad d = \text{nozzle diameter} \qquad Q = \frac{\pi d^2}{4}K_2H^{\frac{1}{2}} = K_5d^2H^{\frac{1}{2}}$$

Substituting all in the formula for N_S we get $N_s = k\frac{d}{D}$

The value of k has to be deduced from the data of the wheel and nozzle. Note that Ns is sometimes defined in terms of water power as $N_s = \dfrac{NP^{\frac{1}{2}}}{\rho^{\frac{1}{2}}(gH)^{\frac{5}{4}}}$

This is just an alternative formula and the same result can be easily obtained other ways. You will need the substitution $P = \rho QgH$

SELF ASSESSMENT EXERCISE No. 1

1. The buckets of a Pelton wheel revolve on a mean diameter of 1.5 m at 1500 rev/min. The jet velocity is 1.8 times the bucket velocity. Calculate the water flow rate required to produce a power output of 2MW. The mechanical efficiency is 80% and the blade friction coefficient is 0.97. The deflection angle is 165^o.
(Ans. 116.3 kg/s)

2. Calculate the diagram power for a Pelton Wheel 2m mean diameter revolving at 3000 rev/min with a deflection angle of 170^o under the action of two nozzles , each supplying 10 kg/s of water with a velocity twice the bucket velocity. The blade friction coefficient is 0.98.
(Ans. 3.88 MW)

   If the coefficient of velocity is 0.97, calculate the pressure behind the nozzles.
(Ans 209.8 MPa)

3. A Pelton Wheel is 1.7 m mean diameter and runs at maximum power. It is supplied from two nozzles. The gauge pressure head behind each nozzle is 180 metres of water. Other data for the wheel is :

   Coefficient of Discharge C_d = 0.99
   Coefficient of velocity C_v = 0.995
   Deflection angle = 165^o.
   Blade friction coefficient = 0.98
   Mechanical efficiency = 87% Nozzle diameters = 30 mm

   Calculate the following.

   i. The jet velocity (59.13 m/s)
   ii. The mass flow rate (41.586 kg/s)
   iii The water power (73.432 kW)
   iv. The diagram power (70.759 kW)
   v. The diagram efficiency (96.36%)
   vi. The overall efficiency (83.8%)
   vii. The wheel speed in rev/min (332 rev/min)

4. Explain the significance and use of 'specific speed' $Ns = NP^{1/2}/\{\rho^{1/2}(gH)^{5/4}\}$
Explain why in the case of a Pelton wheel with several nozzles, P is the power per nozzle.
Explain why a Francis Wheel is likely to be preferred to a Pelton wheel when site conditions suggest that either could be used.
Calculate the specific speed of a Pelton Wheel given the following.

d = nozzle diameter.
D = Wheel diameter.
u = optimum blade speed = 0.46 v_1
v_1= jet speed.
η= 88% C_v = coefficient of velocity = 0.98

Answer Ns = 11.9 d/D

5. Explain the usefulness of specific speed in the selection of pumps and turbines.

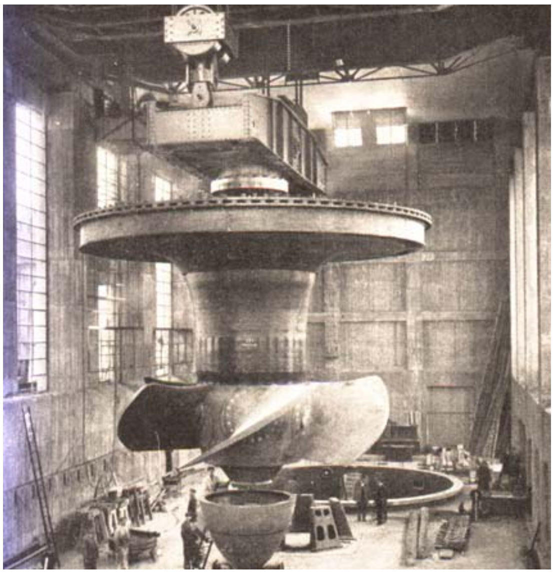
A turbine is to run at 150 rev/min under a head difference of 22 m and an expected flow rate of 85 m^3/s.

A scale model is made and tested with a flow rate of 0.1 m^3/s and a head difference of 5 m. Determine the scale and speed of the model in order to obtain valid results.

When tested at the speed calculated, the power was 4.5 kW. Predict the power and efficiency of the full size turbine.

Answers 0.05 scale 16.17 MW and 88%.

4. KAPLAN TURBINE



The Kaplan turbine is a pure reaction turbine. The main point concerning this is that all the flow energy and pressure is expended over the rotor and not in the supply nozzles. The picture shows the rotor of a large Kaplan turbine. They are most suited to low pressure heads and large flow rates such as on dams and tidal barrage schemes.

The diagram below shows the layout of a large hydroelectric generator in a dam.

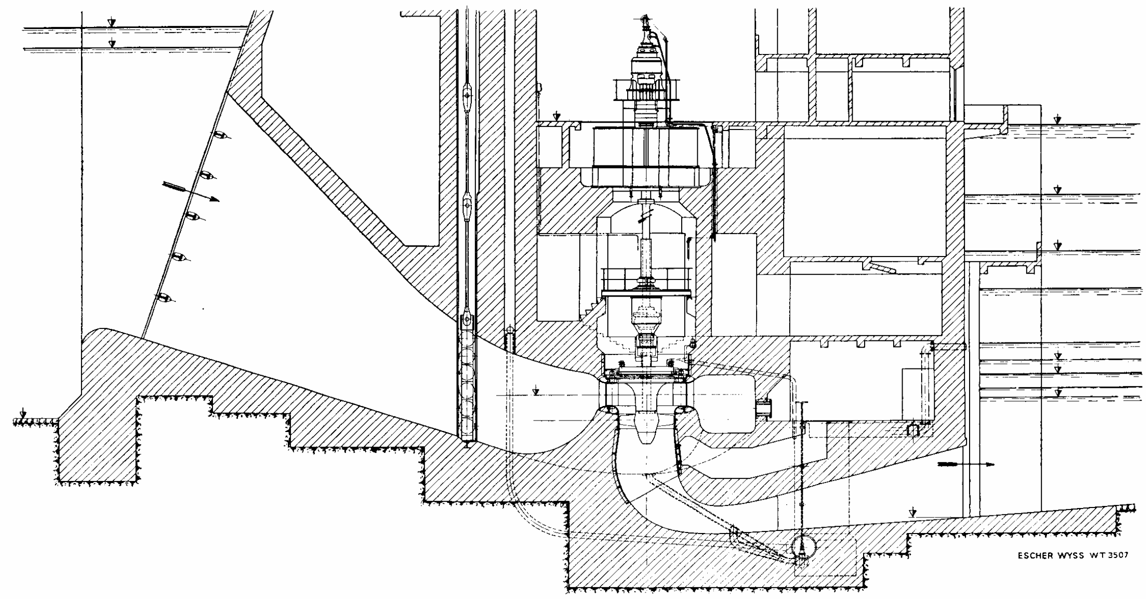


Fig.14 Picture and schematic of a Kaplan Turbine

5. FRANCIS WHEEL

The Francis wheel is an example of a mixed impulse and reaction turbine. They are adaptable to varying heads and flows and may be run in reverse as a pump such as on a pumped storage scheme. The diagram shows the layout of a vertical axis Francis wheel.

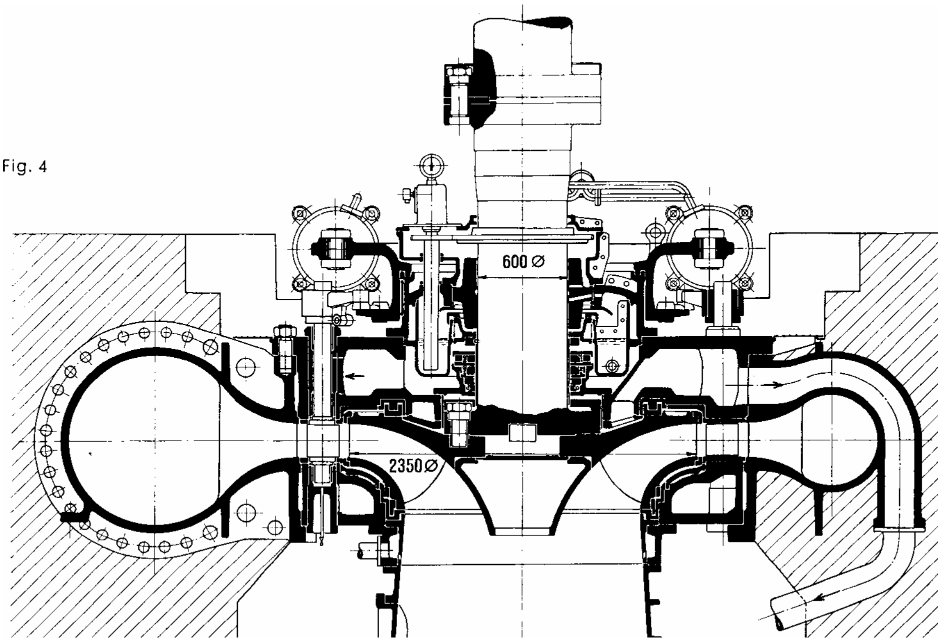


Fig.15

The Francis Wheel is an inward flow device with the water entering around the periphery and moving to the centre before exhausting. The rotor is contained in a casing that spreads the flow and pressure evenly around the periphery.

The impulse part comes about because guide vanes are used to produce an initial velocity v_1 that is directed at the rotor.

Pressure drop occurs in the guide vanes and the velocity is $v_1 = k (\Delta H)^{½}$ where ΔH is the head drop in the guide vanes.

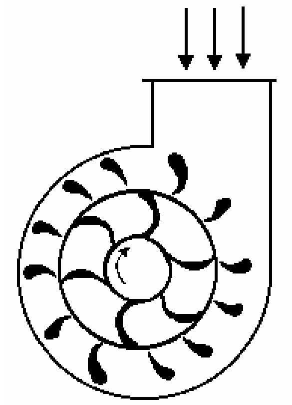


Fig.16

The angle of the guide vanes is adjustable so that the inlet angle β_1 is correct for shockless entry.

The shape of the rotor is such that the vanes are taller at the centre than at the ends. This gives control over the radial velocity component and usually this is constant from inlet to outlet. The volume flow rate is usually expressed in terms of radial velocity and circumferential area.

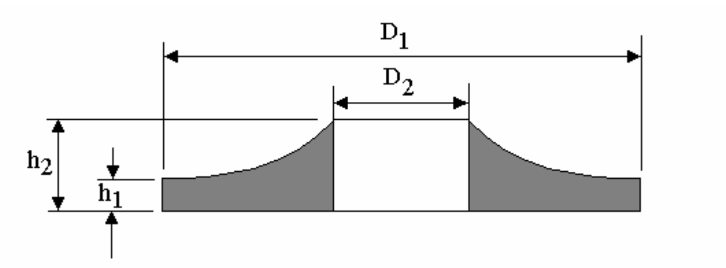


Fig.17

v_R = radial velocity   A = circumferential area = $\pi D\, h\ k$

$Q = v_R\, \pi D\, h\ k$ h = height of the vane.

k is a factor which allows for the area taken up by the thickness of the vanes on the circumference. If v_R is constant then since Q is the same at all circumferences,

$$D_1 h_1 = D_2 h_2.$$

VECTOR DIAGRAMS

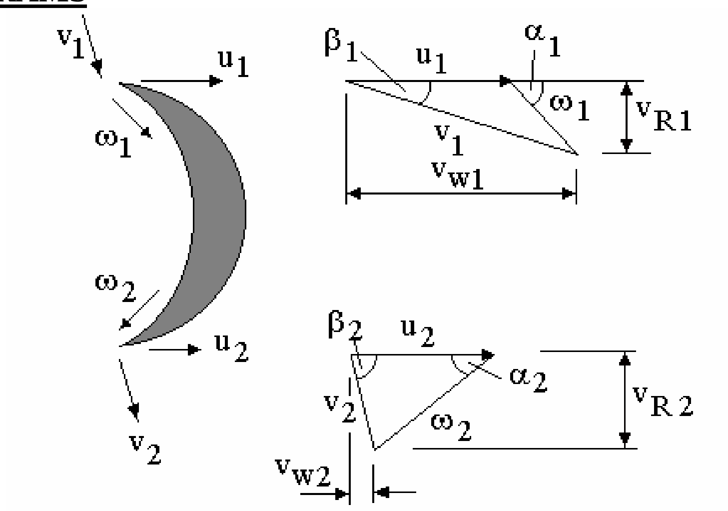


Fig. 18

The diagram shows how the vector diagrams are constructed for the inlet and outlet. Remember the rule is that the vectors add up so that $u + v = \omega$

If u is drawn horizontal as shown, then V_W is the horizontal component of v and v_R is the radial component (vertical).

MORE DETAILED EXAMINATION OF VECTOR DIAGRAM

Applying the sine rule to the inlet triangle we find

$$\frac{v_1}{\sin(180-\alpha_1)} = \frac{u_1}{\sin\{180-\beta_1-(180-\alpha_1)\}}$$

$$\frac{v_1}{\sin(\alpha_1)} = \frac{u_1}{\sin(\alpha_1-\beta_1)} \qquad v_1 = \frac{u_1\sin(\alpha_1)}{\sin(\alpha_1-\beta_1)}.................(1)$$

$$\text{Also} \quad v_1 = \frac{v_{r1}}{\sin(\beta_1)}............(2) \qquad v_{r1} = v_{w1}\tan\beta_1................(3)$$

$$\text{equate (1) and (2)} \qquad \frac{u_1\sin(\alpha_1)}{\sin(\alpha_1-\beta_1)} = \frac{v_{r1}}{\sin(\beta_1)}$$

$$v_{r1} = \frac{u_1\sin(\alpha_1)\sin(\beta_1)}{\sin(\alpha_1-\beta_1)}...........(4) \qquad \text{equate (3) and (4)} \qquad v_{w1} = \frac{u_1\sin(\alpha_1)\sin(\beta_1)}{\sin(\alpha_1-\beta_1)\tan\beta_1}.............(5)$$

If all the angles are known, then v_{w1} may be found as a fraction of u_1.

DIAGRAM POWER

Because u is different at inlet and outlet we express the diagram power as :

$$\text{D.P.} = m\,\Delta(uv_w) = m\,(u_1v_{w1} - u_2v_{w2})$$

The kinetic energy represented by v_2 is energy lost in the exhausted water. For maximum efficiency, this should be reduced to a minimum and this occurs when the water leaves radially with no whirl so that $v_{w2} = 0$. This is produced by designing the exit angle to suit the speed of the wheel. The water would leave down the centre hole with some swirl in it. The direction of the swirl depends upon the direction of v_2 but if the flow leaves radially, there is no swirl and less kinetic energy. Ideally then,

$$\text{D.P.} = m\,u_1\,v_{w1}$$

WATER POWER

The water power supplied to the wheel is $mg\Delta H$ where ΔH is the head difference between inlet and outlet.

HYDRAULIC EFFICIENCY

The maximum value with no swirl at exit is η_{hyd} = D.P./W.P. = $u_1 v_{w1}/g\rho H$

OVERALL EFFICIENCY

$\eta_{o/a}$= Shaft Power/Water Power

$\eta_{o/a} = 2\pi NT/mg\Delta H$

LOSSES

The hydraulic losses are the difference between the water power and diagram power.

Loss = $mg\Delta H - mu_1 v_{w1} = mgh_L$ $\quad h_L = \Delta H - u_1 v_{w1}/g \quad \Delta H - h_L = u_1 v_{w1}/g$

WORKED EXAMPLE No. 2

The following data is for a Francis Wheel.

Radial velocity is constant No whirl at exit.

Flow rate	0.189 m^3/s		
D_1= 0.6 m	D_2=0.4 m	k =0.85	h_1=50 mm
$\alpha_1=110^o$	N=562 rev/min		

Head difference from inlet to outlet is 32 m. Entry is shockless. Calculate

i. the guide vane angle
ii. the diagram power
iii. the hydraulic efficiency
iv. the outlet vane angle
v. the blade height at outlet.

SOLUTION

$u_1=\pi ND_1$ = 17.655 m/s $v_{r1}= Q/(\pi D_1 h_1 k)$= 0.189/($\pi$ x 0.6 x 0.05 x 0.85) = 2.35 m/s

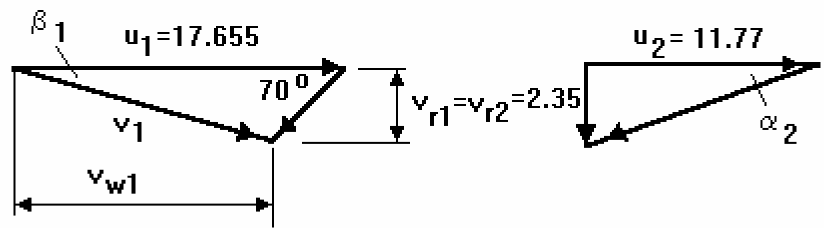


Fig. 19

v_{w1} and β_1 may be found by scaling or by trigonometry.

v_{w1}=16.47m/s $\quad\beta_1=8.12^o$ $\quad u_2=\pi ND_2$ = 11.77 m/s

$\alpha_2= \tan^{-1}$ (2.35/11.77) = 11.29^o

D.P. = m $u_1 v_{w1}$=189 (17.655 x 16.47) = 54 957 Watts

W.P.= $mg\Delta H$ = 189 x 9.81 x 32 = 59 331 Watts

η_{hyd} = 54 957/59 331 = 92.6%

since $v_{r1} = v_{r2}$ then $D_1 h_1 = D_2 h_2$

h_2 = 0.6 x 0.05/0.4 = 0.075 m

WORKED EXAMPLE No. 3

The runner (rotor) of a Francis turbine has a blade configuration as shown. The outer diameter is 0.4 m and the inner diameter is 0.25 m. The vanes are 65 mm high at inlet and 100 mm at outlet. The supply head is 20 m and the losses in the guide vanes and runner are equivalent to 0.4 m. The water exhausts from the middle at atmospheric pressure. Entry is shockless and there is no whirl at exit. Neglecting the blade thickness, determine :

i. the speed of rotation.
ii. the flow rate.
iii. the output power given a mecanical efficiency of 88%.
iv. the overall efficiency.
v. The outlet vane angle.

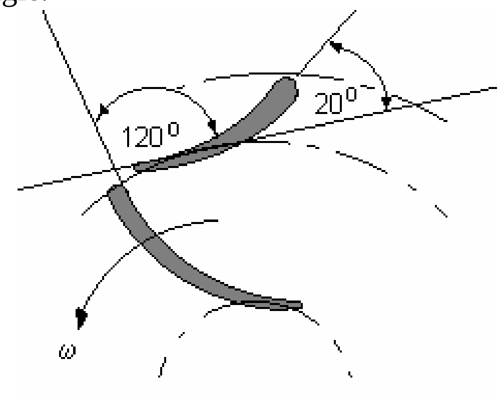


Fig.20

SOLUTION

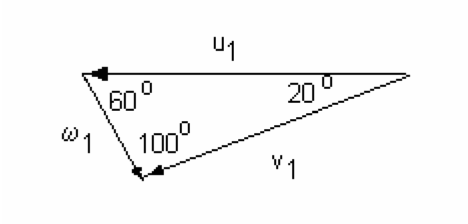


Fig.21

$$v_{w1} = \frac{u_1 \sin(\alpha_1)\sin(\beta_1)}{\sin(\alpha_1 - \beta_1)\tan\beta_1}$$

$$v_{w1} = \frac{u_1 \sin(120)\sin(20)}{\sin(100)\tan(20)} = 0.826u_1$$

The inlet vector diagram is as shown. Values can be found by drawing to scale.
Since all angles are known but no flow rate, find v_{w1} in terms of u_1

$\Delta H - h_L = u_1 v_{w1}/g$

$20 - 0.4 = 19.6 = u_1 v_{w1}/g$

$19.6 = 0.826\ u_1^2/g$

$u_1 = 15.26$ m/s

$u_1 = \pi N D_1/60$

$N = 15.26 \times 60/(\pi \times 0.4) = 728.5$ rev/min

$$v_{r1} = \frac{u_1 \sin(\alpha_1)\sin(\beta_1)}{\sin(\alpha_1 - \beta_1)} = \frac{15.26 \sin(120)\sin(20)}{\sin(100_1)} = 4.589\text{m/s}$$

$Q = v_{r1} \times \pi D_1 h_1 = 12.6 \times \pi \times 0.4 \times 0.065 = 0.375\ m^3/s$
m = 375 kg/s

$v_{w1} = 0.826\ u_1 = 12.6$ m/s

Diagram Power = $m\ u_1\ v_{w1} = 375 \times 15.26 \times 12.6 = 72.1$ kW

Output power = 0.88 x 72.1 = 63.45 kW

OUTLET TRIANGLE
$u_2 = \pi N D_2/60 = \pi \times 728.5 \times 0.25/60 = 9.54$ m/s
$Q = v_{r2} \times \pi D_2 h_2$
$0.375 = v_{r2} \times \pi \times 0.25 \times 0.1$
$v_{r2} = 4.775$ m/s $= v_2$ if no whirl.
$\tan \alpha_2 = 4.775/9.54 = 0.5$
$\alpha_2 = 26.6^o$.

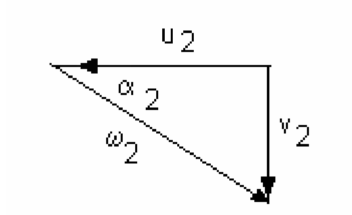


Fig. 22

SELF ASSESSMENT EXERCISE No. 2

1. The following data is for a Francis Wheel
Radial velocity is constant
No whirl at exit.
Flow rate=0.4 m^3/s
D_1=0.4 m
D_2=0.15 m
k =0.95
$\alpha_1=90^o$
N=1000 rev/min
Head at inlet = 56 m
head at entry to rotor = 26 m
head at exit = 0 m
Entry is shockless.

Calculate i. the inlet velocity v_1 (24.26 m/s)
ii. the guide vane angle (30.3^o)
iii. the vane height at inlet and outlet (27.3 mm, 72.9 mm)
iv. the diagram power (175.4 MW)
v. the hydraulic efficiency (80%)

2. A radial flow turbine has a rotor 400 mm diameter and runs at 600 rev/min. The vanes are 30 mm high at the outer edge. The vanes are inclined at 42^o to the tangent to the inner edge. The flow rate is 0.5 m^3/s and leaves the rotor radially. Determine

i. the inlet velocity as it leaves the guide vanes. (19.81 m/s)
ii. the inlet vane angle. (80.8^o)
iii. the power developed. (92.5 kW)

3. The runner (rotor) of a Francis turbine has a blade configuration as shown. The outer diameter is 0.45 m and the inner diameter is 0.3 m. The vanes are 62.5 mm high at inlet and 100 mm at outlet. The supply head is 18 m and the losses in the guide vanes and runner are equivalent to 0.36 m. The water exhausts from the middle at atmospheric pressure. Entry is shockless and there is no whirl at exit. Neglecting the blade thickness, determine :

i. The speed of rotation. (1691 rev/min)
ii. The flow rate. (1.056 m^3/s)
iii. The output power given a mechanical efficiency of 90%. (182.2 MW)
iv. The overall efficiency. (88.2%)
v. The outlet vane angle. (22.97^o)

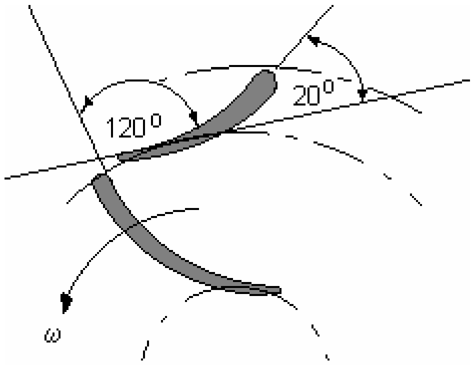


Fig. 23

FLUID MECHANICS

TUTORIAL No.8B

CENTRIFUGAL PUMPS

When you have completed this tutorial you should be able to

- Derive the dimensionless parameters of a pump
  - Flow Coefficient
  - Head Coefficient
  - Power Coefficient
  - Specific Speed.
- Explain how to match a pump to system requirements.
- Explain the general principles of Centrifugal Pumps.
- Construct blade vector diagrams for Centrifugal Pumps.
- Deduce formulae for power and efficiency and Head.
- Solve numerical problems for Centrifugal Pumps.

1. **DIMENSIONAL ANALYSIS**

The power 'P' of any rotary hydraulic pump depends upon the density 'ρ' , the speed 'N', the characteristic diameter 'D', the head change 'ΔH', the volume flow rate 'Q' and the gravitational constant 'g'. The general equation is:

$$P = f(\rho, N, D, \Delta H, Q, g)$$

It is normal to consider gΔH as one quantity. $P = f\{\rho, N, D, (g\Delta H), Q\}$

There are 6 quantities and 3 dimensions so there are three dimensionless groups Π_1, Π_2 and Π_3. First form a group with P and ρND.

$$P = \varphi(\rho ND) = \Pi_1 \rho^a N^b D^c$$

$$M^1L^2T^{-3} = \left(ML^{-3}\right)^a \left(T^{-1}\right)^b \left(D^1\right)^c$$

$$\text{Mass} \quad 1 = a$$

$$\text{Time} \quad -3 = -b \qquad b = 3$$

$$\text{Length} \;\; 2 = -3a + c = -3 + c \quad c = 5$$

$$P = \Pi_1 \rho^1 N^3 D^5 \qquad \Pi_1 = \frac{P}{\rho\ N^3D^5} = \text{ Power Coefficient}$$

Next repeat the process between Q and ρND

$$Q = \varphi(\rho ND) = \Pi_2 \rho^a N^b D^c$$

$$M^3T^{-1} = \left(ML^{-3}\right)^a \left(T^{-1}\right)^b \left(D^1\right)^c$$

$$\text{Time} \quad -1 = -b \qquad b = 1$$

$$\text{Mass} \quad 0 = a$$

$$\text{Length} \;\; 3 = -3a + c \quad c = 3$$

$$Q = \Pi_2 \rho^0 N^1 D^3 \qquad \Pi_2 = \frac{Q}{ND^3} = \text{ Flow Coefficient}$$

Next repeat the process between gΔH and ρND

$$(g\,\Delta H) = \varphi(\rho ND) = \Pi_3 \rho^a N^b D^c$$

$$M^0L^2T^{-2} = \left(ML^{-3}\right)^a \left(T^{-1}\right)^b \left(D^1\right)^c$$

$$\text{Mass} \quad 0 = a \qquad \text{Time} \quad -2 = -b \qquad b = 2 \qquad \text{Length} \;\; 2 = -3a + c \quad c = 2$$

$$Q = \Pi_3 \rho^0 N^2 D^2 \qquad \Pi_3 = \frac{Q}{N^2D^2} = \text{ Head Coefficient}$$

Finally the complete equation is $\dfrac{P}{\rho N^3 D^5} = \varphi\left(\dfrac{Q}{ND^3}\right)\left(\dfrac{g\,\Delta H}{N^2D^2}\right)$

SPECIFIC SPEED Ns

The specific speed is a parameter used for pumps and turbines to determine the best design to match a given pumped system. The formula may be derived from consideration of the pump geometry or by dimensional analysis. The latter will be used here.

$$\frac{P}{\rho N^3 D^5} = \varphi\left(\frac{Q}{ND^2}\right)\left(\frac{g\,\Delta H}{N^2 D^2}\right)$$

The three dimensionless numbers represent the power coefficient, the flow coefficient and the head coefficient respectively. Now consider a family of geometrically similar machines operating at dynamically similar conditions. For this to be the case the coefficients must have the same values for each size. Let the 3 coefficients be Π_1, Π_2 and Π_3 such that

$$\Pi_1 = \frac{P}{\rho N^3 D^5} \qquad \Pi_2 = \frac{Q}{ND^3} \quad D = \left(\frac{Q}{N\Pi_2}\right)^{\frac{1}{3}} \qquad \Pi_3 = \frac{g\,\Delta H}{N^2 D^2} \quad D = \left(\frac{g\,\Delta H}{N^2 \Pi_3}\right)^{\frac{1}{2}}$$

Equating $\left(\frac{Q}{N\Pi_2}\right)^{\frac{1}{3}} = \left(\frac{g\,\Delta H}{N^2 \Pi_3}\right)^{\frac{1}{2}}$

$$\frac{1}{N}\left(\frac{g\,\Delta H}{\Pi_3}\right)^{\frac{1}{2}} = \frac{Q^{\frac{1}{3}}}{\Pi_2^{\frac{1}{3}} N^{\frac{1}{3}}}$$

$$\frac{(\Delta H)^{\frac{1}{2}}}{Q^{\frac{1}{3}} N^{\frac{2}{3}}} = \frac{\Pi_3^{\frac{1}{2}}}{\Pi_2^{\frac{1}{3}} g^{\frac{1}{2}}} = \text{constant}$$

$$\frac{(\Delta H)^{\frac{1}{2}}}{KQ^{\frac{1}{3}}} = N^{\frac{2}{3}}$$

$$\left[\frac{(\Delta H)^{\frac{1}{2}}}{KQ^{\frac{1}{3}}}\right]^{\frac{3}{2}} = N = \frac{(\Delta H)^{\frac{3}{4}}}{K^{\frac{1}{2}} Q^{\frac{1}{2}}} \qquad \frac{NQ^{\frac{1}{2}}}{(\Delta H)^{\frac{3}{4}}} = K^{-\frac{1}{2}} = \text{constant}$$

This constant is called the Specific Speed $Ns = \frac{NQ^{\frac{1}{2}}}{(\Delta H)^{\frac{3}{4}}}$

Ns is a dimensionless parameter that and the units used are normally rev/min for speed, m^3/s for flow rate and metres for head. Other units are often used and care should be taken when quoting Ns values.

It follows that for a given speed, the specific speed is large for large flows and low heads and small for small flows and large heads. The important value is the one that corresponds to the conditions that produce the greatest efficiency.

2. MATCHING PUMPS TO SYSTEM REQUIREMENTS

The diagram shows a typical relationship between the head and flow of a given CF pump at a given speed.

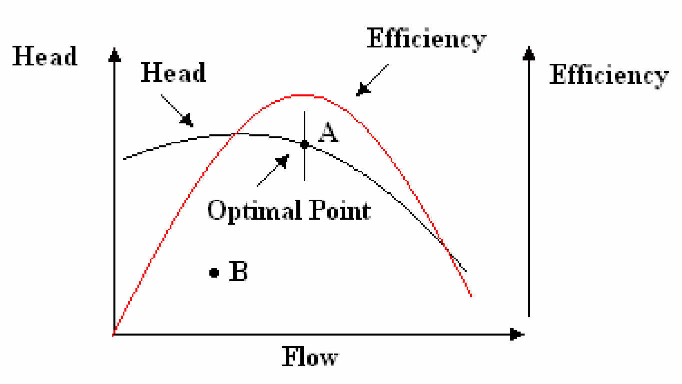


Figure 1

The Ns value may be calculated using the flow and head corresponding to the maximum efficiency at point A.

SELECTING PUMP SIZE

The problem is that the optimal point of any given pump is unlikely to correspond to the system requirements for example at point B. What we should do ideally is find a geometrically similar pump that will produce the required head and flow at the optimal point.

The geometrically similar pump will run under dynamically similar conditions so it follows that the specific speed Ns is the same for both pumps at the optimal point. The procedure is to first calculate the specific speed of the pump using the flow and head at the optimal conditions.

$$N_s = \frac{N_A Q_A^{\frac{1}{2}}}{H_A^{\frac{3}{4}}}$$

Suppose point B is the required operating point defined by the system.

$N_s = \dfrac{N_B Q_B^{\frac{1}{2}}}{H_B^{\frac{3}{4}}}$ Equating, we can calculate N_B, the speed of the geometrically similar pump.

We still don't know the size of the pump that will produce the head and flow at B. Since the head and flow coefficients are the same then:-

Equating Flow Coefficients we get $D_B = D_A\left(\dfrac{Q_B N_A}{Q_A N_B}\right)^{1/3}$

Equating head coefficients we get we get $D_B = \dfrac{N_A}{N_B}\sqrt{\dfrac{H_B}{H_A}}$

If the forgoing is correct then both will give the same answer.

WORKED EXAMPLE No. 1

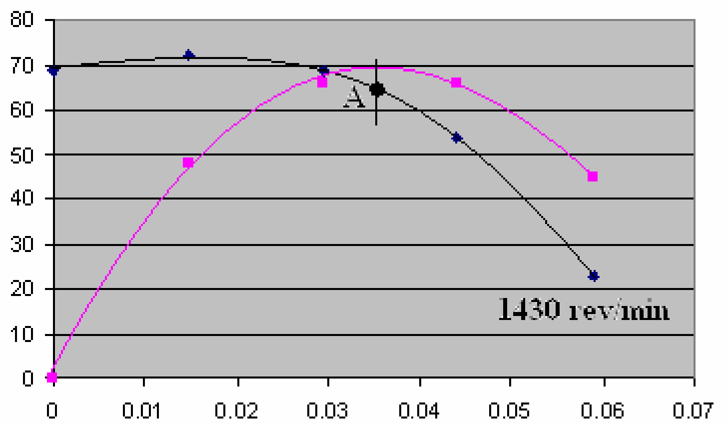
A centrifugal pump is required to produce a flow of water at a rate of 0.0160 m^3/s against a total head of 30.5 m. The operating characteristic of a pump at a speed of 1430 rev/min and a rotor diameter of 125 mm is as follows.

Efficiency	0	48	66	66	45	%
Q_A	0	0.0148	0.0295	0.0441	0.059	m^3/s
H_A	68.6	72	68.6	53.4	22.8	m

Determine the correct size of pump and its speed to produce the required head and flow.

SOLUTION

Plot the data for the pump and determine that the optimal head and flow are 65 m and 0.036m^3/s



Calculate Ns at point A $N_s = \frac{N_A Q_A^{\frac{1}{2}}}{H_A^{\frac{3}{4}}} = \frac{1430 \text{ x } 0.036^{1/2}}{65^{3/4}} = 11.85$

Calculate the Speed for a geometrically similar pump at the required conditions.

$$N_B = Ns\frac{H_B^{\frac{3}{4}}}{Q_B^{\frac{1}{2}}} = \frac{11.85 \text{ x } 30.5^{3/4}}{0.016^{1/2}} = 1216 \text{ rev/min}$$

Next calculate the diameter of this pump.

$$D_B = D_A\left(\frac{Q_B N_A}{Q_A N_B}\right)^{1/3} = 125\left(\frac{0.016 \text{ x } 1430}{0.036 \text{ x } 1216}\right)^{1/3} = 101 \text{ mm}$$

or $D_B = D_A \frac{N_A}{N_B}\sqrt{\frac{H_B}{H_A}} = 125 \text{ x } \frac{1430}{1216}\sqrt{\frac{30.5}{65}} = 101 \text{ mm}$

Answer:- we need a pump 101 mm diameter running at 1216 rev/min.

RUNNING WITH THE WRONG SIZE

In reality we are unlikely to find a pump exactly the right size so we are forced to use the nearest we can get and adjust the speed to obtain the required flow and head. Let B be the required operating point and A the optimal point for the wrong size pump. We make the flow and head coefficients the same for B and some other point C on the operating curve. The diameters cancel because they are the same.

$$\frac{Q_B}{N_B D_B^3} = \frac{Q_C}{N_C D_C^3} \qquad Q_B = Q_C \frac{N_B}{N_C} \qquad \frac{g H_C}{N_C^2 D_C^2} = \frac{g H_A}{N_C^2 D_C^2} \qquad H_B = H_C \frac{N_B^2}{N_C^2}$$

Substitute $\frac{N_B}{N_A} = \frac{Q_B}{Q_A}$ to eliminate the speed

$$H_C = H_B \left(\frac{Q_C}{Q_B}\right)^2$$

This is a family of parabolic curves starting at the origin. If we take the operating point B we can determine point C as the point where it intersects the operating curve at speed A.
The important point is that the efficiency curve is unaffected so at point B the efficiency is not optimal.

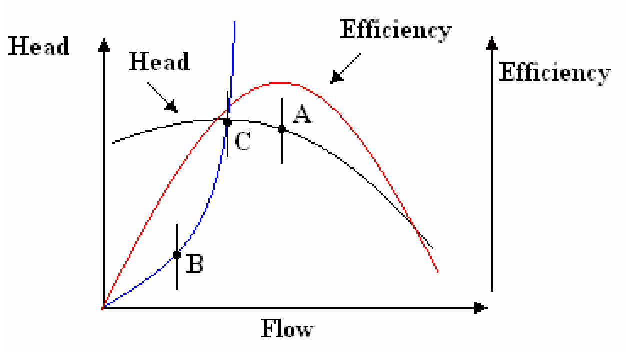


Figure 2

WORKED EXAMPLE No. 2

If only the 125 mm pump in WE 1 is available, what speed must it be run at to obtain the required head and flow? What is the efficiency and input power to the pump?

SOLUTION

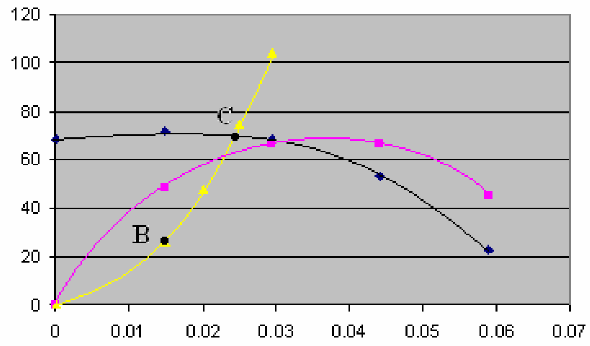
B is the operating point so we must calculate H_C and Q_C

$$H_C = H_B \left(\frac{Q_C}{Q_B}\right)^2 = 30.5\left(\frac{Q_C}{0.016}\right)^2 = 119141 Q_C^2$$

This must be plotted to determine Q_C
From the plot H_C = 74 m
Q_C = 0.025 m^3/s
Equate flow coefficients to find the speed at B

$$\frac{Q_B}{N_B D_B^3} = \frac{Q_C}{N_C D_C^3} \qquad \frac{0.016}{N_B} = \frac{0.025}{1430}$$

N_B = 915 rev/min



Check by repeating the process with the head coefficient.

$$\frac{g H_B}{N_B^2 D_B^2} = \frac{g H_A}{N_A^2 D_A^2} \qquad N_B = N_A\sqrt{\frac{H_B}{H_A}} = 1430\sqrt{\frac{30.5}{74}} = 918 \text{ rev/min}$$

The efficiency at this point is 63% Water Power = mgH = 16 x 9.81 x 30.5 = 4787 W

Power Input = WP/η = 4787/0.63 =7598 W

WORKED EXAMPLE No.3

A pump draws water from a tank and delivers it to another with the surface 8 m above that of the lower tank. The delivery pipe is 30 m long, 100 bore diameter and has a friction coefficient of 0.003. The pump impeller is 500 mm diameter and revolves at 600 rev/min. The pump is geometrically similar to another pump with an impeller 550 mm diameter which gave the data below when running at 900 rev/min.

ΔH (m)	37	41	44	45	42	36	28
$Q(m^3/s)$	0	0.016	0.32	0.048	0.064	0.08	0.096

Determine the flow rate and developed head for the pump used.

SOLUTION

First determine the head flow characteristic for the system.
ΔH = developed head of the pump = $8 + 4fLu^2/2gd$ + minor losses
No details are provided about minor losses so only the loss at exit may be found.
$h_L = 4fLu^2/2gd + u^2/2g$
$\Delta H = = 8 + 4fLu^2/2gd + u^2/2g$

$u = 4Q/\pi d^2 = 127.3\ Q$
$\Delta H = 8 + 4x\ 0.003 \text{ x } 30\ 000(127.3Q)^2/(2g \text{ x } 0.1) + (127.3Q)^2/2g$
$\Delta H = 8 + 3800Q^2$

Produce a table and plot ΔH against Q for the system.

ΔH (m)	8	8.38	14.08	32.3	46
$Q(m^3/s)$	0	0.01	0.04	0.08	0.1

Plot the system head and pump head against flow and find the matching point.
This is at H = 34.5 and Q = 0.084 m^3/s

Next determine the head - flow characteristic for the pump actually used by assuming dynamic and geometric similarity.

Flow Coefficient Q/ND^3 = constant
$Q_2 = Q_2\ (N_1/N_2)(D_1{}^3/D_2{}^3)$ $\qquad$ $Q_2 = (600/900)\ \ (500/550)^3 = 0.5\ Q_1$
$\Delta H/(ND)^2$ = constant
$\Delta H_2 = \Delta H_1(N_2D_2/N_1D_1)^2$ $\qquad$ $\Delta H_2 = \Delta H_1\{600\ \text{ x } 500/900 \text{ x } 550\}^2 = 0.367\ \Delta H_1$

Produce a table for the pump using the coefficients and data for the first pump.

ΔH_2 (m)	13.58	15.05	16.15	16.51	15.41	13.21	10.28
$Q_2(m^3/s)$	0	0.08	0.016	0.024	0.032	0.04	0.048

Plot this graph along with the system graph and pick off the matching point.

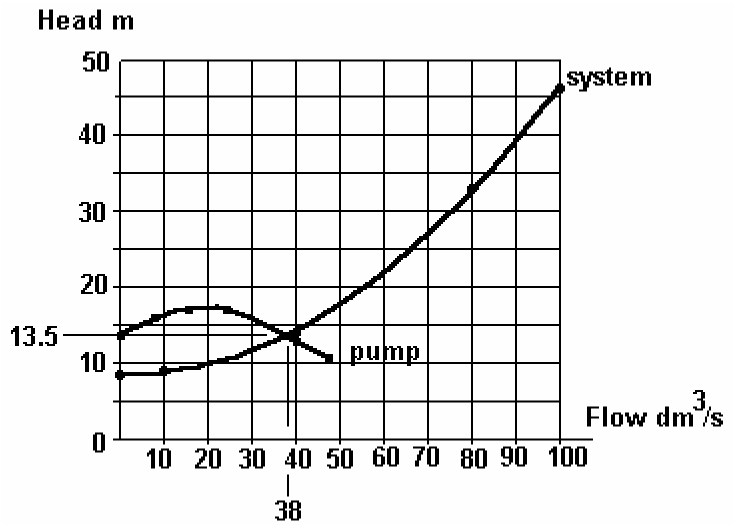


Figure 3

Ans. 13.5 m head and 38 dm^3/s flow rate.

SELF ASSESSMENT EXERCISE No.1

1. A centrifugal pump must produce a head of 15 m with a flow rate of 40 dm^3/s and shaft speed of 725 rev/min. The pump must be geometrically similar to either pump A or pump B whose characteristics are shown in the table below.

   Which of the two designs will give the highest efficiency and what impeller diameter should be used ?

   Pump A D = 0.25 m N = 1 000 rev/min

Q (dm^3/s)	8	11	15	19
H (m)	8.1	7.9	7.3	6.1
η%	48	55	62	56

   Pump B D = 0.55 m N = 900 rev/min

Q (dm^3/s)	6	8	9	11
H (m)	42	36	33	27
η%	55	65	66	58

   Answer Pump B with D= 0.455 m

2. Define the Head and flow Coefficients for a pump.

   Oil is pumped through a pipe 750 m long and 0.15 bore diameter. The outlet is 4 m below the oil level in the supply tank. The pump has an impeller diameter of 508 mm which runs at 600 rev/min. Calculate the flow rate of oil and the power consumed by the pump. It may be assumed $C_f=0.079(Re)^{-0.25}$. The density of the oil is 950 kg/m^3 and the dynamic viscosity is 5×10^{-3} N s/m^2.

   The data for a geometrically similar pump is shown below.

   D = 0.552 m N = 900 rev/min

Q (m^3/min)	0	1.14	2.27	3.41	4.55	5.68	6.86
H (m)	34.1	37.2	39.9	40.5	38.1	32.9	25.9
η%	0	22	41	56	67	72	65

   Answer 2 m^3/min and 7.89 kWatts

3. GENERAL THEORY

A Centrifugal pump is a Francis turbine running backwards. The water between the rotor vanes experiences centrifugal force and flows radially outwards from the middle to the outside. As it flows, it gains kinetic energy and when thrown off the outer edge of the rotor, the kinetic energy must be converted into flow energy. The use of vanes similar to those in the Francis wheel helps. The correct design of the casing is also vital to ensure efficient low friction conversion from velocity to pressure. The water enters the middle of the rotor without swirling so we know v_{w1} is always zero for a c.f. pump. Note that in all the following work, the inlet is suffix 1 and is at the inside of the rotor. The outlet is suffix 2 and is the outer edge of the rotor.

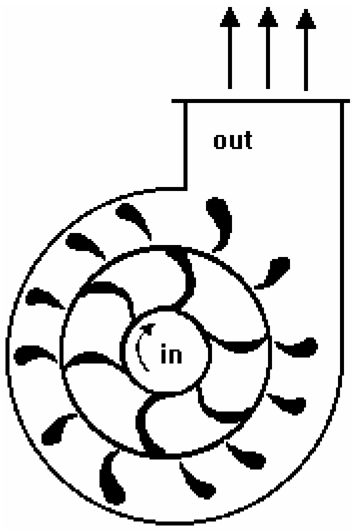


Fig. 1 Basic Design

The increase in momentum through the rotor is found as always by drawing the vector diagrams. At inlet v_1 is radial and equal to v_{r1} and so v_{w1} is zero. This is so regardless of the vane angle but there is only one angle which produces shockless entry and this must be used at the design speed.

At outlet, the shape of the vector diagram is greatly affected by the vane angle. The diagram below shows a typical vector diagram when the vane is swept backwards (referred to the vane velocity u).

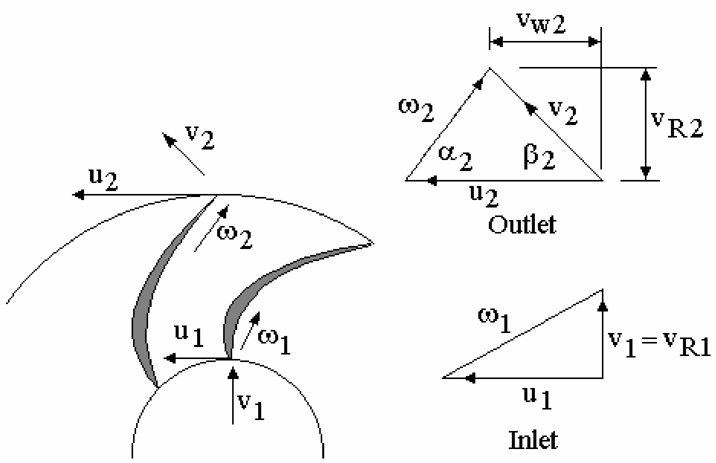


Fig. 2

v_{w2} may be found by scaling from the diagram. We can also apply trigonometry to the diagram as follows.

$$v_{w2} = u_2 - \frac{v_{r2}}{\tan\alpha_2} = u_2 - \frac{Q}{A_2\tan(\alpha_2)} = u_2 - \frac{Q}{\pi k D_2 t_2 \tan(\alpha_2)} \qquad u_2 = \pi N D_2$$

t is the height of the vane and k is the correction factor for the blade thickness.

3.1. DIAGRAM POWER D.P.= $m\Delta u v_W$

since usually v_{W1} is zero this becomes D.P.= $m u_2 v_{W_2}$

3.2. WATER POWER W.P. = $mg\Delta h$

Δh is the pressure head rise over the pump.

3.3. MANOMETRIC HEAD Δh_m

This is the head that would result if all the energy given to the water is converted into pressure head. It is found by equating the diagram power and water power.

$$m u_2 v_{w2} = mg\Delta g_m \qquad \Delta h_m = \frac{u_2 v_{w2}}{g} = \frac{u_2}{g}\left\{u_2 - \frac{Q}{A_2 \tan(\alpha_2)}\right\}$$

3.4. MANOMETRIC EFFICIENCY η_m

$$\eta_m = \frac{\text{Water Power}}{\text{Diagram Power}} = \frac{mg\,\Delta h}{m u_2 v_{w2}} = \frac{mg\,\Delta h}{mg\,\Delta h_m} = \frac{\Delta h}{\Delta h_m}$$

3.5. SHAFT POWER S.P. = $2\pi NT$

3.6. OVERALL EFFICIENCY $\eta_{o/a} = \dfrac{\text{Water Power}}{\text{Shaft Power}}$

3.7. KINETIC ENERGY AT ROTOR OUTLET $\text{K.E.} = \dfrac{m v_2^2}{2}$

Note the energy lost is mainly in the casing and is usually expressed as a fraction of the K.E. at exit.

3.8. NO FLOW CONDITION

There are two cases where you might want to calculate the head produced under no flow condition. One is when the outlet is blocked say by closing a valve, and the other is when the speed is just sufficient for flow to commence.

Under normal operating conditions the developed head is given by the following equation.

$$\Delta h = \frac{u_2 v_{w2}}{g} = \frac{u_2}{g}\left\{u_2 - \frac{Q}{A_2 \tan(\alpha_2)}\right\}$$

When the outlet valve is closed the flow is zero. The developed head is given by the following equation. $\Delta h = \dfrac{u_2}{g}\{u_2 - 0\} = \dfrac{u_2^2}{g}$

When the speed is reduced until the head is just sufficient to produce flow and overcome the static head, the radial velocity v_{r2} is zero and the fluid has a velocity u_2 as it is carried around with the rotor. The kinetic energy of the fluid is $\dfrac{m u_2^2}{2}$ and this is converted into head equal to the static head.

It follows that $h_s = \dfrac{u_2^2}{2g}$. Substituting $u_2 = \dfrac{\pi N D_2}{60}$ we find that $N = 83.5\dfrac{\sqrt{h_s}}{D}$.

WORKED EXAMPLE No.4

A centrifugal pump has the following data :

Rotor inlet diameter	D_1 = 40 mm
Rotor outlet diameter	D_2 = 100 mm
Inlet vane height	h_1 = 60 mm
Outlet vane height	h_2 = 20 mm
Speed	N =1420 rev/min
Flow rate	Q = 0.0022 m^3/s
Blade thickness coefficient	k = 0.95

The flow enters radially without shock.
The blades are swept forward at 30^o at exit.
The developed head is 5 m and the power input to the shaft is 170 Watts.

Determine the following.

i. The inlet vane angle
ii. The diagram power
iii. The manometric head
iv. The manometric efficiency
v. The overall efficiency.
vi. The head produced when the outlet valve is shut.
vii. The speed at which pumping commences for a static head of 5 m.

SOLUTION

$u_1 = \pi ND_1 = 2.97$ m/s
$u_2 = \pi ND_2 = 7.435$ m/s
$v_{r_1} = Q/k\pi D_1 h_1 = 0.307$ m/s
$v_{r2} = Q/k\pi D_2 h_2 = 0.368$ m/s
Since the flow enters radially $v_1 = v_{r_1} = 0.307$ m/s and $v_{w1} = 0$
From the inlet vector diagram the angle of the vane that produces no shock is found as follows:
$\tan\alpha_1 = 0.307/2.97$ hence $\alpha_1 = 5.9^o$.

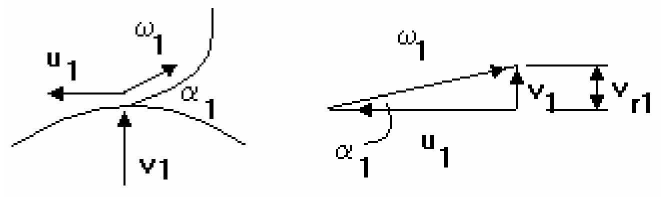


Fig. 41
Inlet vector diagram

From the outlet vector diagram we find :

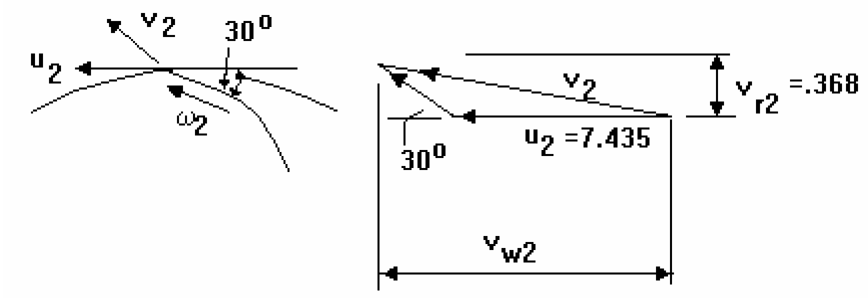


Fig. 42
Outlet vector diagram

v_{w_2} = 7.435 + 0.368/tan 30° = 8.07 m/s
D.P.= $mu_2v_{w_2}$
D.P.= 2.2 x 7.435 x 8.07 = 132 Watts
W.P. = mgΔh = 2.2 x 9.81 x 5 = 107.9 Watts
Δh_m = W.P./D.P.= 107.9/132 = 81.7%
$\Delta h_m = u_2v_{w_2}/g$ = 7.435 x 8.07/9.81 =6.12 m
$\Delta_m = \Delta h/\Delta h_m$ = 5/6.12 = 81.7%
$\eta_{o/a}$ = W.P./S.P. = 107.9/170 = 63.5%

When the outlet valve is closed the static head is $\Delta h = \frac{u_2^2}{g} = \frac{7.435^2}{9.81} = 5.63\,\text{m}$

The speed at which flow commences is $N = 83.5\frac{\sqrt{h_s}}{D} = 83.5\frac{\sqrt{5}}{0.1} = 1867$ rev/min

SELF ASSESSMENT EXERCISE No. 2

1. The rotor of a centrifugal pump is 100 mm diameter and runs at 1 450 rev/min. It is 10 mm deep at the outer edge and swept back at 30°. The inlet flow is radial. the vanes take up 10% of the outlet area. 25% of the outlet velocity head is lost in the volute chamber. Estimate the shut off head and developed head when 8 dm^3/s is pumped. (5.87 m and 1.89 m)

2. The rotor of a centrifugal pump is 170 mm diameter and runs at 1 450 rev/min. It is 15 mm deep at the outer edge and swept back at 30°. The inlet flow is radial. the vanes take up 10% of the outlet area. 65% of the outlet velocity head is lost in the volute chamber. The pump delivers 15 dm^3/s of water.

   Calculate
   - i. The head produced. (9.23 m)
   - ii. The efficiency. (75.4%)
   - iii. The power consumed. (1.8 kW)

FLUID MECHANICS

TUTORIAL 8C

PUMPED PIPED SYSTEMS

On completion of this tutorial you should be able to do the following.

- Examine the conditions that produce cavitation.
- Calculate pressure surges due to flow changes.
- Calculate the stress in pipes due to pressure changes.
- Describe how pressure surges are damped.
- Describe how pressure oscillations occur in pipes.

Let's start by examining the meaning of specific speed of pumps.

1. SPECIFIC SPEED Ns

The specific speed is a parameter used for pumps and turbines to determine the best design to match a given pumped system. The formula may be derived from consideration of the pump geometry or by dimensional analysis. The derivation is covered in the next tutorial. The specific speed is defined as

$$N_s = \frac{NQ^{\frac{1}{2}}}{(\Delta H)^{\frac{3}{4}}}$$

Traditionally the units used are rev/min for speed, m^3/s for flow rate and metres for head. The value that corresponds to the most efficient operating point of the pump or turbine is the one of greatest importance.

1. 2. DYNAMIC HEAD AND SUCTION HEAD

Consider a pump delivering liquid from a tank on the suction side into a tank on the outlet side through a pipe.

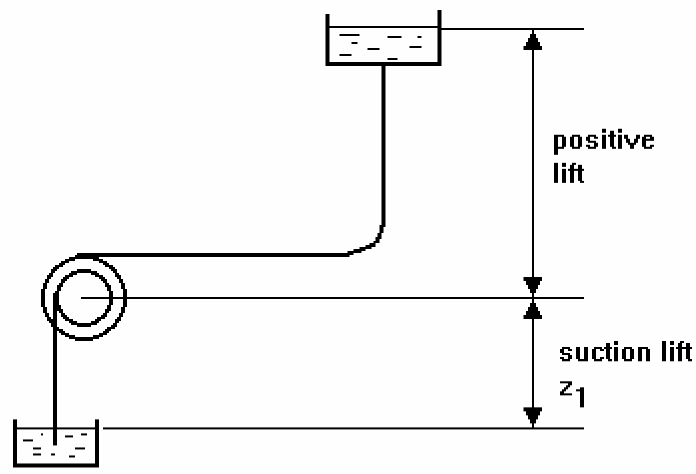


Figure 1

Dynamic head = h_d

h_d = positive lift + head loss

Suction head = h_{suc} = suction lift + head loss

The head loss could include loss at entry, loss in fittings and bends as well as pipe friction.

$$\mathbf{h_{suc} = z_1 + h_{f1} + v_1^2/2g}$$

1.3. CAVITATION

When a liquid cavitates, it turns into a vapour and then suddenly changes back into a liquid with a load cracking sound. The bubbles of vapour cause damage to the metalwork by eroding it away. The main reason for cavitation is due to the local pressure falling below the vapour pressure of the liquid. The vapour pressure is raised with temperature and is more likely to occur in hot liquids. In pumps and turbines, the drop in pressure is often due to the wake set up behind the impeller. The system design is also important to prevent a vacuum forming due to restrictions on the suction side of the pump or negative heads on the outlet side of the turbine. An important parameter used for determining the likelihood of cavitation in pumps is the Nett Positive Suction Head.

1.4. NET POSITIVE SUCTION HEAD (N.P.S.H.)

The Net Positive Suction Head is the amount by which the absolute pressure on the suction side is larger than the vapour pressure (saturation pressure) of the liquid.

NPSH = absolute inlet head - vapour pressure head

Absolute inlet head = $p_a/\rho g - h_s$

where p_a= atmospheric pressure and $h_s = p_s/\rho g$

The vapour pressure varies with temperature and for water is found in thermodynamic temperatures under the heading p_s. (for saturation pressure).
Vapour pressure as a head is $p_s/\rho g$

$$NPSH = (p_a/\rho g - h_{suc}) - p_s/\rho g$$
$$NPSH = (p_a - p_s)/\rho g - h_{suc}$$

WORKED EXAMPLE No.1

A water pump has a suction lift of 5 m. The friction head in the suction pipe is 0.3m. The kinetic head is negligible. The water temperature is 16°C. Atmospheric pressure is 1.011 bar (10.31 m water). Determine the NPSH .

SOLUTION

h_{suc} = 5 + 0.3 = 5.3 m
p_s = 0.01817 bar (from tables)
NPSH = $(1.011 - 0.01817) \times 10^5/(1000 \times 9.81) - 5.3$
NPSH = 4.821 metres of water

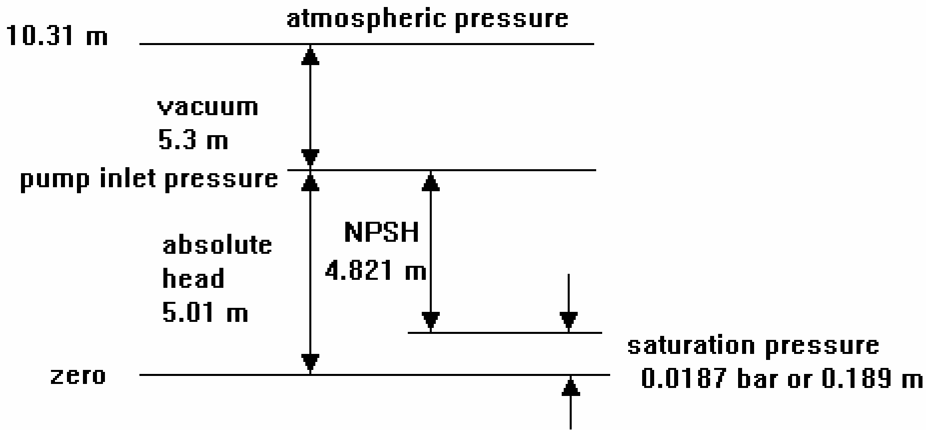


Figure 2

1.5. CAVITATION PARAMETER

A further useful parameter is the cavitation parameter σ. This is defined as

$$\sigma = \text{NPSH}/h_d$$

Values at which cavitation occur are sometimes quoted by manufacturers but as a rough guide they are related to the specific speed and typical values are

$\sigma = 0.05$ when $N_S = 1000$
$\sigma = 1.0$ when $N_S = 8000$

WORKED EXAMPLE No.2

If $\sigma = 0.4$ for the previous example find the minimum delivery head which prevents cavitation.

SOLUTION

$0.4 = 4.821/h_d$ hence $h_d = 12.05$ m.

SELF ASSESSMENT EXERCISE No. 1

1. A pump has a suction pipe and a delivery pipe. The head required to pass water through them varies with flow rate as shown.

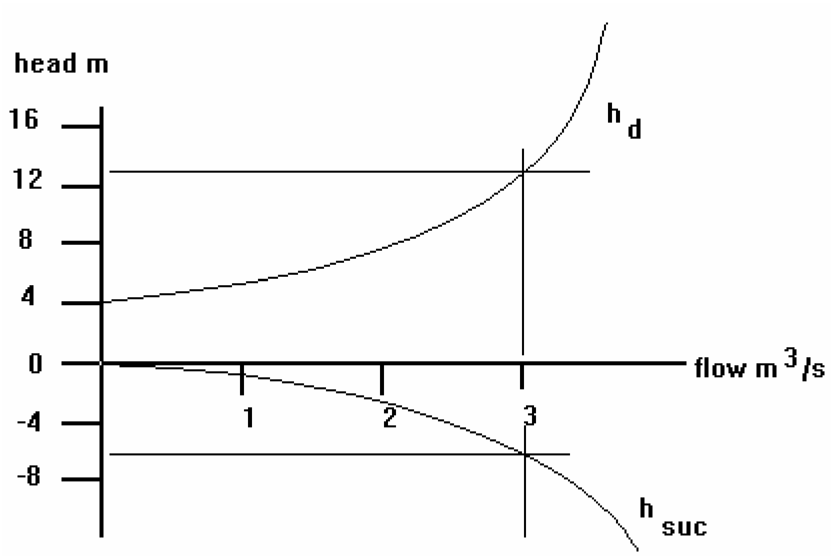


Figure 3

The pump must deliver 3 m^3/s at 2 000 rev/min. Determine the specific speed.

The vapour pressure is 0.025 bar and atmospheric pressure is 1.025 bar. Calculate the NPSH and the cavitation parameter.
Answers NPSH = 4.19 m $\sigma = 0.323$

2. Define the term "Nett Positive Suction Head" and explain its significance in pump operation.

   1.2 kg/s of acetone is to be pumped from a tank at 1 bar pressure. The acetone is at 40^{o}C and the pump is 1.5 m below the surface. The suction pipe is 25 mm bore diameter. Calculate the NPSH at the pump inlet.

   Losses in the suction pipe are equal to three velocity heads.

   The vapour pressure of acetone is 55 kPa. The density is 780 kg/m^3.

   Answer 5.37 m

3. A centrifugal pump delivers fluid from one vessel to another distant vessel. The flow is controlled with a valve. Sketch and justify appropriate positions for the pump and valve when the fluid is a) a liquid and b) a gas.

Let's move on to examine the transient pressure changes in pipes when the fluid is accelerated or decelerated.

2. WATER HAMMER

In this section, we will examine the causes of water hammer. The sudden acceleration or deceleration of fluids in pipes is accompanied by corresponding changes in pressure that can be extremely large. In the extreme, the pressure surge can split the pipe. The phenomenon is often accompanied by load hammer noises, hence the name.

First, we must examine the Bulk Modulus (K) and the derivation of the acoustic velocity in an elastic fluid.

2.1 BULK MODULUS (K)

Bulk modulus was discussed in Chapter 1 and defined as follows.

$$K = \frac{\text{Change in pressure}}{\text{Volumetric strain}} = \frac{V\Delta p}{\Delta V} = \frac{V\delta p}{\delta V}$$

V is volume and p is pressure. The following work shows how this may be changed to the form **K = ρdp/dρ**

Consider a volume V_1 that is compressed to volume V_2 by a small increase in pressure δp. The reduction in volume is δV. The initial density is ρ and this increases by δρ

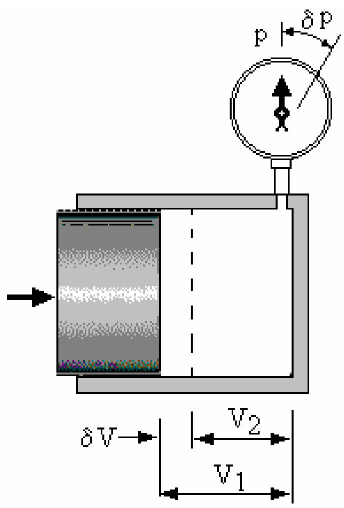


Figure 4

The mass of δV is $\delta m = \rho\,\delta V$ ....(2.1)
The initial mass of V_2 is $m_1 = \rho\,V_2$ ....(2.2)
The final mass of V_2 is $m_2 = (\rho + \delta\rho)\,V_2$..............(2.3)

The increase in mass is due to the mass of δV being compressed into the volume V_2.
Hence (2.1) = (2.3) - (2.2)

$$\rho\,\delta V = (\rho + \delta\rho)\,V_2 - \rho V_2 = \rho\,V_2 + \delta\rho\,V_2 - \rho V_2$$
$$\rho\,\delta V = \delta\rho\,V_2$$
$$\rho\,\delta V = \delta\rho\,(V_1 - \delta V)$$
$$\rho\,\delta V = V_1\delta\rho - \delta\rho\,\delta V$$

The product of two small quantities (δρ δV) is infinitesimally small so it may be ignored.

$$\rho\delta V = V_1\delta\rho$$
$$\frac{\delta V}{V_1} = \frac{\delta\rho}{\rho}$$
$$\frac{V_1}{\delta V} = \frac{\rho}{\delta\rho} \quad \text{substitute this into the formula for K}$$
$$K = \frac{V\delta p}{\delta V} = \frac{\rho\delta p}{\delta\rho}$$

In the limit as $\delta V \to 0$, we may revert to calculus notation.

Hence **K = ρdp/dρ**

2.2 SPEED OF SOUND IN AN ELASTIC MEDIUM

Most students don't need to know the derivation of the formula for the speed of sound but for those who are interested, here it is.

Consider a pipe of cross sectional area A full of fluid. Suppose a piston is pushed into the end with a velocity u m/s. Due to the compressibility of the fluid, further along the pipe at distance L, the fluid is still stationary. It has taken t seconds to achieve this position. The velocity of the interface is hence a = L/t m/s. In the same time the piston has moved x metres so u = x/t.

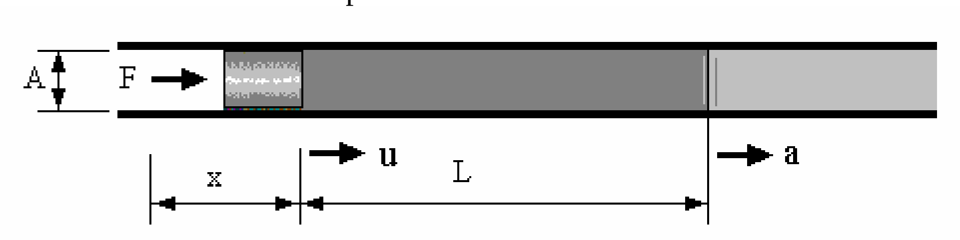


Figure 5

The moving fluid has been accelerated from rest to velocity a. The inertia force needed to do this is in the form of pressure so the moving fluid is at a higher pressure than the static fluid and the interface is hence a pressure wave travelling along the pipe at velocity a.

The volume Ax has been compacted into the length L. The initial density of the fluid is ρ.

The mass compacted into length L is $dm = \rho Ax$.

substitute x = ut $dm = \rho A u t$ ....(2.4)

The density of the compacted fluid has increased by dρ so the mass in the length L has increased by $dm = A L d\rho$

Substitute L = at $dm = A a t d\rho$......(2.5)

Equate (3.10. 4) and (3.10.5) $\rho A u t = A a t d\rho$ $a = u \rho/d\rho$............(2.6)

The force to accelerate the fluid from rest to a m/s is given by Newton's 2nd law

F = mass x acceleration = A dp

mass = ρAL acceleration = u/t A dp = ρ A L u/t

dp = ρ L u/t

Substitute L = at then dp = ρ a u $a = (dp/u\rho)$ ....(2.7)

The velocity of the pressure wave a is by definition the acoustic velocity. Multiplying (2.8) by (2.7) gives a^2.

Hence $a^2=(u\rho/d\rho)(dp/u\rho)$ $a = (dp/d\rho)^{½}$ ....(2.9)

Previously it was shown that $K = \rho\, dp/d\rho$ $\mathbf{a = (K/\rho)^{½}}$

Students who have studied fundamental thermodynamics will understand the following extension of the theory to gases. The following section is not needed by those following the basic module.

Two important gas constants are the adiabatic index γ and the characteristic gas constant R. For a gas, the pressure change is adiabatic and if dp is small then the adiabatic law applies.

$$pV^{\gamma} = \text{Constant}$$

Dividing through by m^{γ} we get $p(V/m)^{\gamma} = \text{constant}/m^{\gamma} = \text{constant}$ $\quad p/\rho^{\gamma} = C$

Differentiating we get
$$dp/d\rho = C\,(\gamma\rho^{\gamma-1})$$
$$dp/d\rho = (p/\rho^{\gamma})(\gamma\rho^{\gamma-1})$$
$$dp/d\rho = p\gamma/\rho$$

From (2.8) it follows that $a = (p\gamma/\rho)^{½}$

From the gas law we have
$$pV = mRT$$
$$p = (m/V)RT$$
$$p = \rho RT$$

The velocity of a sound wave is that of a weak pressure wave. If the pressure change is large then $dp/d\rho$ is not a constant and the velocity would be that of a shock wave which is larger than the acoustic velocity.

For air $\gamma = 1.4$ and R = 287 J/kg K. Hence at 20°C (293 K) the acoustic velocity in air is as follows.

$a = (\gamma RT)^{1/2} = (1.4 \times 287 \times 293)^{½} = 343$ m/s

2.3 PRESSURE SURGES DUE TO GRADUAL VALVE CLOSURE

Consider a pipe line with a fluid flowing at a steady velocity of u m/s. A stop valve is gradually closed thus decelerating the fluid uniformly from u to zero in t seconds.

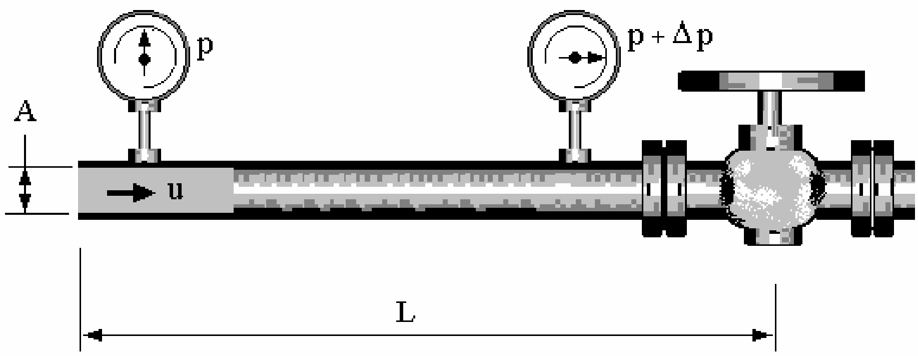


Figure 6

Volume of fluid = AL
Mass of fluid = ρAL
Deceleration = u/t
Inertia force required F = mass x deceleration = ρAL u/t

To provide this force the pressure of the fluid rises by Δp and the force is A Δp.

Equating forces we have
$$A\,\Delta p = \rho AL\, u/t$$
$$\mathbf{\Delta p = \rho L\, u/t}$$

2.4 PRESSURE SURGES DUE TO SUDDEN VALVE CLOSURE

If the valve is closed suddenly then as t is very small the pressure rise is very large. In reality, a valve cannot close instantly but very rapid closure produces very large pressures. When this occurs, the compressibility of the fluid and the elasticity of the pipe is an important factor in reducing the rise in pressure. First, we will consider the pipe as rigid.

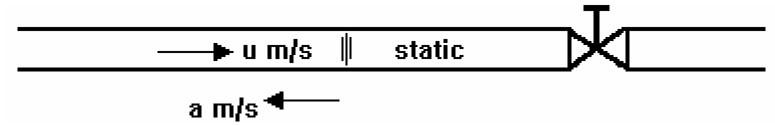


Figure 7

When the fluid stops suddenly at the valve, the fluid further up the pipe is still moving and compacting into the static fluid. An interface between moving and static fluid (a shock wave) travels up the pipe at the acoustic velocity. This is given by the equation:

$a = (K/\rho)^{½}$ $\qquad$ $K = Vdp/dV$

If we assume that the change in volume is directly proportional to the change in pressure then we may change this to finite changes such that

$K = V\,\delta p/\delta V$ $\qquad$ $\delta V = V\delta p/K$

The mean pressure rise is $\delta p/2$
The strain energy stored by the compression = $\delta p\,\delta V/2$
The change in kinetic energy = ½ mu^2
Equating for energy conservation we get

$mu^2/2 = \delta p\,\delta V/2 = (\delta p)V(\delta p)/2K$ $\qquad$ $mu^2 = V(\delta p)^2/K$
$mKu^2/V = (\delta p)^2$
$(\delta p)^2 = (m/V)K\,u^2 = \rho\,K\,u^2$
$\delta p = u(K\rho)^{½}$

Since $\quad a^2 = K/\rho$ then $K = a^2\rho$ $\qquad$ $\delta p = u(a^2\rho^2)^{½}$
Then $\quad \delta p = u\,a\,\rho$
For a large finite change, this becomes $\mathbf{\Delta p = a\,u}\,\rho$

SELF ASSESSMENT EXERCISE No. 2

The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa throughout.

1. A pipe 50 m long carries water at 1.5 m/s. Calculate the pressure rise produced when

   a) the valve is closed uniformly in 3 seconds. (25 kPa)
   b) when it is shut suddenly. (3 MPa)

2. A pipe 2000 m long carries water at 0.8 m/s. A valve is closed. Calculate the pressure rise when

   a) it is closed uniformly in 10 seconds. (160 kPa)
   b) it is suddenly closed. (1.6 MPa)

2.5 THE EFFECT OF ELASTICITY IN THE PIPE

A pressure surge in an elastic pipe will cause the pipe to swell and some of the energy will be absorbed by straining the pipe wall. This reduces the rise in pressure. The more elastic the wall is, the less the pressure rise will be. Consider the case shown.

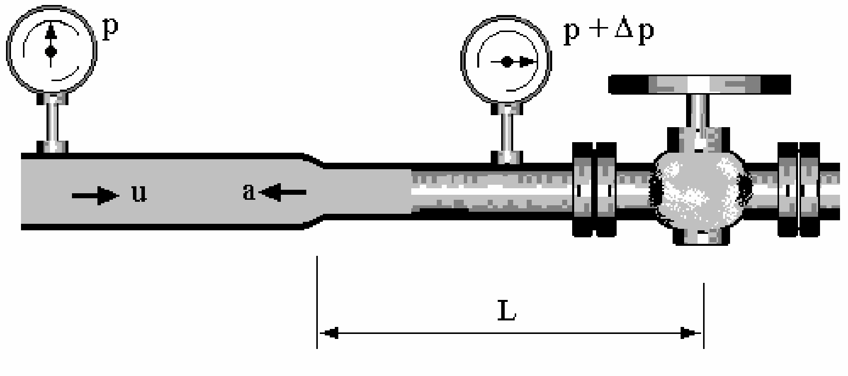


Figure 8

Kinetic Energy lost by fluid = ½ mu^2

The mass of fluid is ρAL so substituting K.E. = ½ ρALu^2

Strain Energy of fluid = $\Delta p^2 AL/2K$ (from last section)

Now consider the strain energy of the pipe wall. The strain energy of an elastic material with a direct stress σ is given by

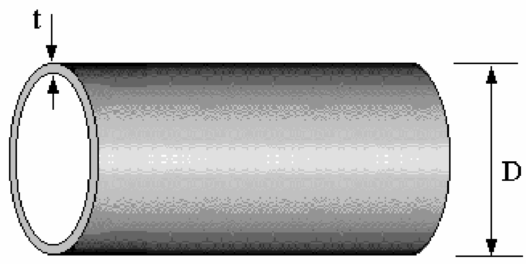


Figure 9

S.E. = $(\sigma^2/2E)$ x volume of material

The pipe may be regarded as a thin cylinder and suitable references will show that stress stretching it around the circumference is given by the following formula.

$\sigma = \Delta pD/2t$

Volume of metal = πDtL

Hence $$\text{S.E.} = \left(\frac{\Delta p\, D}{2t}\right)^2 \text{ x } \frac{\pi DtL}{2E} = \frac{(\Delta p)^2 DAL}{2tE}$$

Equating KE lost to the total S.E. gained yields

$$\frac{\rho ALu^2}{2} = \frac{(\Delta p)^2 DAL}{2tE} + \frac{(\Delta p)^2 AL}{2K}$$

$$\rho u^2 = \frac{(\Delta p)^2 D}{tE} + \frac{(\Delta p)^2}{K} = (\Delta p)^2 \left\{\frac{D}{tE} + \frac{1}{K}\right\}$$

$$\Delta p = u\sqrt{\frac{\rho}{\left\{\frac{D}{2tE} + \frac{1}{K}\right\}}}$$

The solution is usually given in terms of the effective bulk modulus K' which is defined as follows.

$$K' = \left\{\frac{D}{tE} + \frac{1}{K}\right\}^{-1}$$

The pressure rise is then given by **$\Delta p = u[\rho/K']^{½}$**

The acoustic velocity in an elastic pipe becomes a' and is given as a' = $(K'/\rho)^{½}$

Hence **$\Delta p = \rho\, u\, a'$**

WORKED EXAMPLE No.3

The density of water is 1000 kg/m^3 and the bulk modulus is 4 GPa. The modulus of elasticity for steel E is 200 Gpa.

A steel pipe carries water at 2 m/s. The pipe is 0.8 m bore diameter and has a wall 5 mm thick. Calculate the pressure rise produced when the flow is suddenly interrupted.

SOLUTION

$$K' = \left\{\frac{D}{tE} + \frac{1}{K}\right\}^{-1} = \left\{\frac{0.8}{0.005 \times 200 \times 10^9} + \frac{1}{4 \times 10^9}\right\}^{-1} = 952.4\ \text{MPa}$$

Sudden closure $a' = (K'/\rho)^{1/2} = (952.4 \times 10^6/1000)^{1/2} = 976$ m/s

$\Delta p = a' u \rho = 976 \times 2 \times 1000 = 1.95$ MPa

2.6 DAMPING OUT PRESSURE SURGES

Pressure surges or water hammer occurs whenever there is a change in flow rate. There are many causes for this besides the opening and closing of valves. Changes in pump speeds may cause the same effect. Piston pumps in particular cause rapid acceleration and deceleration of the fluid. In power hydraulics, changes in the velocity of the ram cause the same effect. The problem occurs both on large scale plant such as hydroelectric pipelines and on small plant such as power hydraulic systems. The principles behind reduction of the pressure surges are the same for each, only the scale of the equipment is different.

For example, on power hydraulic systems, accumulators are used. These are vessels filled with both liquid and gas. On piston pumps, air vessels attached to the pipe are used. In both cases, a sudden rise in pressure produces compression of the gas that absorbs the strain energy and then releases it as the pressure passes.

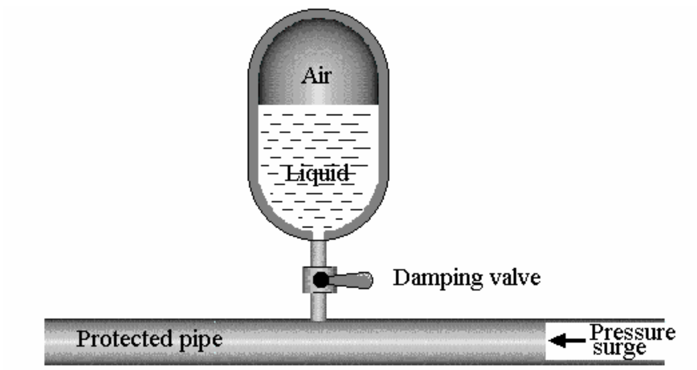


Figure 10

On hydroelectric schemes or large pumped systems, a surge tank is used. This is an elevated reservoir attached as close to the equipment needing protection as possible. When the valve is closed, the large quantity of water in the main system is diverted upwards into the surge tank. The pressure surge is converted into a raised level and hence potential energy. The level drops again as the surge passes and an oscillatory trend sets in with the water level rising and falling. A damping orifice in the pipe to the surge tank will help to dissipate the energy as friction and the oscillation dies away quickly.

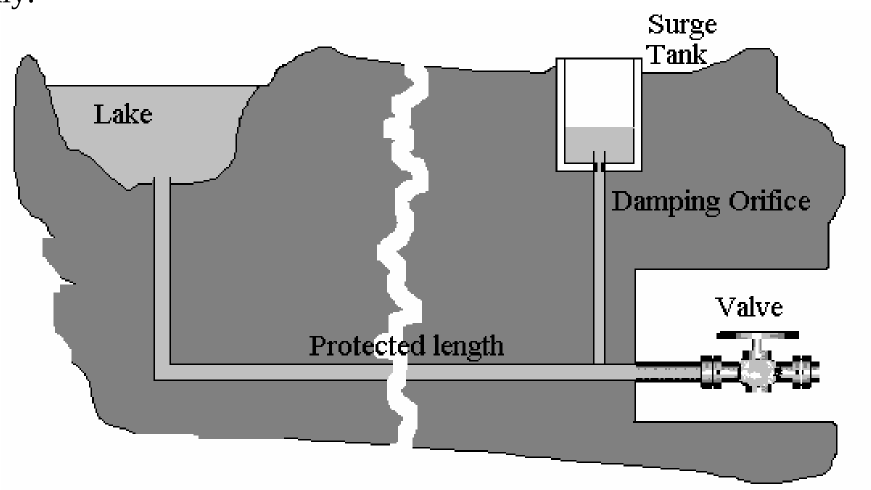


Figure 11

2.7 ANALYSIS OF SURGE TANK

Let the area of the surge tank be A_T and the area of the main pipe be A_p. The length of the pipe is L. Let the volume flow rate during normal operation of the turbines be Q. In the simplest analysis we will consider that there is no friction anywhere and that when an emergency stop is made, all the water is diverted into the surge tank.

Mean velocity in surge tank $u_T = \frac{dz}{dt} = \frac{Q}{A_T} \qquad Q = A_T \frac{dz}{dt}$

Mean velocity in the pipe $u_p = \frac{Q}{A_p}$

Substitute for Q $\quad u_p = \frac{dz}{dt}\frac{A_T}{A_p}$ ....(1)

The diversion of the flow into the surge tank raises the level by z. This produces an increased pressure at the junction point of $\Delta p = \rho gz$

The pressure force produced $F = A_p \Delta p = A_p \Delta gz$

The inertia force required to decelerate the water in the pipe is

F = mass x deceleration = - mass x acceleration = - $\rho A_p L\, du/dt$

Equating forces we have the following.

$A_p \rho gz = -\rho A_p L \frac{du}{dt} \qquad gz = -L\frac{du}{dt} \qquad z = -\frac{L}{g}\frac{du}{dt}$..................(2)

Putting (1) into (2) we get

$z = -\frac{L}{g}\frac{A_T}{A_p}\frac{d^2z}{dt^2} \qquad \frac{d^2z}{dt^2} = -\frac{gA_p}{LA_T}z$ ....(3)

By definition this is simple harmonic motion since the displacement z is directly proportional to the acceleration and opposite in sense. It follows that the frequency of the resulting oscillation is $f = \frac{1}{2\pi}\sqrt{\frac{g}{L}\frac{A_p}{A_T}}$

The periodic time will be T=1/f

The amplitude and periodic time are referred to as the APO (amplitude and period of oscillation).

A good mathematician would solve the 2nd order differential equation to produce both the frequency and amplitude.

Equation (3) maybe re-written as follows.

$$\frac{d^2z}{dt^2} = -\frac{gA_p}{LA_T}z = -\omega^2 z \qquad \frac{1}{\omega^2}\frac{d^2z}{dt^2} + z = 0$$

This is a special case of the standard 2nd order differential equation with no friction.

$$\left(\frac{1}{\omega}\right)^2\left(\frac{d^2z}{dt^2}\right) + 2\frac{\delta}{\omega}\left(\frac{dz}{dt}\right) + z = 0$$

δ is called the damping ratio and this appears when frictional damping is considered. ω is the angular frequency ($\omega=2\pi f$). This equation appears in many forms including the following.

$$\frac{d^2z}{dt^2} + 2\delta\delta\left(\frac{dz}{dt}\right) + \omega^2 z = 0$$

and for no friction

$$\frac{d^2z}{dt^2} + \omega^2 z = 0$$

The standard solution to this equation is $z = z_o \sin(\omega t)$

z_o is the amplitude, that is, the amount by which the height in the tank will move up and down from the mean level. The following is a direct way of finding the amplitude.

$$\text{The mean change in height} = \frac{z_o}{2}$$

$$\text{The weight of water entering the surge tank} = \rho g A_T z_o$$

$$\text{The potential energy stored in the tank} = \rho g A_T z_o \frac{z_o}{2} = \rho g A_T \frac{z_o^2}{2}$$

$$\text{The kinetic energy lost} = \text{Mass x } \frac{u^2}{2} = \rho L A_p \frac{u^2}{2}$$

$$\text{Equate the energies.} \quad \rho L A_p \frac{u^2}{2} = \rho g A_T \frac{z_o^2}{2}$$

$$z_o = u_o\sqrt{\frac{LA_p}{gA_T}}$$

The equation for the motion in full is $z = u_o\sqrt{\frac{LA_p}{gA_T}}\sin(\omega\, t)$

The peak of the surge occurs at T/4 seconds from the disturbance.

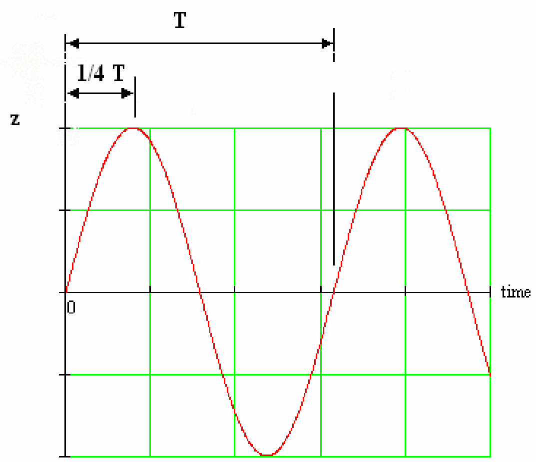


Figure 12

2.8 THE EFFECT OF PIPE FRICTION

If pipe friction is taken into account, then the normal level of the surge tank will not be the same as the level of the storage lake but will be less by an amount equal to the frictional loss in the system. This is given by D'Arcys equation as $h_f = 4fLu^2/2g$

If f is regarded as a constant then we may say $h_f = Cu^2$

This must be now included into equation (3) so that

z (plus or minus h_f)= $-(L/g)(A_T/A_p)(d^2z/dt^2)$

The solution of this may be done by step by step integration and it would show that the amplitude of the oscillation dies down (as for any damped oscillation in mechanical systems). In reality the problem is more complicated because f is not constant and varies from 0 at zero flow conditions to a maximum at maximum flow conditions. Reflected waves also complicate the story.

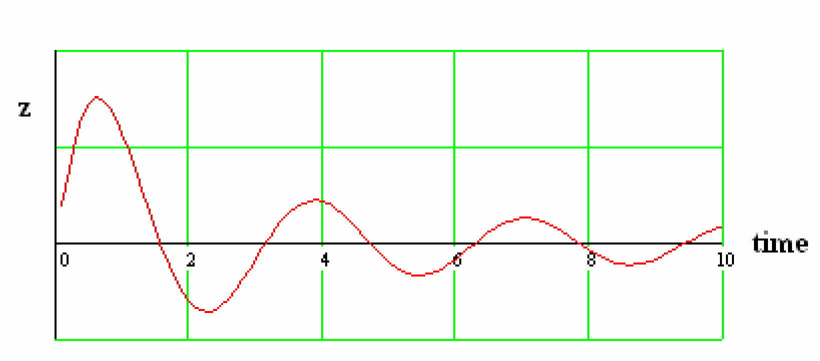


Figure 13

2.9 OSCILLATIONS

The pressure surge travel along the pipe lines at the local acoustic velocity. They can be reflected so that they travel back again to where they started. Reflections may occur from a dead end (such as a valve, a ram or a pump) , or from an open end (such as a free surface). When they are reflected from an open end, they are reversed into a rarefaction (negative pressure with respect to the normal level).

When a valve is suddenly closed at the turbine of a hydro-electric plant, a pressure surge is set up which travels to the lake and is reflected back as a rarefaction to the valve where it is again reflected as pressure. The pressure waves will pass back and forth in the pipe gradually dying away as the energy is dissipated. The use of a surge tank with a damping orifice reduces the effect to two or three oscillations. The same thing may occur when a large pump is suddenly switched off only in this case the fluid on the suction side will cause a pressure wave and on the delivery side will cause a rarefaction because the fluid is travelling away from the pump.

In some systems such as power hydraulics, the time taken for the pressure surge to travel away from the disturbance and to be reflected back again may coincide with the natural frequency of the item causing the disturbance. This may be a spring loaded valve for example. The result is a unstable oscillation with interaction between the dynamics of the valve and the dynamics of the system causing positive feedback and sustaining the oscillation. This results in valve squeal.

This is a complex area of study and the student should consult advanced text for full details. Attenuation due to friction is also involved. The theory is similar to that of A.C. electric power transmission. The student would also need to study the dynamics of mechanisms, especially forced and natural oscillations.

SELF ASSESSMENT EXERCISE No. 3

1. Derive the water hammer equation for a long elastic pipe carrying water from a large upstream reservoir with a constant water level to a lower downstream reservoir. Flow is controlled by a valve at the downstream end.

   Sketch the variation in pressure with time for both ends and the middle of the pipe. following sudden closure of the valve. Sketch these variations for when friction is negligible and for when both friction and cavitation occur.

   Assuming the effective bulk modulus is given by K' = $\{(D/tE) + 1/K\}^{-1}$

   and that the maximum stress in the pipe is σ, derive a formula for the maximum allowable discharge.

2a. Explain the purpose and features of a surge tank used to protect hydroelectric installations.

b. Derive an expression for the amplitude of oscillation of the water surface in a surge tank of cross sectional area A_T connected to a pipe of cross sectional area A_p and length L following a complete stoppage of the flow. The normal mean velocity in the pipe is u_o and friction may be ignored.

   The general solution to the standard second order differential equation

$$\frac{d^2z}{dt^2} + m^2 z = c^2$$

   is

$$z = E\sin(mt) + F\cos(mt) + \frac{c^2}{m^2}$$

3.a. A hydroelectric turbine is supplied with 0.76 m^3/s of water from a dam with the level 51 m above the inlet valve. The pipe is 0.5 m bore diameter and 650 m long.

   Calculate the pressure at inlet to the turbine given that the head loss in the pipe is 8.1 m. (0.41 MPa).

   Calculate the maximum pressure on the inlet valve if it is closed suddenly. The speed of sound in the pipe is 1200 m/s. (5.05 MPa)

b. The pipe is protected by a surge tank positioned close to the inlet valve.

   Calculate the maximum change in level in the surge tank when the valve is closed suddenly (ignore friction). (3.97 m)

   Calculate the periodic time of the resulting oscillation.

4. A pipe 2 m bore diameter and 420 m long supplies water from a dam to a turbine. The turbine is located 80 m below the dam level. The pipe friction coefficient f is 0.01 ($f = 4C_f$).

   Calculate the pressure at inlet to the turbine when 10 m^3/s of water is supplied. (0.772 MPa)

   Calculate the pressure that would result on the inlet valve if it was closed suddenly. The speed of sound in the pipe is 1432 m/s. (4.55 MPa)

   Calculate the fastest time the valve could be closed unormly if the pressure rise must not exceed 0.772 MPa). (1.72 s)

5. a) Sketch the main features of a high-head hydro-electric scheme.

   b) Deduce from Newton's laws the amplitude and period of oscillation (APO) in a cylindrical surge tank after a sudden stoppage of flow to the turbine. Assume there is no friction.

   c) State the approximate effect of friction on the oscillation.

   d) An orifice of one half the tunnel diameter is added in the surge pipe near to the junction with the tunnel. What effect does this have on the APO ?

FLUID MECHANICS

TUTORIAL 9

COMPRESSIBLE FLOW

On completion of this tutorial you should be able to

- define entropy
- derive expressions for entropy changes in fluids
- derive Bernoulli's equation for gas
- derive equations for compressible ISENTROPIC flow
- derive equations for compressible ISOTHERMAL flow
- solve problems involving compressible flow
- derive equations for shock waves
- solve problems involving shock waves

Let's start by revising entropy.

1. ENTROPY

1.1 DEFINITION

You should already be familiar with the theory of work laws in closed systems. You should know that the area under a pressure-volume diagram for a reversible expansion or compression gives the work done during the process.

In thermodynamics there are two forms of energy transfer, work (W) and heat (Q). By analogy to work, there should be a property which if plotted against temperature, then the area under the graph would give the heat transfer. This property is entropy and it is given the symbol S. Consider a p-V and T-s graph for a reversible expansion.

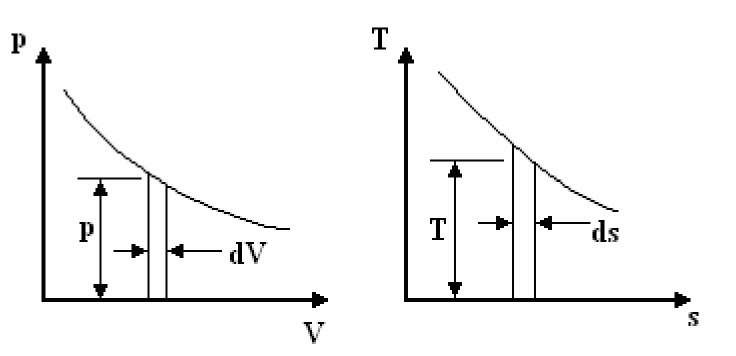


Figure 1

From the p-V graph we have $W = \int p dV$

From the T-S graph we have $Q = \int T dS$

This is the way entropy was developed for thermodynamics and from the above we get the definition

$$dS = dQ/T$$

The units of entropy are hence J/K.

Specific entropy has a symbol s and the units are J/kg K

It should be pointed out that there are other definitions of entropy but this one is the most meaningful for thermodynamics. A suitable integration will enable you to solve the entropy change for a fluid process.

2. ISENTROPIC PROCESSES

The word *Isentropic* means constant entropy and this is a very important thermodynamic process. It occurs in particular when a process is reversible and adiabatic. This means that there is no heat transfer to or from the fluid and no internal heat generation due to friction. In such a process it follows that if dQ is zero then dS must be zero. Since there is no area under the T-S graph, then the graph must be a vertical line as shown.

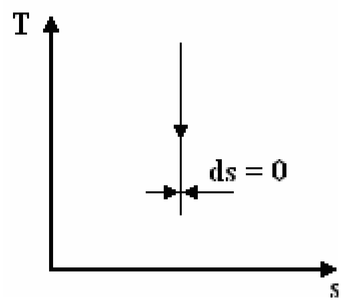


Figure 2

There are other cases where the entropy is constant. For example, if there is friction in the process generating heat but this is lost through cooling, then the nett result is zero heat transfer and constant entropy. You do not need to be concerned about this at this stage.

Entropy is used in the solution of gas and vapour problems. We should now look at practical applications of this property and study the entropy changes which occur in closed and steady flow systems for perfect gases and vapours. These derivations should be learned for the examination.

3. ENTROPY CHANGES FOR A PERFECT GAS IN A CLOSED SYSTEMS

Consider a closed system expansion of a fluid against a piston with heat and work transfer taking place.

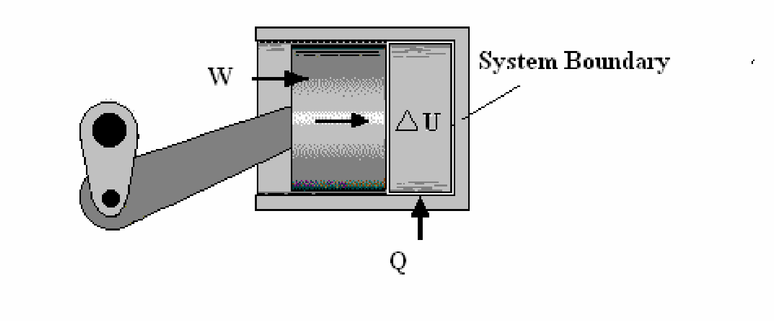


Figure 3

Applying the non-flow energy equation we have

$$Q + W = \Delta U$$

Differentiating we have $dQ + dW = dU$

Since $dQ = TdS$ and $dW = -pdV$ then $TdS - pdV = dU$

$$TdS = dU + pdV$$

This expression is the starting point for all derivations of entropy changes for any fluid (gas or vapour) in closed systems. It is normal to use specific properties so the equation becomes

$$Tds = du + pdv$$

but from the gas law $pv = RT$ we may substitute for p and the equation becomes

$$Tds = du + RTdv/v$$

rearranging and substituting $du = c_V\, dT$ we have

$$ds = c_V\, dT/T + Rdv/v ....(1)$$

s is specific entropy

v is specific volume.

u is specific internal energy and later on is also used for velocity.

3.1 ISOTHERMAL PROCESS

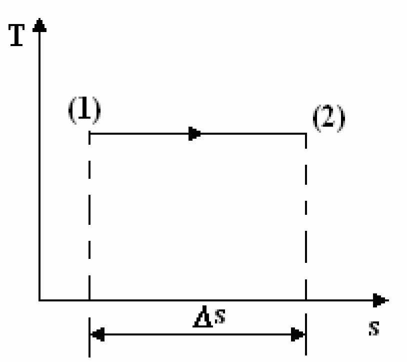


Figure 4

In this case temperature is constant. Starting with equation (1)

$$ds = c_V\, dT/T + R\,dv/v.$$

since dT = 0 then

$$s_2 - s_1 = \Delta s = R \ln(v_2/v_1)$$

A quicker alternative derivation for those familiar with the work laws is:

$Q + W = \Delta U$ but $\Delta U = 0$ then $Q = -W$ and $W = -mRT \ln\frac{V_2}{V_1}$

$Q = \int T ds = T\Delta S$ but T is constant.

$$\Delta S = \frac{Q}{T} = -\frac{W}{T} = mR\ln\frac{V_2}{V_1}$$

$$\Delta S = mR\ln\frac{V_2}{V_1}$$

$\Delta s = R\ln\frac{v_2}{v_1}$ and since $\frac{v_2}{v_1} = \frac{p_1}{p_2}$

$$\Delta s = R\ln\frac{p_1}{p_2}$$

3.2 CONSTANT VOLUME PROCESS

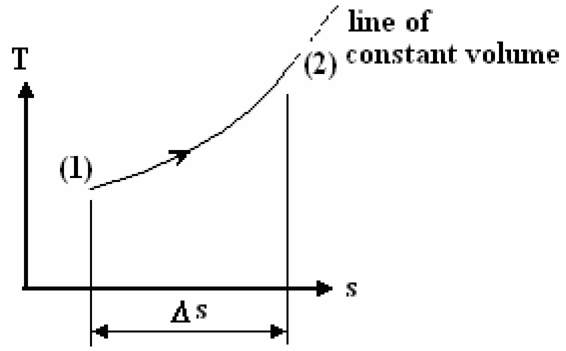


Figure 5

Starting again with equation (1) we have $ds = c_v dT/T + R dv/v$

In this case dv=0 so $ds = c_v dT/T$

Integrating between limits (1) and (2) $\Delta s = c_v \ln(T_2/T_1)$

3.3 CONSTANT PRESSURE PROCESS

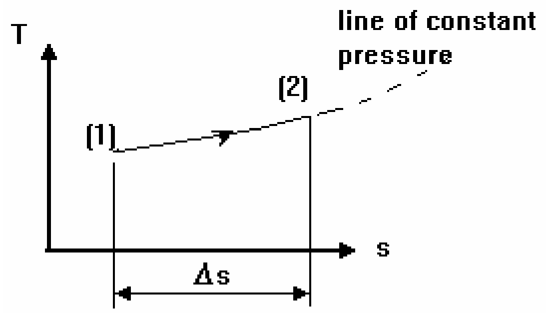


Figure 6

Starting again with equation (1) we have

$ds = C_v \frac{dT}{T} + R\frac{dv}{v}$ In this case we integrate and obtain

$\Delta s = C_v \ln\frac{T_2}{T_1} R\ln\frac{v_2}{v_1}$ For a constant pressure process, v/T = constant

$\frac{v_2}{v_1} = \frac{T_2}{T_1}$ so the expression becomes $\Delta s = C_v \ln\frac{T_2}{T_1} + R\ln\frac{T_2}{T_1} = (C_v + R)\ln\frac{T_2}{T_1}$

It was shown in an earlier tutorial that $R = c_p - c_v$ hence

$$\Delta s = C_p \ln\frac{T_2}{T_1}$$

3.4 POLYTROPIC PROCESS

This is the most difficult of all the derivations here. Since all the forgoing are particular examples of the polytropic process then the resulting formula should apply to them also.

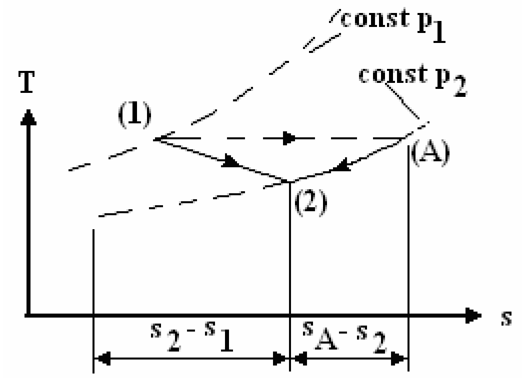


Figure 7

The polytropic expansion is from (1) to (2) on the T-s diagram with different pressures, volumes and temperatures at the two points. The derivation is done in two stages by supposing the change takes place first at constant temperature from (1) to (A) and then at constant pressure from (A) to (2). You could use a constant volume process instead of constant pressure if you wish.

$$s_2 - s_1 = (s_A - s_1) - (s_A - s_2)$$

$$s_2 - s_1 = (s_A - s_1) + (s_2 - s_A)$$

For the constant temperature process

$$(s_A - s_1) = R \ln(p_1/p_A)$$

For the constant pressure process

$$(s_2 - s_A) = (c_p) \ln(T_2/T_A)$$

Hence

$$\Delta s = R \ln \frac{p_1}{p_A} + C_p \ln \frac{T_2}{T_A} + s_2 - s_1$$ Since $p_A = p_2$ and $T_A = T_1$

Then

$$\Delta s = s_2 - s_1 = R \ln \frac{p_1}{p_2} + C_p \ln \frac{T_2}{T_1}$$ Divide through by R

$$\frac{\Delta s}{R} = \ln \frac{p_1}{p_2} + \frac{C_p}{R} \ln \frac{T_2}{T_1}$$

From the relationship between c_p, c_v, R and γ we have $c_p/R = \gamma/(\gamma - 1)$

Hence $$\frac{\Delta s}{R} = \ln \frac{p_1}{p_2} + \frac{\gamma}{\gamma - 1} \ln \frac{T_2}{T_1} \qquad \frac{\Delta s}{R} = \ln \frac{p_1}{p_2} \left(\frac{T_2}{T_1} \right)^{\frac{\gamma}{\gamma - 1}}$$

This formula is for a polytropic process and should work for isothermal, constant pressure, constant volume and adiabatic processes also. In other words, it must be the derivation for the entropy change of a perfect gas for any closed system process. This derivation is often requested in the exam.

WORKED EXAMPLE No. 1

A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.2} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.
R = 287 J/kg K γ =1.4.

SOLUTION

$$T_2 = 473\left(\frac{1}{5}\right)^{1-\frac{1}{1.2}} = 361.7\text{K}$$

$$\frac{\Delta s}{R} = \ln\left(\frac{p_1}{p_2}\right)\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\Delta s}{R} = (\ln 5)\left(\frac{361.7}{472}\right)^{3.5} = 0.671$$

$$\Delta s = 0.671 \text{ x } 287 = 192.5 \text{ J/kgK}$$

SELF ASSESSMENT EXERCISE No. 1

1. Calculate the specific entropy change when a perfect gas undergoes a reversible isothermal expansion from 500 kPa to 100 kPa. R = 287 J/kg K.
(Answer +461.9 J/kg K).

2. Calculate the total entropy change when 2 kg of perfect gas is compressed reversibly and isothermally from 9 dm^3 to 1 dm^3. R=300 J/kg K. (Answer -1.32 kJ/k)

3. Calculate the change in entropy when 2.5 kg of perfect gas is heated from 20°C to 100°C at constant volume. Take c_V= 780 J/kg K (Answer 470 J/K)

4. Calculate the total entropy change when 5 kg of gas is expanded at constant pressure from 30°C to 200°C. R = 300 J/kg K c_V= 800 J/kg K (Answer 2.45 kJ/K)

5. Derive the formula for the specific change in entropy during a polytropic process using a constant volume process from (A) to (2).

6. A perfect gas is expanded from 5 bar to 1 bar by the law $pV^{1.6} = C$. The initial temperature is 200°C. Calculate the change in specific entropy.
R = 287 J/kg K γ =1.4. (Answer -144 J/kg K)

7. A perfect gas is expanded reversibly and adiabatically from 5 bar to 1 bar by the law $pV^{\gamma} = C$. The initial temperature is 200°C. Calculate the change in specific entropy using the formula for a polytropic process. R = 287 J/kg K γ =1.4. (The answer should be zero since the process is constant entropy).

Let's go on to apply the knowledge of entropy to the flow of compressible fluids starting with isentropic flow.

4. ISENTROPIC FLOW

Isentropic means constant entropy. In this case we will consider the flow to be ADIABATIC also, that is, with no heat transfer.

Consider gas flowing in a duct which varies in size. The pressure and temperature of the gas may change.

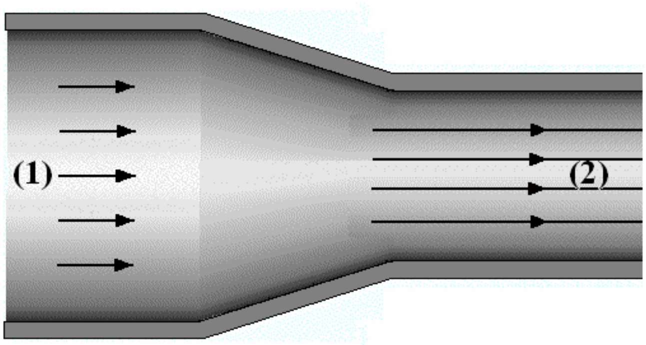


Figure 8

Applying the steady flow energy equation between (1) and (2) we have :

$$\Phi - P = \Delta U + \Delta F.E. + \Delta K.E. + \Delta P.E.$$

For Adiabatic Flow, $\Phi = 0$ and if no work is done then $P = 0$

$$\Delta U + \Delta F.E. = \Delta H$$

hence :

$$0 = \Delta H + \Delta K.E. + \Delta P.E.$$

In specific energy terms this becomes :

$$0 = \Delta h + \Delta k.e. + \Delta p.e.$$

rewriting we get:

$$h_1 + u_1^2/2 + g z_1 = h_2 + u_2^2/2 + g z_2$$

For a gas, $h = C_p T$ so we get Bernoulli's equation for gas which is :

$$C_p T_1 + u_1^2/2 + g z_1 = C_p T_2 + u_2^2/2 + g z_2$$

Note that T is absolute temperature in Kelvins $T = {}^oC + 273$

4.1 STAGNATION CONDITIONS

If a stream of gas is brought to rest, it is said to STAGNATE. This occurs on leading edges of any obstacle placed in the flow and in instruments such as a Pitot Tube. Consider such a case for horizontal flow in which P.E. may be neglected.

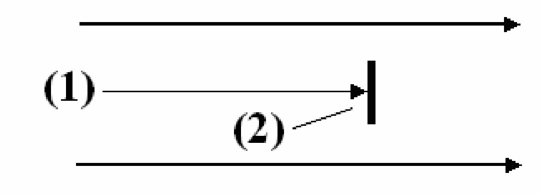


Figure 9

$u_2 = 0$ and $z_1 = z_2$ so $C_pT_1 + u_1^2/2 = C_pT_2 + 0$

$T_2 = u_1^2/2C_p + T_1$

T_2 is the stagnation temperature for this case.

$$\text{Let } T_2 - T_1 = \Delta T = u_1^2/2C_p$$

$$\Delta T = u_1^2/2C_p$$

Now $C_p - C_v = R$ and $C_p / C_v = \gamma$ γ is the adiabatic index .

hence $C_p = R / (\gamma - 1)$ and so :

$$\Delta T = u_1^2 (\gamma - 1) / (2\gamma R)$$

It can be shown elsewhere that the speed of sound a is given by :

$$a^2 = \gamma RT$$

hence at point 1:

$$\Delta T / T_1 = u_1^2 (\gamma - 1) / (2\gamma RT_1) = u_1^2 (\gamma - 1) / 2a_1^2$$

The ratio u/a is the Mach Number M_a so this may be written as :

$$\Delta T / T_1 = M_a^2 (\gamma - 1)/2$$

If M_a is less than 0.2 then M_a^2 is less than 0.04 and so $\Delta T/T_1$ is less than 0.008. It follows that for low velocities, the rise in temperature is negligible under stagnation conditions.

The equation may be written as :

$$\frac{T_2 - T_1}{T_1} = \frac{M_a^2(\gamma - 1)}{2}$$

$$\frac{T_2}{T_1} = \left\{\frac{M_a^2(\gamma - 1)}{2}\right\} + 1$$

Since pV/T = constant and pV^{γ} = constant then :

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

Hence :

$$\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{M_a^2(\gamma - 1)}{2} + 1$$

$$\left(\frac{p_2}{p_1}\right) = \left[\frac{M_a^2(\gamma - 1)}{2} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

p_2 is the stagnation pressure. If we now expand the equation using the binomial theorem we get :

$$\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}\left\{1 + \frac{Ma^2}{4} + \frac{Ma^4}{8} + \ldots\ldots\ldots\right\}$$

If M_a is less than 0.4 then : $\frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}$

Now compare the equations for gas and liquids :

LIQUIDS $\quad u = (2\Delta p/\rho)^{0.5}$

GAS $\quad \frac{p_2}{p_1} = 1 + \frac{\gamma Ma^2}{2}$

Put $p_2 = p_1 + \Delta p$ so : $\Delta p = \frac{\gamma Ma^2}{2} p_1 = \frac{\gamma v_1^2 p_1}{2\gamma RT} = \frac{\rho_1 u_1^2}{2}$

where $\rho_1 = p_1/RT$ and $M_a^2 = u_1^2/(\gamma RT)$

hence $\quad u = (2\Delta p/\rho_1)^{0.5}$ which is the same as for liquids.

SELF ASSESSMENT EXERCISE No. 2

Take $\gamma = 1.4$ and R = 283 J/kg K in all the following questions.

1. An aeroplane flies at Mach 0.8 in air at 15^o C and 100 kPa pressure. Calculate the stagnation pressure and temperature. (Answers 324.9 K and 152.4 kPa)

2. Repeat problem 1 if the aeroplane flies at Mach 2. (Answers 518.4 K and 782.4 kPa)

3. The pressure on the leading edges of an aircraft is 4.52 kPa more than the surrounding atmosphere. The aeroplane flies at an altitude of 5 000 metres. Calculate the speed of the aeroplane.(Answer 109.186 m/s)

   Note from fluids tables, find that a = 320.5 m/s p_1 = 54.05 kPa γ = 1.4

4. An air compressor delivers air with a stagnation temperature 5 K above the ambient temperature. Determine the velocity of the air. (Answer 100.2 m/s)

Let's now extend the work to pitot tubes.

5. PITOT STATIC TUBE

A Pitot Static Tube is used to measure the velocity of a fluid. It is pointed into the stream and the differential pressure obtained gives the stagnation pressure.

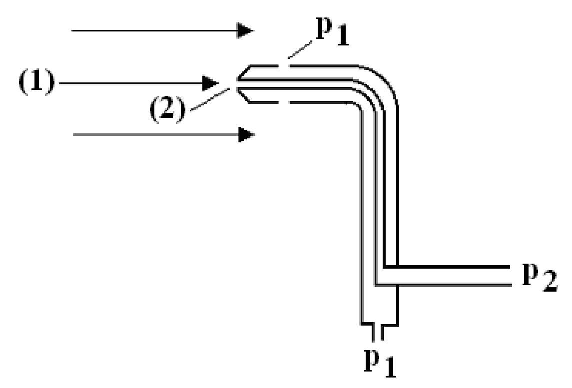


Figure 10

$p_2 = p_1 + \Delta p$

Using the formula in the last section, the velocity v may be found.

WORKED EXAMPLE No.2

A pitot tube is pointed into an air stream which has a pressure of 105 kPa. The differential pressure is 20 kPa and the air temperature is 20°C. Calculate the air speed.

SOLUTION

$p_2 = p_1 + \Delta p = 105 + 20 = 125$ kPa

$$\frac{p_2}{p_1} = \left[\left\{\frac{Ma^2(\gamma - 1)}{2}\right\} + 1\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{125}{105} = \left[\left\{\frac{Ma^2(\gamma - 1)}{2}\right\} + 1\right] \text{hence Ma} = 0.634$$

$a = (\gamma RT)^{0.5} = (1.4 \times 287 \times 293)^{0.5} = 343$ m/s

$M_a = u/a$ hence $u = 217.7$ m/s

Let's further extend the work now to venturi meters and nozzles.

6. VENTURI METERS AND NOZZLES

Consider the diagrams below and apply Isentropic theory between the inlet and the throat.

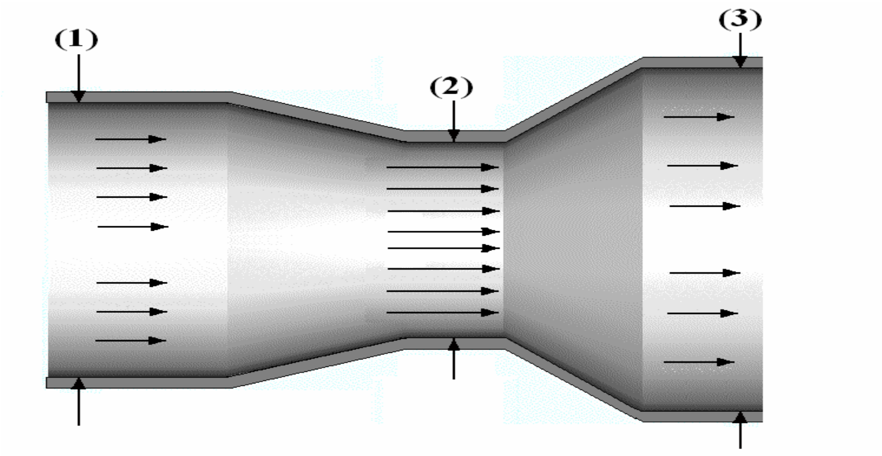


Figure 11

$u_2^2 - u_1^2 = h_1 - h_2$

If the Kinetic energy at inlet is ignored this gives us $u_2^2 = h_1 - h_2$

For a gas $h = C_pT$ so: $u_2^2 = C_P\left[T_1 - T_2\right]$

Using $C_p = \gamma R/(\gamma-1)$ we get $u_2^2 = \frac{2\gamma R}{\gamma - 1}\left[T_1 - T_2\right]$

$RT = pV/m = p/\rho$ so

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]$$

$p_1V_1^\gamma = p_2V_2^\gamma$ so it follows that $\frac{p_1}{\rho_1^\gamma} = \frac{p_2}{\rho_2^\gamma}$

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \frac{p_2\rho_1}{p_1\rho_2}\right]$$

$$u_2^2 = \frac{2\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right]$$

The mass flow rate $m = \rho_2 A_2 u_2 C_d$ where C_d is the coefficient of discharge which for a well designed nozzle or Venturi is the same as the coefficient of velocity since there is no contraction and only friction reduces the velocity.

$\rho_2 = \rho_1\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}$ hence $m = C_d A_2\sqrt{\left[\frac{2\gamma}{\gamma - 1}\right]\left\{\left[p_1\rho_1\right]\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]\right\}}$

If a graph of mass flow rate is plotted against pressure ratio (p_2/p_1) we get:

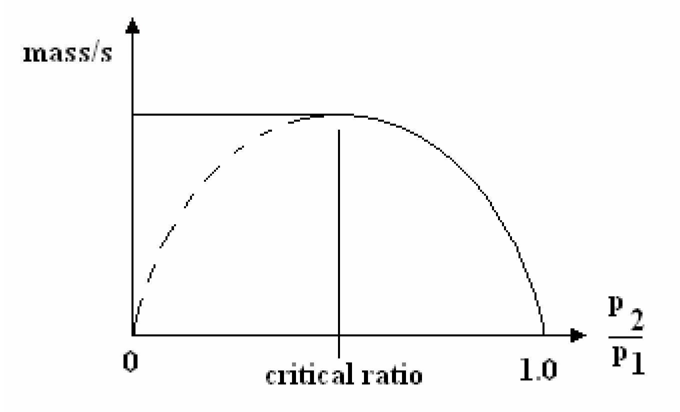


Figure 12

Apparently the mass flow rate starts from zero and reached a maximum and then declined to zero. The left half of the graph is not possible as this contravenes the 2nd law and in reality the mass flow rate stays constant over this half.

What this means is that if you started with a pressure ratio of 1, no flow would occur. If you gradually lowered the pressure p_2 , the flow rate would increase up to a maximum and not beyond. The pressure ratio at which this occurs is the CRITICAL RATIO and the nozzle or Venturi is said to be choked when passing maximum flow rate. Let $\frac{p_2}{p_1} = r$

For maximum flow rate, $\frac{dm}{dr} = 0$

The student should differentiate the mass formula above and show that at the maximum condition the critical pressure ratio is :

$$r = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$$

6.1 MAXIMUM VELOCITY

If the formula for the critical pressure ratio is substituted into the formula for velocity, then the velocity at the throat of a choked nozzle/Venturi is :

$$u_2^2 = \left\{\frac{\gamma p_2}{\rho_2}\right\} = \gamma RT = a^2$$

Hence the maximum velocity obtainable at the throat is the local speed of sound.

6.2 CORRECTION FOR INLET VELOCITY

In the preceding derivations, the inlet velocity was assumed negligible. This is not always the case and especially in Venturi Meters, the inlet and throat diameters are not very different and the inlet velocity should not be neglected. The student should go through the derivation again from the beginning but this time keep v_1 in the formula and show that the mass flow rate is

$$m = \frac{C_d A_2 \sqrt{\left[p_1 \rho_1 \frac{2\gamma}{\gamma - 1}\right]\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{1+\frac{1}{\gamma}}\right]}}{\sqrt{1 - \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}}}$$

The critical pressure ratio can be shown to be the same as before.

6.3 MORE ON ISENTROPIC FLOW

When flow is isentropic it can be shown that all the stagnation properties are constant. Consider the conservation of energy for a horizontal duct :

$$h + u^2/2 = \text{constant} \qquad h = \text{specific enthalpy}$$

If the fluid is brought to rest the total energy must stay the same so the stagnation enthalpy h_o is given by :

$h_o = h + u^2/2$ and will have the same value at any point in the duct.

since $h_o = C_p T_o$ then To (the stagnation temperature) must be the same at all points. It follows that the stagnation pressure p_o is the same at all points also. This knowledge is very useful in solving questions.

6.4 ISENTROPIC EFFICIENCY (NOZZLE EFFICIENCY)

If there is friction present but the flow remains adiabatic, then the entropy is not constant and the nozzle efficiency is defined as :

$$\eta = \text{actual enthalpy drop/ideal enthalpy drop}$$

For a gas this becomes : $(T_1 - T_2)/(T_1 - T_2')$

T_2' is the ideal temperature following expansion. Now apply the conservation of energy between the two points for isentropic and non isentropic flow :

$$C_p T_1 + u_1^2/2 = C_p T_2 + u_2^2/2 \quad \text{....... for isentropic flow}$$

$$C_p T_1 + u_1^2/2 = C_p T_{2'} + u_{2'}^2/2 \quad \text{.........for non isentropic}$$

Hence

$$\eta = (T_1 - T_2)/(T_1 - T_2') = (u_2^2 - u_1^2)/(u_{2'}^2 - u_1^2)$$

If v_1 is zero (for example Rockets) then this becomes :

$$\eta = u_2^2 / u_{2'}^2$$

SELF ASSESSMENT EXERCISE No. 3

1. A Venturi Meter must pass 300g/s of air. The inlet pressure is 2 bar and the inlet temperature is 120^oC. Ignoring the inlet velocity, determine the throat area. Take C_d as 0.97. Take γ =1.4 and R = 287 J/kg K (assume choked flow)
(Answer 0.000758 m^2)

2. Repeat problem 1 given that the inlet is 60 mm diameter and the inlet velocity must not be neglected. (Answer 0.000747 m^2)

3. A nozzle must pass 0.5 kg/s of steam with inlet conditions of 10 bar and 400^oC. Calculate the throat diameter that causes choking at this condition. The density of the steam at inlet is 3.263 kg/m^3. Take γ for steam as 1.3 and C_d as 0.98.
(Answer 23.2 mm)

4. A Venturi Meter has a throat area of 500 mm^2. Steam flows through it, and the inlet pressure is 7 bar and the throat pressure is 5 bar. The inlet temperature is 400^oC. Calculate the flow rate. The density of the steam at inlet is 2.274 kg/m^3.

   Take γ= 1.3. R = 462 J/kg K. Cd=0.97. (Answer 383 g/s)

5. A pitot tube is pointed into an air stream which has an ambient pressure of 100 kPa and temperature of 20^oC. The pressure rise measured is 23 kPa. Calculate the air velocity. Take γ = 1.4 and R = 287 J/kg K. (Answer 189.4 m/s)

6. A fast moving stream of gas has a temperature of 25^oC. A thermometer is placed into it in front of a small barrier to record the stagnation temperature. The stagnation temperature is 28^oC. Calculate the velocity of the gas. Take γ= 1.5 and R = 300 J/kg K. (Answer 73.5 m/s)

Let's do some further study of nozzles of venturi shapes now.

7. CONVERGENT - DIVERGENT NOZZLES

A nozzle fitted with a divergent section is in effect a Venturi shape. The divergent section is known as a diffuser.

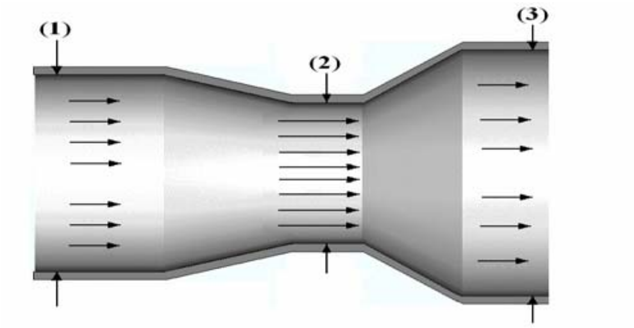


Figure 13

If p_1 is constant and p_3 is reduced in stages, at some point p_2 will reach the critical value which causes the nozzle to choke. At this point the velocity in the throat is sonic.

If p_3 is further reduced, p_2 will remain at the choked value but there will be a further pressure drop from the throat to the outlet. The pressure drop will cause the volume of the gas to expand. The increase in area will tend to slow down the velocity but the decrease in volume will tend to increase the velocity. If the nozzle is so designed, the velocity may increase and become supersonic at exit.

In rocket and jet designs, the diffuser is important to make the exit velocity supersonic and so increase the thrust of the engine.

7.1 NOZZLE AREAS

When the nozzle is choked, the velocity at the throat is the sonic velocity and the Mach number is 1. If the Mach number at exit is M_e then the ratio of the throat and exit area may be found easily as follows.

$u_t = (\gamma R T_t)^{0.5}$ $u_e = M_e(\gamma R T_e)^{0.5}$ mass/s = $\rho_t A_t v_t = \rho_e A_e v_e$.

$\frac{A_t}{A_e} = \frac{\rho_e u_e}{\rho_t u_t}$ but earlier it was shown that $\frac{\rho_e}{\rho_t} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}}$

$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} \frac{M_e(\gamma R T_e)^{0.5}}{(\gamma R T_t)^{0.5}}$ It was also shown earlier that $\frac{T_e}{T_t} = \left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}$

$$\frac{A_t}{A_e} = \left(\frac{p_e}{p_t}\right)^{\frac{1}{\gamma}} M_e \left\{\left(\frac{p_e}{p_t}\right)^{1-\frac{1}{\gamma}}\right\}^{0.5}$$

$$\frac{A_t}{A_e} = M_e \left(\frac{p_e}{p_t}\right)^{\frac{1+\gamma}{2\gamma}}$$

There is much more which can be said about nozzle design for gas and steam with implications to turbine designs. This should be studied in advanced text books.

WORKED EXAMPLE No.3

Solve the exit velocity for the nozzle shown assuming isentropic flow:

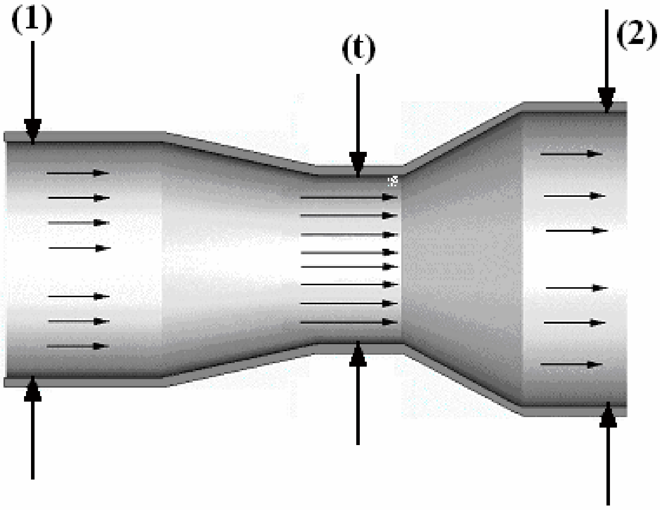


Figure 14

$T_1 = 350$ K $P_1 = 1$ MPa $p_2 = 100$ kPa

The nozzle is fully expanded (choked). Hence $M_t = 1$ (the Mach No.)
The adiabatic index $\gamma = 1.4$

SOLUTION

The critical pressure $p_t = p_1 \{2/(\gamma - 1)\}^{\gamma/(\gamma-1)} = 0.528$ MPa

$T_t/T_1 = (p_t/p_1)^{(\gamma-1)/\gamma}$ hence $T_t = 291.7$ K
$T_o/T_t = \{1 + M^2(\gamma - 1)/2\}$ hence $T_o = 350$ K

It makes sense that the initial pressure and temperature are the stagnation values since the initial velocity is zero.

$T_2 = T_t (p_2/p_t)^{(\gamma-1)/\gamma} = 181.3$ K $a_2 = (\gamma R T_2)^{0.5} = 270$ m/s

$p_o/p_2 = \{1 + M_2{}^2(\gamma - 1)/2\}^{\gamma/(\gamma-1)}$

Hence $M_2 = 2.157$ and $u_2 = 2.157 \times 270 = 582.4$ m/s

SELF ASSESSMENT EXERCISE No. 4

1. Air discharges from a pipe into the atmosphere through an orifice. The stagnation pressure and temperature immediately upstream of the orifice is 10 bar and 287 K at all times.

   Determine the diameter of the orifice which regulates the flow rate to 0.03 kg/s. (Answer 4 mm)

   Determine the diameter of the orifice which regulates the flow rate to 0.0675 kg/s. (Answer 6 mm)

   Atmospheric pressure is 1 bar, the flow is isentropic and the air should be treated as a perfect gas. The following formulae are given to you.

   $T_o = T\{1 + M^2(\gamma-1)/2\}$ $\qquad$ $p_1/p_2 = (T_1/T_2)^{\gamma/(\gamma-1)}$

   The relationship between areas for the flow of air through a convergent- divergent nozzle is given by

   $A/A^* = (1/M)\{(M^2 + 5)/6\}^3$

   where A and A* are cross sectional areas at which the Mach Numbers are M and 1.0 respectively.

   Determine the ratio of exit to throat areas of the nozzle when the Mach number is 2.44 at exit. (Answer 2.49/1)

   Confirm that an exit Mach number of 0.24 also gives the same area ratio.

2. Air discharges from a vessel in which the stagnation temperature and pressure are 350 K and 1.3 bar into the atmosphere through a convergent-divergent nozzle. The throat area is 1×10^{-3} m^2. The exit area is 1.2×10^{-3} m^2. Assuming isentropic flow and no friction and starting with the equations

   $a = (\gamma RT)^{1/2}$
   $C_p T_o = C_p T + v_2/2$
   $p\rho^{-\gamma} = \text{constant}$

   Determine the mass flow rate through the nozzle , the pressure at the throat and the exit velocity. (Answers 0.28 kg/s, 0.686 bar, 215.4 m/s)

3. Show that the velocity of sound in a perfect gas is given by $a = (\gamma RT)^{1/2}$

Show that the relationship between stagnation pressure, pressure and Mach number for the isentropic flow of a perfect gas is

$p_o/p = \{1 + (\gamma-1)M^2/2\}^{\gamma/(\gamma-1)}$

It may be assumed that $ds = C_p\, d(l_n v) + C_v\, d(l_n p)$

where v is the specific volume.

A convergent - divergent nozzle is to be designed to produce a Mach number of 3 when the absolute pressure is 1 bar. Calculate the required supply pressure and the ratio of the throat and exit areas.

(Answers 36.73 bar 0.236/1)

Let's now examine the flow of gases in long pipes and ducts in which the temperature stays constant.

8. ISOTHERMAL FLOW

Isothermal flow normally occurs in long pipes in which the temperature of the gas has time to normalise with the surroundings. Consider a section of such a pipe :

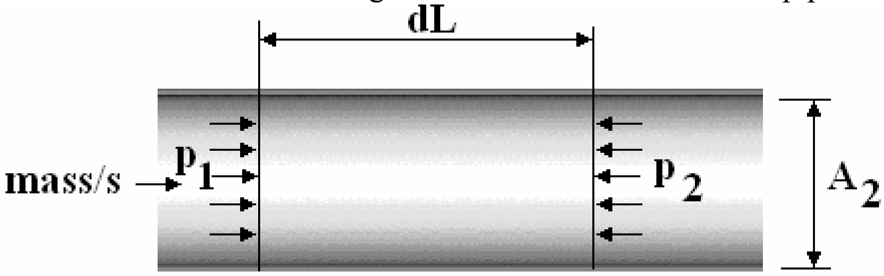


Figure 15

The friction coefficient is C_f as defined by D'Arcy's formula.

$p_2 - p_1 = dp$ Pressure force = $A(p_1 - p_2) = -A\,dp$

Resisting shear force = $\tau_w \pi D dL$ where τ_w is the wall shear stress and D is the diameter.

Since the pressure drops along the length, the volume expands and so the velocity increases since the area is constant du = increase in velocity.

If the velocity increases, then the mass must be accelerated. the inertia force required to this is equal to the change in momentum per second $m(u_2 - u_1)$

Balancing forces produces this equation :

$$-A\,dp - \pi\,D\,dL\,\tau_w = m(u_2 - u_1) = \rho_2 A u_2 \quad \text{divide by A and put } A = \frac{\pi D^2}{4}$$

$$-dp = \frac{4\pi D\,dL\,\tau_w}{\pi D^2} + \rho_2 u_2 (u_2 - u_1) \quad \text{let the change in velocity be very small so that } (u_2 - u_1) = du$$

$$-dp = \frac{4\,dL\,\tau_w}{D} + \rho_2 u_2 (du)$$

The friction coefficient C_f is defined as :

C_f = Wall shear Stress/dynamic pressure = $2\,\tau_w / \rho\,u^2$

$$-dp = \frac{4\,dL\rho_2\,u_2^2 C_f}{2D} + \rho_2 u_2 (du) \quad \text{divide through by } \rho_2\,u_2^2$$

$$\frac{-dp}{\rho_2\,u_2^2} = \frac{2\,dLC_f}{D} + \frac{du}{u}$$

We may drop the suffix so that for any given point in the pipe

$$\frac{-dp}{\rho\,u^2} = \frac{2\,dLC_f}{D} + \frac{du}{u}$$

$-dp / (\rho u^2) = (2\,C_f\,dL / D) + (du / u)$

Usually the change in velocity is negligible and dv is approximately zero. This reduces the equation to :

$-dp / (\rho u^2) = (2\ C_f\ dL / D)$

Hence $\quad -dp / dL = (\rho\ u^2)(2\ C_f / D)$

Since $V = mRT/p = Av$ then $v = mRT/pA = 4mRT/p\pi D^2$ where m is the mass flow rate

and so $\quad u^2 = 16m^2R^2T^2/p^2\pi^2D^4$

and $\quad \rho u^2 = 16\ \rho m^2R^2T^2/p^2\pi^2D^4$

but $\rho = p/RT$ so $\rho u^2 = 16\rho m^2 RT/p\pi^2 D^4$

so $-dp/dL = (32 C_f m^2 RT)/(p\pi^2 D^5)$

$-pdp = (32 C_f m^2 RT)(dL)/\pi^2 D^5)$

Integrating with corresponding limits of L=0 when $p = p_1$ and L=L when $p = p_2$

Then $(1-p_2^2/p_1^2) = (64 m^2 RT C_f L)/(\pi^2 D^5 p_1^2)$

SELF ASSESSMENT EXERCISE No. 5

1. An air storage vessel contains air at 6.5 bar and 15°C. Air is supplied from the vessel to a machine through a pipe 90 m long and 50 mm diameter. The flow rate is 2.25 m^3/min at the pipe inlet. The friction coefficient C_f is 0.005. Neglecting kinetic energy, calculate the pressure at the machine assuming isothermal flow. (Answer 5.98 bar)

8.1 FRICTION COEFFICIENT

The friction coefficient C_f has been comprehensively explained in other tutorials for non - compressible flow.

For smooth bore pipes the following is found to be accurate.

BLAZIUS found $C_f = 0.079\ Re^{-0.25}$

LEE found that $C_f = 0.0018 + Re^{-0.35}$

Otherwise use the Moody chart to find f in which case you need to remember that

$Re = \rho u D/\mu$ but since $u = V/A$ and $\rho = m/V$ then $Re = 4m/\rho u D$

ASSIGNMENT 6

1. A natural gas pipeline is 1000 m long and 100 mm bore diameter. It carries 0.7 kg/s of gas at a constant temperature of 0^o C. The viscosity is 10.3×10^{-6} N s/m^2 and the gas constant R = 519.6 J/kg K. The outlet pressure is 105 kPa. Calculate the inlet pressure. using the Blazius formula to find f. (Answer 357 kPa.)

2. A pipeline is 20 km long and 500 mm bore diameter. 3 kg/s of natural gas must be pumped through it at a constant temperature of 20^oC. The outlet pressure is 200 kPa. Calculate the inlet pressure using the same gas constants as Q.1.
(Answer 235 kPa)

3. Air flows at a mass flow rate of 9.0 kg/s isothermally at 300 K through a straight rough duct of constant cross sectional area of 1.5×10^{-3} m^2. At end A the pressure is 6.5 bar and at end B it is 8.5 bar. Determine

   a. the velocities at each end. (Answers 794.8m/s and 607.7 m/s)

   b. the force on the duct. (Answer 1 380 N)

   c. the rate of heat transfer through the walls. (Answer 1.18 MJ)

   d. the entropy change due to heat transfer. (Answer 3.935 KJ/k)

   e. the total entropy change. (Answer 0.693 kJ/K)

It may be assumed that $ds = C_p\, dT/T + R\, dp/p$

4. A gas flows along a pipe of diameter D at a rate of m kg/s. Show that the pressure gradient is
$-dp/dL = (32\ fm^2RT)/(p\pi^2D^5)$

Methane gas is passed through a pipe 500 mm diameter and 40 km long at 13 kg/s. The supply pressure is 11 bar. The flow is isothermal at 15°C. Given that the molecular mass is 16 kg/kmol and the friction coefficient C_f is 0.005 determine

a. the exit pressure. (Answer 3.99 bar)

b. the inlet and exit velocities. (Answers 9.014 m/s and 24.85 m/s)

c. the rate of heat transfer to the gas. (Answer 3.48 kW)

d. the entropy change resulting from the heat transfer. (Answer 12.09 kJ/K)

e. the total entropy change calculated from the formula

$ds = C_p \ln(T_2/T_1) - R \ln(p_2/p_1)$ (Answer 1.054 kJ/K)

Let's now go on to look at shock waves that occur in compressible flow when it goes supersonic.

9. NORMAL SHOCK WAVES

Shock waves occur in compressible fluids and are due to a sudden rise in pressure from p_1 to p_2 for example resulting from an explosion or from a sudden change in flow.

Consider a sudden rise in pressure travelling through a stream tube of fluid of constant cross sectional area A. The conditions before the change are denoted by suffix (1) and after the change by suffix (2). In particular the Mach Numbers are M_1 and M_2. We will look at the laws governing the changes one at a time starting with momentum.

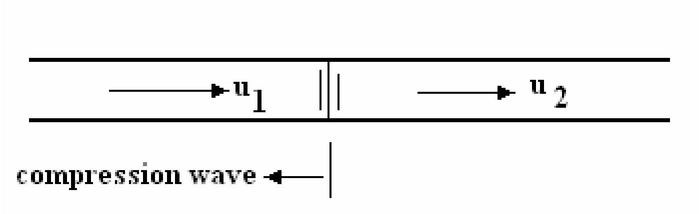


Figure 16

9.1 MOMENTUM CHANGE

From the fundamental law Force = rate of change in momentum we get

$$(p_1 - p_2) A = m (u_2 - u_1)$$

Substitute mass flow rate = m = ρA u and divide both sides by A

$$(p_1 - p_2) = (\rho_2 v_2^2 - \rho_1 v_1^2)$$

Substitute the sonic velocity $a = \sqrt{\gamma RT}$ and $RT = p/\rho$ It follows that $a^2 = \gamma p/\rho$

Mach Number is defined as $M = u/a$ so $u^2 = \gamma p M^2/\rho$

Hence $(p_1 - p_2) = (p_2\gamma M_2^2 - p_1\gamma M_1^2)$

$$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} \text{........................(i)}$$

9.2 ENERGY CONSERVATION

The change in pressure is so rapid that there is no time for heat to transfer out of the gas so the pressure rise is adiabatic. In this case we may use Bernoulli.

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2} \quad \text{assume } c_p \text{ is constant and substitute } u^2 = \gamma RTM^2$$

$$c_p T_1 + \frac{\gamma R T_1 M_1^2}{2} = c_p T_2 + \frac{\gamma R T_2 M_2^2}{2}$$

$$\frac{T_2}{T_1} = \frac{c_p + \frac{\gamma R M_1^2}{2}}{c_p + \frac{\gamma R M_2^2}{2}} \quad \text{substitute the relationship } R = \frac{c_p(\gamma - 1)}{\gamma} \text{ and simplify}$$

$$\frac{T_2}{T_1} = \frac{1+(\gamma-1)\frac{M_1^2}{2}}{1+(\gamma-1)\frac{M_2^2}{2}} \text{.....................(ii)}$$

9.3 CONSERVATION OF MASS

$m = \rho_1 A u_1 = \rho_2 A u_2$ so $\rho_1/\rho_2 = u_2/u_1$ but $u = a M$

$$\frac{\rho_1}{\rho_2} = \frac{a_2 M_2}{a_1 M_1} \text{ where } a = \sqrt{\gamma RT}$$

so $$\frac{\rho_1}{\rho_2} = \frac{\sqrt{\gamma RT_2}}{\sqrt{\gamma RT_1}} \text{ x } \left(\frac{M_2}{M_1}\right) = \sqrt{\frac{T_2}{T_1}} \text{ x } \left(\frac{M_2}{M_1}\right) \text{ substitute equation (ii)}$$

$$\frac{\rho_1}{\rho_2} = \sqrt{\frac{1+(\gamma-1)\frac{M_1^2}{2}}{1+(\gamma-1)\frac{M_2^2}{2}}} \text{ x } \left(\frac{M_2}{M_1}\right) \text{................(iii)}$$

9.4 GAS LAWS

$$\frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1} \quad \text{so dividing top and bottom by m}$$

$$\frac{T_2}{T_1} = \frac{p_2 \rho_1}{p_1 \rho_2} \quad \text{and substituting equation (iii) gives :}$$

$$\frac{T_2}{T_1} = \sqrt{\frac{1+(\gamma-1)\frac{M_1^2}{2}}{1+(\gamma-1)\frac{M_2^2}{2}}} \text{ x } \left(\frac{M_2}{M_1}\right)\left(\frac{p_2}{p_1}\right) \text{......................(iv)}$$

9.4 OVERALL RESULT

By combining the previous work the following equation can be obtained.

$$M_2^2 = \frac{\frac{2}{\gamma-1} + M_1^2}{\frac{2\gamma M_1^2}{\gamma-1} - 1} \text{........................(v)}$$

This equation can now be used to solve problems where the Mach number before the change is known.

Note if $M_1 = 1$ then $M_2 = 1$ and if $M_1 > 1$ then $M_2 < 1$

WORKED EXAMPLE No. 4

A gas has a temperature of 300 K, pressure of 1.5 bar and velocity of 450 m/s. Calculate the velocity, pressure and temperature after a shock wave passes into it. Take $\gamma = 1.3$ and the mean molar mass is 44.

SOLUTION

First calculate the gas constant R

$R = R_o$/molar mass = 8314/44 = 188.95 J/kg K

Note the universal gas constant R_o is on the back page of the fluids tables or should be remembered as 8314 J/kmol K

Next calculate $a_1 = \sqrt{(\gamma RT)} = \sqrt{(1.3 \times 188.95 \times 300)} = 271.46$ m/s

Hence the initial velocity is supersonic with a Mach No.

$M_1 = 450/271.46 = 1.6577$

Now calculate M_2 from equation (v)
Show for yourself that $M_2 = 0.642$

Now use equation (i) to find p_2 $$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2}$$

Show for yourself that $p_2 = 4.47$ bar

Next use equation (ii) to find T_2

$$\frac{T_2}{T_1} = \frac{1+\frac{(\gamma-1)M_1^2}{2}}{1+\frac{(\gamma-1)M_2^2}{2}}$$

Show for yourself that $T_2 = 399$ K

Now find the sonic velocity after compression $a_2 = \sqrt{(\gamma RT)} = 313$ m/s

Hence $u_2 = a_2M_2 = 201$ m/s

SELF ASSESSMENT EXERCISE No. 7

1. Write down the equations representing the conservation of mass, energy and momentum across a normal shock wave.

   Carbon dioxide gas enters a normal shock wave at 300 K and 1.5 bar with a velocity of 450 m/s. Calculate the pressure, temperature and velocity after the shock wave. The molecular mass is 44 kg/kmol and the adiabatic index is 1.3. (Answers 446 kPa, 399 K and 201 m/s)

2. Air discharges from a large container through a convergent - divergent nozzle into another large container at 1 bar. the exit mach number is 2.0. Determine the pressure in the container and at the throat. (Answers 7.82 bar and 4.13 bar).

   When the pressure is increased in the outlet container to 6 bar, a normal shock wave occurs in the divergent section of the nozzle. Sketch the variation of pressure, stagnation pressure, stagnation temperature and Mach number through the nozzle.

   Assume isentropic flow except through the shock. The following equations may be used.

   $$\frac{\gamma RT}{\gamma - 1} + \frac{u^2}{2} = \text{constant}$$

   $$u = Ma\sqrt{\gamma RT}$$

   $$\frac{p_1}{p_2} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}$$

HYDROLOGY - TUTORIAL 1

UNIFORM FLOW IN CHANNELS

In this tutorial you will

- Derive formula for flow through notches.
- Solve problems involving flow through notches.
- Define uniform channel flow.
- Derive formulae relating channel dimensions and flow rate.
- Define the Froude Number.
- Define sub-critical and super critical flow.

The student is advised to study Tutorial 1 from the Fluid mechanics D203 section before starting this tutorial.

1. FLOW THROUGH NOTCHES

A notch is placed in a channel to measure the flow by restricting it. The flow rate is related to the depth of water behind the notch and a calibrated depth gauge is all that is needed to indicate the flow rate.

RECTANGULAR NOTCH

The velocity of water due to a pressure head only is u = √2gh. This assumes there is negligible velocity approaching the notch.

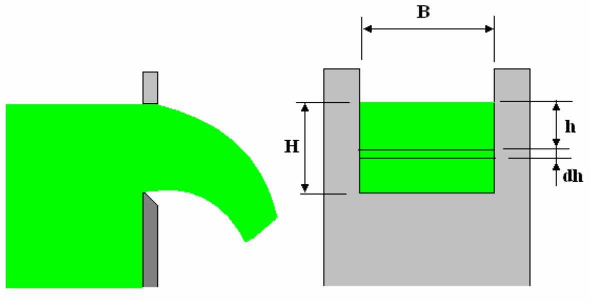


Figure 1

The flow through the elementary strip is

dQ = u B dh

$$Q = B\int_0^H u\,dh = B\sqrt{2g}\int_0^H h^{1/2}dh = \frac{2B}{3}\sqrt{2g}H^{3/2}$$

Where the flow approaches the edge of a notch, there is a contraction because the velocity at the edge is not normal to the plane of the notch. This produces a reduction in the cross section of flow and some friction in the flow. Depending on the design of the edges a coefficient of discharge C_d is needed to correct the formula.

$$Q = C_d\frac{2B}{3}\sqrt{2g}\,H^{3/2}$$

Further study will yield formula for C_d based on the various shapes of the edges.

SUBMERGED RECTANGULAR NOTCH and SLUICE GATE

If the notch is a rectangular hole, the integration must be between the two depths H_1 and H_2 yielding

$$Q = C_d\frac{2B}{3}\sqrt{2g}\left(H_2^{3/2} - H_1^{3/2}\right)$$

If the bottom of the notch is the floor of the downstream channel, we have a sluice gate and the same formula applies.

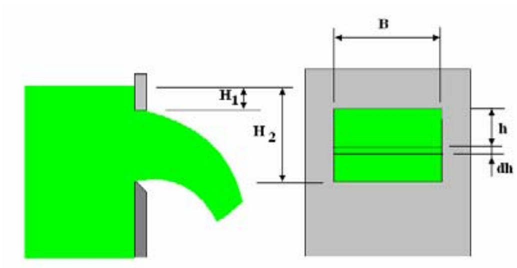


Figure 2

VEE NOTCH

The width of the elementary strip varies depth such that

b = 2(H - h) tan(θ/2)

$$Q = \int_0^H ub\,dh = 2\sqrt{2g}\tan\left(\frac{\theta}{2}\right)\int_0^H (H-h)h^{1/2}dh$$

$$Q = 2\sqrt{2g}\tan\left(\frac{\theta}{2}\right)\int_0^H \left(Hh^{1/2} - h^{3/2}\right)dh$$

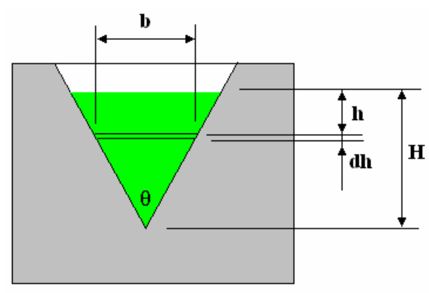


Figure 3

$$Q = 2\sqrt{2g}\tan\left(\frac{\theta}{2}\right)\left[\frac{2}{3}H^{5/2} - \frac{2}{5}H^{5/2}\right] = 2\sqrt{2g}\tan\left(\frac{\theta}{2}\right)\left[\frac{4}{15}H^{5/2}\right]$$ and introducing C_d we have

$$Q = C_d\frac{8}{15}\sqrt{2g}\tan\left(\frac{\theta}{2}\right)H^{5/2}$$

VELOCITY OF APPROACH

If the velocity approaching the notch is not negligible say u_1 then the velocity through the elementary strip is $u = \sqrt{\left(u_1^2 + 2gh\right)}$. If a notch is fitted into a channel not much bigger than the notch, the velocity of the water approaching the notch is not negligible and a correction needs to be made.

WORKED EXAMPLE No.1

The depth of water above the sill of a rectangular notch is 0.23 m and the notch is 0.5 m wide. The coefficient of discharge is 0.6. Calculate the flow rate of water.

SOLUTION

$$Q = C_d \frac{2B}{3}\sqrt{2g}H^{3/2} = 0.6\left(\frac{2 \text{ x } 0.5}{3}\right)\sqrt{2g}\ 0.25^{3/2} = 0.1107 \text{ m}^3/\text{s}$$

WORKED EXAMPLE No.2

The depth of water above the sill of a vee notch is 0.4 m and has an included angle of 90°. The coefficient of discharge is 0.65. Calculate the flow rate of water.

SOLUTION

$$Q = C_d \frac{8}{15}\sqrt{2g}\tan\left(\frac{\theta}{2}\right)H^{5/2} = 0.65 \text{ x } \frac{8}{15}\sqrt{2g}\ \tan 45^\circ \text{ x } 0.4^{5/2} = 0.155 \text{ m}^3/\text{s}$$

WORKED EXAMPLE No.3

The depth of water behind a sluice gate in a horizontal rectangular channel is 3 m and the sluice is 0.8 m high. The coefficient of discharge is 0.75. Calculate the flow rate of water in the channel downstream.

SOLUTION

$$Q = C_d \frac{2B}{3}\sqrt{2g}\left(H_2^{3/2} - H_2^{3/2}\right) = 0.75\frac{2 \text{ x } 3}{3}\sqrt{2g}\left(3^{3/2} - 2.2^{3/2}\right) = 12.84 \text{ m}^3$$

SELF ASSESSMENT EXERCISE No.1

1 The depth of water above the sill of a rectangular notch is 0.4 m and the notch is 0.75 m wide. The coefficient of discharge is 0.62. Calculate the flow rate of water. (0.347 m^3/s)

2 The depth of water above the sill of a Vee notch is 0.2 m and has an included angle of 60°. The coefficient of discharge is 0.6. Calculate the flow rate of water. (0.228 m^3/s)

3. A sluice controls the flow in a rectangular channel 2.5 m wide. The depth behind the sluice is 2 m and the sluice is 0.5 m high. What is the discharge? Take C_d =0.8. (5.85 m^3/s)

2. UNIFORM FLOW IN CHANNEL

Channel flow is characterised by constant pressure (usually atmosphere) at all points on the surface. This means that flow can only be induced by gravity so the bed of the channel must slope downwards. There is no pressure gradient in the fluid pushing it along.

If the cross section is uniform and the depth is uniform then the flow rate is uniform at all points along the length. This can only occur if the change of potential height is balanced by the friction losses.
This is UNIFORM FLOW.

DEFINITIONS
Flow rate= Q (m^3/s)   Flow rate per unit width q m^2/s
Cross sectional area = A (m^2)
Wetted perimeter = P (m)
Mean velocity = u_o = Q/A (m/s)
Slope of bed = S which is otherwise called the energy gradient.

The hydraulic gradient is i and this is the friction head loss per unit length of the bed.
The hydraulic gradient is the same as the slope if the flow has a constant depth (uniform flow).

The hydraulic radius is defined as $R_h = A/P$ and this is also often called the hydraulic mean depth with symbol m.

The wetted area is $A_w = PL$

τ_w is the wall shear stress. This is the force per unit surface area resisting flow at the surface of contact between the fluid and the wall.

CHEZY FORMULA

Consider part of a flow of regular cross section A and length L.

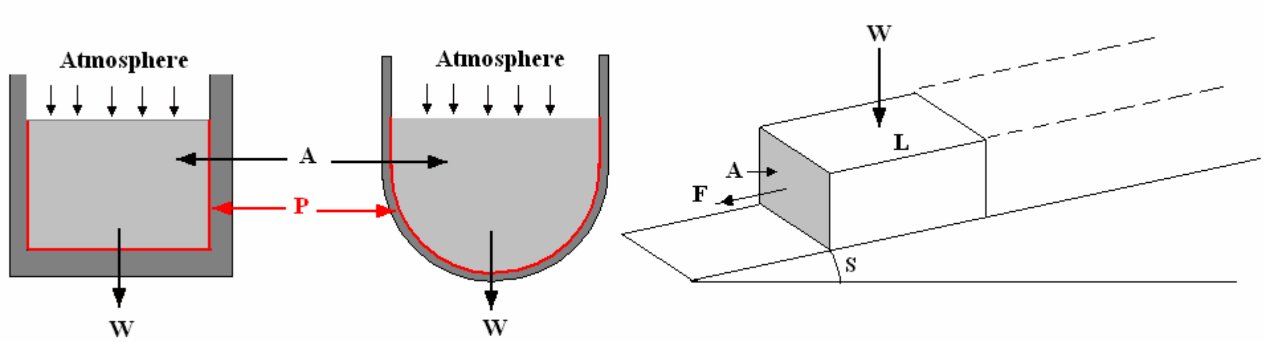


Figure 4

If the slope is small the weight of the section considered is $W = \rho gAL$
Resolving the weight parallel to the bed the force causing flow is $F = W \sin(S)$
If S is small sin S = S radians so $F = W S = \rho gALS$
If the flow is steady there is no inertia involved so the force resisting motion must be equal to this force.

The resisting force per unit surface area $= F/A_w = \tau_w = F/PL = \rho gALS/PL = \rho gAS/P = \rho g R_h S$

Chezy thought that $\tau_w \propto u_o^2$ and so $\tau_w = C_1 u_o^2$ Hence $C_1 u_o^2 = \rho g R_h S$

The Chezy formula is $u_o = \mathbf{C (R_h S)^{1/2}}$

$C = (\rho g/C_1)^{1/2}$ and C is the Chezy constant.

WORKED EXAMPLE No. 4

An open channel has a rectangular section 2 m wide. The flow rate is 0.05 m^3/s and the depth is 0.4 m. Calculate the slope of the channel using the Chezy formula for steady flow. Take the constant C = 50 $m^{½}$ /s

SOLUTION

$A = 2 \times 0.4 = 0.8\ m^2$
$P = 2 + 0.4 + 0.4 = 2.8$ m
$R_h = A/P = 0.2857$ m
$u_o = Q/A = 0.05/0.8 = 0.0625$ m/s
$u_o = 0.0625 = C\,(R_h\,S)^{½}$
$u_o = 0.0625 = 50\,(0.2857\,S)^{½}$
$S = 5.469 \times 10^{-6}$

SELF ASSESSMENT EXERCISE No.2

1. An open channel has a triangular section with sides at 45° to the vertical. The flow rate is 0.0425 m^3/s and the depth is 0.225 m. Calculate the slope of the channel using the Chezy formula for steady flow. Take the constant C = 49 $m^{½}$ /s
(Answer 0.00369)

2. A channel with a section as shown carries 1.1 m^3/s of water with the depth as shown. The slope of the bed is 1/2000. Calculate the constant C in the Chezy formula.

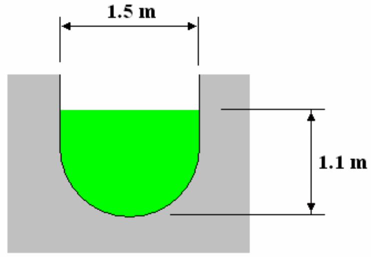


Figure 5

(Answer 51.44)

THE CHEZY - MANNING FORMULA

Manning extended Chezy's formula. Based on research he stated that $C = \frac{R_h^{1/6}}{n}$

n is a dimensionless constant based on the surface roughness of the channel. Substituting this into the Chezy formula yields

$$u_o = \frac{R_h^{2/3} S^{1/2}}{n}$$ This is the Chezy - Manning formula.

WORKED EXAMPLE No. 5

An open channel has a rectangular section 5 m wide. The flow rate is 1.2 m^3/s and the depth is 1.4 m. Calculate the slope of the channel using the Manning formula for steady flow. Take the constant n = 0.019 $m^{½}$ /s

SOLUTION

$A = 5 \times 1.4 = 7\ m^2$
$P = 5 + 1.4 + 1.4 = 7.8$ m
$R_h = 7/7.8 = 0.897$ m
$u_o = Q/A = 0.171$ m/s

$$u_o = \frac{R_h^{2/3} S^{1/2}}{n} \text{ rearrange } \quad S = \left(\frac{nu_o}{R_h^{2/3}}\right)^2 = \left(\frac{0.019 \times 0.171}{0.897^{2/3}}\right)^2 = 12.256 \times 10^{-6}$$

SELF ASSESSMENT EXERCISE No.3

1. A rectangular channel is 2 m wide and runs 1.5 m deep. The slope of the bed is 1/4000. Using the Manning formula with n = 0.022, calculate the flow rate.
(Answer 1.534 m^3/s)

2. An open channel has a rectangular section 3 m wide. The flow rate is 1.4 m^3/s and the depth is 0.8 m. Calculate the slope of the channel using the Manning formula for steady flow. Take the constant n = 0.02 $m^{½}$ /s
(Answer 292.5 x 10^{-6})

3. Water flows down a half full circular pipeline of diameter 1.4m. The pipeline is laid at a gradient if 1/250. If the constant n in the Manning formula is n = 0.015 what is the discharge.
(1.612 m^3/s)

DARCY FORMULA APPLIED TO CHANNELS

The Chezy formula may be related to the Darcy formula for flow in round pipes.

The Darcy formula (not derived here) is $h_f = \dfrac{4C_f L u_o^2}{2gd}$

Sometimes this is stated as $h_f = \dfrac{f\,L u_o^2}{2gd}$ where $4C_f = f$

h_f is the friction head and C_f is the friction coefficient which is related to the Reynolds's number and the relative surface roughness.

If a round pipe runs full but with constant pressure along the length, then the Chezy and Darcy formulae may be equated.

From the Darcy formula we have $u_o^2 = \dfrac{2gdh_f}{4C_f L}$

For constant pressure, $h_f/L = S$ $u_o^2 = \dfrac{2gdS}{4C_f}$

From the Chezy formula we have $u_o^2 = C^2 R_h S$

For a round pipe diameter d running full $R_h = d/4$ $u_o^2 = C^2 Sd/4$

Equating we have $\dfrac{C^2 Sd}{4} = \dfrac{2gdS}{4C_f}$

$$C_f = \frac{2g}{C^2}$$

From the Chezy equation we have $u_o^2 = \dfrac{C^2 R h_f}{L}$

$$h_f = \frac{L u_o^2}{C^2 R_h} = \frac{C_f L u_o^2}{2gR_h}$$

This version of the Darcy formula may be used for pipes and channels of any shape with no pressure gradient. Discussion of the Darcy formula show that C_f is related to the surface roughness and this compares with Manning's work.

In the case of LAMINAR FLOW Poiseuille's equation is also relevant and this gives the friction head as

$$h_f = \frac{32\,\mu\,L\,u_o}{\rho g d^2}$$

Equating this to the Darcy formula gives:

$$\frac{32\,\mu\,L\,u_o}{\rho g d^2} = \frac{4C_f L u_o^2}{2gd} \text{ hence } C_f = \frac{16\mu}{\rho u_o d} = \frac{16}{R_e}$$

The complete relationship between the Reynolds' number R_e and the relative surface roughness is given on the Moody Chart. The chart has several regions, laminar flow, turbulent flow and a region between where it is in transition. The turbulent flow varies between smooth surfaces and fully rough surfaces that produce fully developed turbulent flow. ***Relative surface roughness*** is defined as $\varepsilon = k/D$ where k is the mean surface roughness and D the bore diameter. The chart is a plot of C_f vertically against R_e horizontally for various values of ε. In order to use this chart you must know two of the three co-ordinates in order to pick out the point on the chart and hence pick out the unknown third co-ordinate.

For the laminar region $C_f = \dfrac{16}{R_e}$

For smooth pipes, (the bottom curve on the diagram), various formulae have been derived such as those by Blasius and Lee.

BLASIUS $C_f = 0.0791\ R_e^{0.25}$

LEE $C_f = 0.0018 + 0.152\ R_e^{0.35}$.

The Moody diagram shows that the friction coefficient reduces with Reynolds number but at a certain point, it becomes constant. When this point is reached, the flow is said to be fully developed turbulent flow. This point occurs at lower Reynolds numbers for rough pipes.

A formula that gives an approximate answer for any surface roughness is that given by Haaland.

$$\frac{1}{\sqrt{C_f}} = -3.6 \log_{10}\left\{\frac{6.9}{R_e} + \left(\frac{\varepsilon}{3.71}\right)^{1.11}\right\}$$

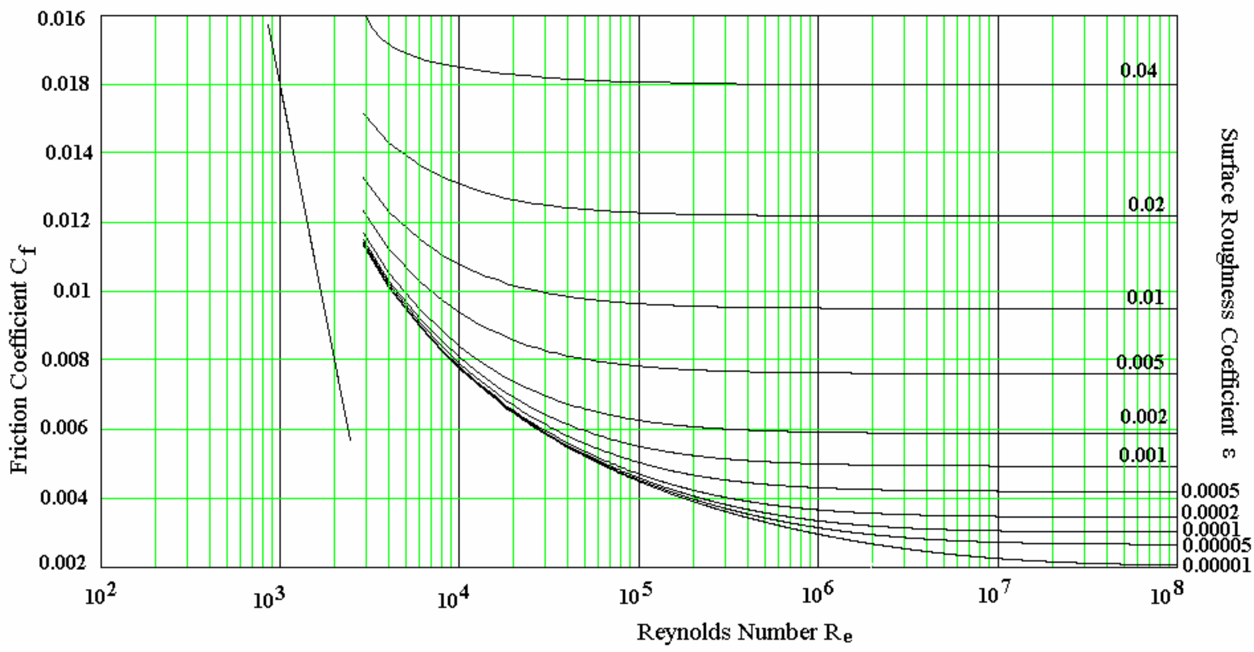


Figure 6

SELF ASSESSMENT EXERCISE No. 4

1. The Darcy - Weisbach formula for a round pipe running full states that $h_f = 4\ C_f L u^2/2gd$ where L is the length, d the diameter and u the mean velocity.

a. Show that for laminar flow $C_f = 16/R_e$

b. Relate the Chezy formula $u = C\ (RS)^{1/2}$ and the Manning formula $u = (R^{2/3}\ S^{1/2})/n$ to the Darcy - Weisbach formula and list the ranges of applicability of all three formula.

c. Sketch the relationship between C_f and R_e for the range $R_e = 10^0$ to $R_e = 10^6$ in a pipe of circular cross section for typical values of surface roughness k.

d. If ageing causes the surface roughness of a pipe to increase, what affect would this have on the flow carrying capacity of the pipe?

3. CRITICAL FLOW

SPECIFIC ENERGY HEAD - h_s

At any point in the length of the channel the fluid has three forms of energy relative to the bed, kinetic, gravitational (potential) and flow (pressure) energy.

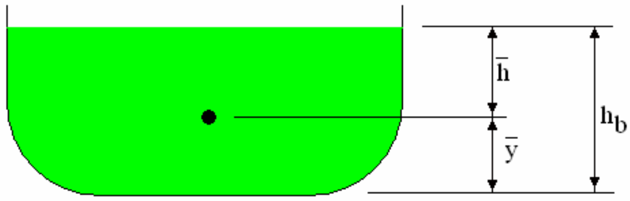


Figure 7

Strictly, all energy terms should be the mean values. The mean depth is $\bar{h}$ and the mean gravitational (potential) head is $\bar{y}$ (the distance to the centroid). The depth at the bottom is $h_b = \bar{h} + \bar{y}$ and the mean velocity is u_o

From the Bernoulli Equation $h_s = h + \bar{y} + \frac{u_o^2}{2g} = h_b + \frac{u_o^2}{2g}$

Text books jump straight to this formula wrongly giving h_b as the pressure head.

Rearrange the formula and $u_o = \{2g(h_s - h_b)\}^{1/2}$

Consider a channel with an unspecified cross section of area A.

$Q = Au_o$ $\qquad\qquad Q = A\{2g(h_s - h_b)\}^{1/2}$

CRITICAL DEPTH – h_C

It will be shown that for a given value of h_s there is a depth h_c that produces maximum flow rate but the value of h_c depends on the shape of the channel since the width is a function of depth and hence the area is a function of depth. Let's examine a rectangular cross section.

RECTANGULAR SECTION

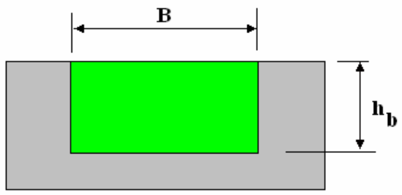
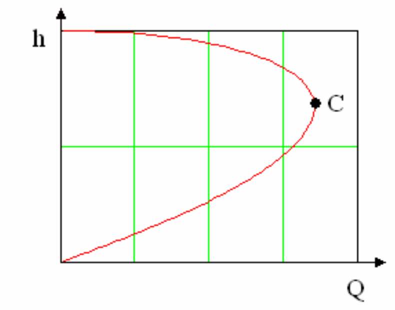


Figure 8

$$Q = A\{2g(h_s - h_b)\}^{1/2} = Bh_b\{2g(h_s - h_b)\}^{1/2}$$

$$Q = B\sqrt{2g}\left\{\left(h_b{}^2 h_s - h_b{}^3\right)\right\}^{1/2}$$

If we plot h – Q for a given value of B and h_s we get figure 9a and if we plot h – h_s for a given value of B and Q we get figure 9b.



Figures 9a

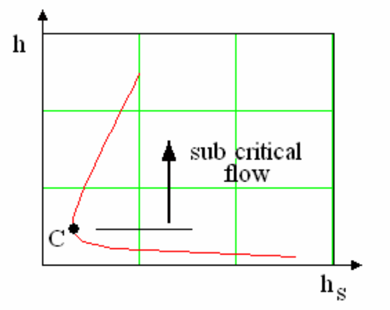


Figure 9 b

The plots reveal some interesting things. Point C is called the critical point and this gives the minimum energy head for a given flow rate or a maximum flow rate for a given energy head.

For a flow rate other than the critical value, there are two possible depths of flow. This is logical since for a given amount of energy the flow can be slow and deep or fast and shallow. Flow at the shallow depth is super-critical and flow at the larger depth is sub-critical. The critical depth is denoted h_c.

To find the critical depth we use max and min theory. At point C $dQ/dh_b = 0$
Differentiate and we get:

$$\frac{dQ}{dh_b} = \left\{(2g)^{1/2}\left(h_s - \frac{3h_b}{2}\right)\right\}^{1/2} \frac{B}{(h_s - h_b)^{1/2}} \qquad 0 = \left(h_s - \frac{3h_b}{2}\right) \qquad h_b = \frac{2h_s}{3} = h_c$$

Since $u_o = \{2g(h_s - h_b)\}^{1/2}$ then substituting for h_s will produce the critical velocity.

$$u_c = \sqrt{2g\left(\frac{3h_c}{2} - h_c\right)} = \sqrt{2g\left(\frac{h_c}{2}\right)} = \sqrt{gh_c}$$

It follows that the critical flow rate is $Q_c = Au_c = B\sqrt{g}\, h_c^{3/2}$

Here is an alternative derivation for the rectangular channel.

$A = B\, h_b$ $\qquad$ $u_o = Q/(A) = Q/(B\, h_b)$

$$h_s = h_b + \frac{Q^2}{2g(B^2h_b^2)}$$ For a given flow rate the minimum value of h_s is found by differentiating.

$$\frac{dh_s}{h_b} = 1 - \frac{2Q^2}{2g(B^2h_b^3)} = 1 - \frac{Q^2}{g(B^2h_b^3)}$$ For a minimum value equate to zero.

$$0 = 1 - \frac{Q^2}{g(B^2h_b^3)} \qquad Q = \sqrt{g(B^2h_b^3)} \text{ or } h_b = \left(\frac{Q^2}{gB^2}\right)^{1/3}$$

These are the critical values so it follows that $Q_c = B\sqrt{g}\, h_c^{3/2}$ or $h_c = \left(\frac{Q_c^2}{gB^2}\right)^{1/3}$

$$u_c = \frac{Q_c}{Bh_c} = \frac{B\sqrt{g}h_c^{3/2}}{Bh_c} = \sqrt{gh_c} \text{ or } h_c = \frac{u_c^2}{g}$$

$$h_s = h_b + \frac{u_o^2}{2g} = h_c + \frac{u_c^2}{2g} = \frac{u_c^2}{g} + \frac{u_c^2}{2g} = \frac{3u_c^2}{2}$$

$$h_s = h_c + \frac{u_c^2}{2g} = h_c + \frac{gh_c}{2g} = \frac{3}{2}h_c \qquad h_c = \frac{2}{3}h_s$$

The critical flow in terms of h_s is $Q_c = B\sqrt{g}\, h_c^{3/2} = B\sqrt{g}\left(\frac{2}{3}h_s\right)^{3/2} = B\sqrt{g\left(\frac{8}{27}\right)}h_s^{3/2}$

The critical velocity in terms of h_s is $u_c = \sqrt{gh_c} = \sqrt{\frac{2gh_s}{3}}$

FROUDE NUMBER

You may have studied this in dimensional analysis. The Froude Number is a dimensionless number important to channel flow as well as to surface waves. It is defined as :

$F_r = \frac{u}{\sqrt{gh}}$ For critical flow $F_r = \frac{u_c}{\sqrt{gh_c}}$ Substitute $u_c = \sqrt{gh_c}$ into this and $F_r = \frac{\sqrt{gh_c}}{\sqrt{gh_c}} = 1$

The Froude number is always 1 when the flow is critical in a RECTANGULAR CHANNEL but not for other shapes. Another name for super-critical flow is SHOOTING or RAPID FLOW and sub–critical is called TRANQUIL FLOW.

Summary for a rectangular channel

The critical depth is $h_c = \frac{2}{3}h_s$ $\qquad$ The critical velocity is $u_c = \sqrt{gh_c} = \sqrt{\frac{2gh_s}{3}}$

The critical flow is $Q_c = B\sqrt{g}\, h_c^{3/2} = B\sqrt{g\left(\frac{8}{27}\right)}h_s^{3/2}$ $\qquad$ Froude Number $F_r = 1$

WORKED EXAMPLE No. 6

A rectangular channel 1.6 m wide must carry water at depth of 1 m. What would be the maximum possible flow rate and what would be the mean velocity?

SOLUTION

For maximum flow rate the depth must be the critical depth so $h_c = 1$m.

The critical velocity is $u_c = (g\,h_c)^{½} = (9.81 \text{ x } 1)^{½} = 3.132$ m/s

The critical flow is $Q_c = A\,u_c = 1.6 \text{ x } 1 \text{ x } 3.132 = 5.01 \text{ m}^3/\text{s}$

Check the Froude number $F_r = \dfrac{u_c}{\sqrt{gh_c}} = \dfrac{3.132}{\sqrt{g \text{ x } 1}} = 1$

If the constant n in the Manning formula is 0.019 $m^{½}$ /s what must the slope of the bed be for constant depth at maximum flow rate?

$A = 1.6 \text{ x } 1 = 1.6 \text{ m}^2$ $\quad P = 1.6 + 1 + 1 = 3.6$ m $\quad R_h = 1.6/3.6 = 0.444$ m

$u_o = u_c = 3.132$ m/s

$u_o = \dfrac{R_h^{2/3} S^{1/2}}{n}$ rearrange $\quad S = \left(\dfrac{n u_o}{R_h^{2/3}}\right)^2 = \left(\dfrac{0.019 \text{ x } 3.132}{0.444^{2/3}}\right)^2 = 0.0104$

WORKED EXAMPLE No. 7

Water flows in a rectangular channel 3 m wide with a mean velocity of 1.5 m/s and a depth of 1.2 m. Determine whether the flow is tranquil or shooting. Calculate the following.

The actual flow rate
The specific energy head
The critical depth
The maximum flow possible

SOLUTION

$F_r = \dfrac{u_o}{\sqrt{gh}} = \dfrac{1.5}{\sqrt{g \text{ x } 1.2}} = 0.437$ It follows that the flow is tranquil.

Actual flow rate = $A\,u_o = (3 \text{ x } 1.2) \text{ x } 1.5 = 5.5 \text{ m}^3/\text{s}$

Energy Head $h_s = h + u_o^2/2g = 1.2 + 1.5^2/2g = 1.315$ m

$h_c = 2h_s/3 = 2 \text{ x } 1.315/3 = 0.876$ m

For maximum flow rate $F_r = 1$

$F_r = 1 = \dfrac{u_c}{\sqrt{gh_c}} \quad u_c = \sqrt{gh_c} = \sqrt{9.81 \text{ x } 0.876} = 2.931$ m/s

$A = 3 \text{ x } 0.876 = 2.629 \text{ m}^2$

$Q_c = Au_c = 2.629 \text{ x } 2.931 = 7.71 \text{ m}^3/\text{s}$

If the depth changed to the critical depth, the flow rate would increase.

SELF ASSESSMENT EXERCISE No.5

1. A rectangular channel is 3.2 m wide and must carry 5 m^3/s of water with the minimum specific head. What would the depth and mean velocity be? (1.563 m and 3.915 m/s)
2. If the channel in question 1 must carry flow at a constant depth and n in the manning formula is 0.022, what is the slope of the bed? (0.013)
3. The flow in a horizontal, rectangular channel, 6m wide is controlled by a sluice gate. The depths of flow upstream and downstream of the gate are 1.5m and 0.300 m respectively. Determine:
   (a) the discharge (b) the specific energy of the flow (c) the critical depth.

VEE OR TRIANGULAR SECTION

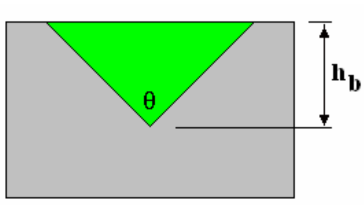


Figure 10

$$Q = A\{2g(h_s - h_b)\}^{1/2} \qquad A = \tfrac{1}{2} h_b \times 2 h_b \tan(\theta/2)$$

$$Q = h_b^2 \tan(\theta/2)\{2g(h_s - h_b)\}^{1/2}$$

$$Q = \tan(\theta/2)\{2g(h_b^4 h_s - h_b^5)\}^{1/2}$$

$$\frac{dQ}{dh_b} = \tan(\theta/2)(2g)^{1/2}(4h_b^3 h_s - 5h_b^4)^{1/2}$$

For maximum $(4h_b^3 h_s = 5h_b^4)$ $\quad (4h_s = 5h_b)$ $\quad h_c = \dfrac{4h_s}{5}$

Since $u_o = \{2g(h_s - h_b)\}^{1/2}$ then substituting for h_s will produce the critical velocity.

$$u_o = \left\{2g\left(\frac{5}{4}h_c - h_c\right)\right\}^{1/2} = (2g)^{1/2}\left(\frac{h_c}{4}\right)^{1/2} = \left(\frac{gh_c}{2}\right)^{1/2} \qquad u_c = \left(\frac{gh_c}{2}\right)^{1/2}$$

$$Q_c = Au_c = h_c^2 \tan(\theta/2)\left(\frac{gh_c}{2}\right)^{1/2} \qquad Q_c = \left(\frac{g}{2}\right)^{1/2} \tan(\theta/2) h_c^{5/2}$$

FROUDE NUMBER

$F_r = \dfrac{u}{\sqrt{gh}}$ For critical flow $F_r = \dfrac{u_c}{\sqrt{gh_c}}$ Substitute for u_c $F_r = \dfrac{\sqrt{\frac{gh_c}{2}}}{\sqrt{gh_c}} = \dfrac{1}{\sqrt{2}} = 0.707$

In terms of h_s

$$Q_c = \left(\frac{g}{2}\right)^{1/2} \tan(\theta/2) h_c^{5/2} = \left(\frac{g}{2}\right)^{1/2} \tan(\theta/2)\left(\frac{4h_s}{5}\right)^{5/2} = \sqrt{\left(\frac{g}{2}\right)}\left(\frac{4h_s}{5}\right)^{5/2} \tan(\theta/2)$$

$$u_c = \left(\frac{gh_c}{2}\right)^{1/2} = \left(\frac{g}{2} \times \frac{4}{5} h_s\right)^{1/2} = \sqrt{\frac{2g}{5} h_s}$$

Summary for triangular section

The critical depth is $\quad h_c = \dfrac{4h_s}{5}$

The critical velocity is $\quad u_c = \sqrt{\dfrac{gh_c}{2}} = \sqrt{\dfrac{2g}{5} h_s}$

The critical flow is $\quad Q_c = \left(\dfrac{g}{2}\right)^{1/2} \tan(\theta/2) h_c^{5/2} = \sqrt{\left(\dfrac{g}{2}\right)}\left(\dfrac{4h_s}{5}\right)^{5/2} \tan(\theta/2)$

Froude Number $\quad F_r = \dfrac{1}{\sqrt{2}} = 0.707$

WORKED EXAMPLE No. 8

A triangular channel 3 m wide with an included angle of 90° must carry water with a depth of 3 m. What would be the maximum possible flow rate the mean velocity at this flow rate?

SOLUTION

For maximum flow rate the depth must be the critical depth so $h_c = 3$ m.

The critical velocity is $u_c = \left(\frac{gh_c}{2}\right)^{1/2} = \left(\frac{3g}{2}\right)^{1/2} = 3.836 \text{ m/s}$

$A = h_c^2 \tan(\theta/2) = 3^2 \tan(45) = 9 \text{ m}^2$

The critical flow is $Q_c = A\,u_c = 9 \text{ x } 3.836 = 34.524 \text{ m}^3\text{/s}$

Check the Froude number $F_r = \frac{u_c}{\sqrt{gh_c}} = \frac{3.836}{\sqrt{g \text{ x } 3}} = 0.707$

If the flow must remain at constant depth and n in the manning formula is 0.025, calculate the slope of the bed.

$P = 2h_c/\cos(\theta/2) = 8.485$ $R_h = A/P = 1.061$ $S = \left(\frac{nu_o}{R_h^{2/3}}\right)^2 = \left(\frac{0.025 \text{ x } 3.836}{1.061^{2/3}}\right)^2 = 0.0085$

WORKED EXAMPLE No. 9

A triangular channel 3 m wide with an included angle of 120° must carry 0.75 m³/s with the minimum specific head. What would be the maximum flow rate the mean velocity?

SOLUTION

For minimum specific head, the flow rate and velocity must be the critical values.

$Q_c = \left(\frac{g}{2}\right)^{1/2} \tan(\theta/2) h_c^{5/2}$ rearranging

$$\frac{Q_c^{2/5}}{\left(\left(\frac{g}{2}\right)^{1/2} \tan(\theta/2)\right)^{2/5}} = h_c = \frac{0.75^{2/5}}{\left(\left(\frac{g}{2}\right)^{1/2} \tan(60)\right)^{2/5}} = 0.521 \text{ m}$$

$A = h_c^2 \tan\theta = 0.469 \text{ m}^2$ $u_c = Q_c/A = 1.6 \text{ m}$

SELF ASSESSMENT EXERCISE No.6

1. A uniform channel has a vee cross section with a symmetrical included angle of 100°. If it carries 1.25 m³/s of water with minimum specific energy head, what would be the depth and mean velocity.
(0.742 m and 1.907 m/s)

2. The same channel described in question 1 must carry the flow at a constant depth. If n in the Manning formula is 0.022, what must be the slope of the bed.
(0.00943)

HYDROLOGY - TUTORIAL 2

TRAPEZOIDAL CHANNELS

In this tutorial you will

- Derive equations associated with flow in a trapezoidal channel.
- Derive equations for optimal dimensions.
- Solve slope of bed using Chezy and manning formulae.
- Solve questions from past papers.

This tutorial is a continuation of tutorial 1 which should be studied first.

TRAPEZOIDAL SECTION

This topic occurs regularly in the Engineering Council Exam. The trapezoidal section is widely used in canals to accommodate the shape of boats and reduce the erosion of the sides.

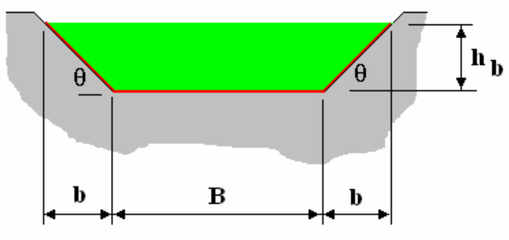


Figure 1

BEST DIMENSION

The channel dimensions that give the maximum flow rate for a fixed cross sectional area is the one with the least amount of friction. This means that it must have the minimum wetted surface area and hence the minimum wetted perimeter P. If this value is then used in any formulae for the flow rate, we will have the maximum discharge possible. Using the notation shown on the diagram we proceed as follows.

Area $A = (B + b)\, h_b$ from which $B = (A/ h_b) - b = (A/ h_b) - h_b/\tan\theta$

Wetted Perimeter $P = B + 2h_b /\sin\theta$

Substitute for B $$P = \frac{A}{h_b} - \frac{h_b}{\tan\theta} + \frac{2h_b}{\sin\theta} = \frac{A}{h_b} + h_b\left(\frac{2}{\sin\theta} - \frac{1}{\tan\theta}\right)$$

For a given cross sectional area the minimum value of P occurs when $dp/dh_b = 0$

$$\frac{dP}{dh_b} = -\frac{A}{h_b^{\,2}} + \left(\frac{2}{\sin\theta} - \frac{1}{\tan\theta}\right) \text{Equate to zero and } A = h_b^{\,2}\left(\frac{2}{\sin\theta} - \frac{1}{\tan\theta}\right) \text{and substitute for A}$$

$$\left(B + \frac{h_b}{\tan\theta}\right)h_b = h_b^{\,2}\left(\frac{2}{\sin\theta} - \frac{1}{\tan\theta}\right) \qquad \left(B + \frac{h_b}{\tan\theta}\right) = h_b\left(\frac{2}{\sin\theta} - \frac{1}{\tan\theta}\right)$$

$$B = 2h_b\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right) \text{ or } B = 2h_b K \text{ where } K = \left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)$$

It can be shown that when this is the case, the bottom and sides are both tangents to a circle of radius h_b.

When $\theta = 90^\circ$ $K = 1$ and when $\theta = 45^\circ$ $K = \sqrt{2} - 1 = 0.414$ and in fact K is almost a linear function such that $K \approx \theta/90$

WORKED EXAMPLE No.1

Calculate the dimensions of a trapezoidal channel with sides at 45° if it must carry 2.5 m^3/s of water with minimum friction given that C = 50 in the Chezy formula and the bed has a gradient of 1 in 1000

SOLUTION

The Chezy formula is $u_o = C\,(R_hS)^{1/2}$ or $Q = A\,C\,(R_hS)^{1/2}$

$$B = 2h_b\left(\frac{1}{\sin 45} - \frac{1}{\tan 45}\right) = 0.828h_b \qquad b = h_b/\tan 45^\circ = h_b$$

$A = (B + b)\, h_b = (0.828\, h_b + h_b)\, h_b = 1.828\, h_b^2$

$$P = B + 2h_b\left(\frac{1}{\sin 45}\right) = 0.828h_b + 2.828h_b = 3.656h_b$$

$R_h = A/P = 0.5\, h_b$

$Q = 2.5 = 1.828\, h_b^2 \times 50(0.5\, h_b/1000)^{1/2}$

$0.000748 = h_b^4\, (0.5\, h_b/1000)$

$0.748 = 0.5\, h_b^5$

$h_b = 1.084$ m $\qquad$ $B = 0.828\, h_b = 0.897$ m

SELF ASSESSMENT EXERCISE No.1

1. Calculate the dimensions of a trapezoidal channel with sides at 60^o to the horizontal if it must carry 4 m^3/s of water with minimum friction given that C = 55 in the Chezy formula and the bed has a gradient of 1 in 1200.
   (h_b = 1.334 m   B = 1.541 m)

2. Calculate the dimensions of a trapezoidal channel with sides at 30^o to the horizontal if it must carry 2 m^3/s of water with minimum friction given that C = 49 in the Chezy formula and the bed has a gradient of 1 in 2000.
   (h_b = 1.053 m   B = 0.564 m)

CRITICAL DEPTH

It requires a lot of Algebra to get to the critical values. Start as before $h_s = h_b + u_o^2/2g$

Rearrange to make u the subject $u_o^2 = \{2g(h_s - h_b)\}$

$Q = Au_o$ $\quad Q^2 = A^2u_o^2$

$A = (B + b)h_b$ $\quad Q^2 = (B + b)^2h_b^2\, u_o^2$

Substitute for u_o $$\frac{Q^2}{2g} = (h_s - h_b)(B + b)^2 h_b{}^2$$

We cannot differentiate this expression because b is a function of h so we make a substitution first.

$$b = h_b/\tan\theta$$

$$\frac{Q^2}{2g} = (h_s - h_b)\left(B + \frac{h_b}{\tan\theta}\right)^2 h_b{}^2 = (h_s - h_b)\left(Bh_b + \frac{h_b^2}{\tan\theta}\right)^2$$ Now we need to multiply out.

$$\frac{Q^2}{2g} = (h_s - h_b)\left(B^2h_b^2 + \frac{h_b^4}{\tan^2\theta} + \frac{2Bh_b^3}{\tan\theta}\right) = \left(B^2h_b^2h_s + \frac{h_b^4h_s}{\tan^2\theta} + \frac{2Bh_b^3h_s}{\tan\theta}\right) - \left(B^2h_b^3 + \frac{h_b^5}{\tan^2\theta} + \frac{2Bh_b^4}{\tan\theta}\right)$$

Now differentiate with respect to h_b to find the maximum flow rate for a given specific energy head.

$$\frac{2QdQ}{2gdh_b} = 2B^2h_bh_s + \frac{4h_b^3h_s}{\tan^2\theta} + \frac{6Bh_b^2h_s}{\tan\theta} - 3B^2h_b^2 - \frac{5h_b^4}{\tan^2\theta} - \frac{8Bh_b^3}{\tan\theta}$$

For maximum Flow rate equate dQ/dh_b to zero.

$$0 = 2B^2h_bh_s + \frac{4h_b^3h_s}{\tan^2\theta} + \frac{6Bh_b^2h_s}{\tan\theta} - 3B^2h_b^2 - \frac{5h_b^4}{\tan^2\theta} - \frac{8Bh_b^3}{\tan\theta}$$

We can simplify by substituting back $h_b/\tan\theta = b$

$$0 = 2B^2h_bh_s + 4b^2h_bh_s + 6Bbh_bh_s - 3B^2h_b^2 - 5b^2h_b^2 - 8Bbh_b^2$$

$$0 = h_b(2B^2h_s + 4b^2hh_s + 6Bbh_s) - h_b^2(3B^2 + 5b^2 + 8Bb)$$

$$0 = (2B^2h_s + 4b^2hh_s + 6Bbh_s) - h_b(3B^2 + 5b^2 + 8Bb)$$

Rearrange to get the critical depth $$h_b = h_c = \frac{(2B^2 + 4b^2 + 6Bb)}{(3B^2 + 5b^2 + 8Bb)}h_s = Ch_s$$

$$C = \frac{(2B^2 + 4b^2 + 6Bb)}{(3B^2 + 5b^2 + 8Bb)} = \frac{(2B + 4b)(B + b)}{(3B + 5b)(B + b)} = \frac{(2B + 4b)}{(3B + 5b)}$$

$h_c = \frac{(2B+4b)}{(3B+5b)}h_s$ or $h_s = \frac{(3B+5b)}{(2B+4b)}h_c$

If $B = 0$ we have a Vee section $h_b = h_c = \frac{4h_s}{5}$ as before.

If $b = 0$ we have a rectangular section $h_b = h_c = \frac{2h_s}{3}$ as before.

There are computer programs for making the calculations such as the one at
http://www.lmnoeng.com/Channels/trapezoid.htm

To find the critical velocity flow rate substitute $h_s = \frac{(3B+5b)}{(2B+4b)}h_c$ into $u_o^2 = u_c^2 = \{2g(h_s - h_c)\}$

$$u_o^2 = u_c^2 = \left\{2g\left(\frac{(3B+5b)}{(2B+4b)}h_c - h_c\right)\right\} = \left\{2gh_c\left(\frac{(3B+5b)}{(2B+4b)} - 1\right)\right\}$$

$$u_c = \sqrt{\left\{2gh_c\left(\frac{(3B+5b)}{(2B+4b)} - 1\right)\right\}} = \sqrt{\left\{2gh_c\left(\frac{(3B+5b)-(2B+4b)}{(2B+4b)}\right)\right\}}$$

$$u_c = \sqrt{\left\{2gh_c\left(\frac{B+b}{(2B+4b)}\right)\right\}}$$

If $B = 0$ we have a Vee section $u_c = \sqrt{\left\{\frac{gh_c}{2}\right\}}$ as before.

If $b = 0$ we have a rectangular section we have $u_c = \sqrt{\{gh_c\}}$ as before.

To find the critical flow rate substitute use $Q_c = A\,u_c$ $\quad A = (B + b)h_c$

$$Q_c = (B+b)h_c\sqrt{\left\{2gh_c\left(\frac{B+b}{(2B+4b)}\right)\right\}} \qquad Q_c = (B+b)h_c^{3/2}\sqrt{\left\{2g\left(\frac{B+b}{(2B+4b)}\right)\right\}}$$

If $B = 0$ we have a Vee section $Q_c = bh_c^{3/2}\sqrt{\left\{\frac{g}{2}\right\}}$ as before in a slightly different form

If $b = 0$ we have a rectangular section we have $Q_c = Bh_c^{3/2}\sqrt{g}$ as before.

Summary for trapezoidal section

The critical depth is $\quad h_c = \frac{(2B+4b)}{(3B+5b)}h_s$

The critical velocity is $\quad u_c = \sqrt{\left\{2gh_c\left(\frac{B+b}{(2B+4b)}\right)\right\}}$

The critical flow is $\quad Q_c = (B+b)h_c^{3/2}\sqrt{\left\{2g\left(\frac{B+b}{(2B+4b)}\right)\right\}}$

The major problem exists that solving with these formulae requires a value for b and this depends on the answer itself.

WORKED EXAMPLE No. 2

A canal has a trapezoidal section with a base 5 m wide and sides inclined at 50° to the horizontal. It is required to have a depth of 2 m, what would the flow rate be if the specific energy head is a minimum? Calculate the depth, flow rate and mean velocity for this condition. What is the Froude Number?

SOLUTION

For minimum specific energy, the flow and depth must be critical so $h_c = 2$ m.
$b = 2/\tan 50^\circ = 1.678$ $\qquad$ $B = 5$

$$Q_c = (B+b)h_c^{3/2}\sqrt{\left\{2g\left(\frac{B+b}{(2B+4b)}\right)\right\}} = (6.678)2^{3/2}\sqrt{\left\{2g\left(\frac{6.678}{16.713}\right)\right\}} = 52.89 \text{ m}^3/\text{s}$$

$A = (B + b)h_c = 6.678 \text{ x } 2 = 13.356 \text{ m}^2$ $\qquad$ $u_c = Q_c/A = 3.96$ m $\qquad$ $Fr = u_c/\sqrt{(gh_c)} = 0.89$

WORKED EXAMPLE No. 3

A channel has a trapezoidal section with a base 0.5 m wide and sides inclined at 45° to the horizontal. It must carry 0.3 m^3/s of water at the critical depth. Calculate the depth and mean velocity.

SOLUTION

There is no simple way to solve this problem because of the complexity of the formula.

$$Q_c = (B+b)h_c^{3/2}\sqrt{\left\{2g\left(\frac{B+b}{(2B+4b)}\right)\right\}} \text{ where } b = h_c/\tan\theta$$

Evaluate and plot Q_c for various values of h_c and we get the following graphs.

From the graph we see that when $Q_c = 0.3$ m^3/s, $h_c = 0.27$
$A = (0.5 + 0.275)(0.275) = 0.213 \text{ m}^2$ $\qquad$ $u_c = Q_c/A = 1.$

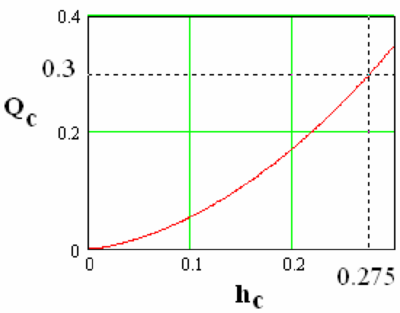


Figure 2

SELF ASSESSMENT EXERCISE No.2

1. A channel has a trapezoidal section with a base 2 m wide and sides inclined at 60° to the horizontal. It must carry 0.4 m^3/s of water with the minimum specific energy head. Calculate the depth and mean velocity for this condition.
(0.157 m and 1.22 m/s)

2. A canal has a trapezoidal section with a base 4 m wide and sides inclined at 40° to the horizontal. It is required to have a depth of 1.5 m, what would the flow rate be if the specific energy head is a minimum? Calculate the flow rate and mean velocity for this condition.
(29.1 m^3/s and 3.353 m/s)

WORKED EXAMPLE No. 4

An open channel has a trapezoidal cross section with sides inclined at 45^o to the vertical. The channel must carry 21 m^3/s with a velocity of 3 m/s with minimum friction. Determine the smallest slope of the bed for these conditions and the corresponding depth and dimensions of the channel. The constant n in the Manning formula is 0.012. Show that this is a sub critical flow.

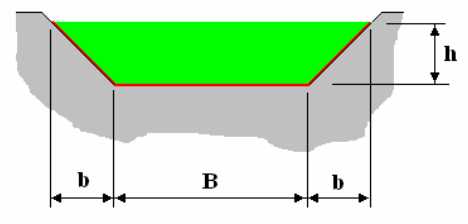


Figure 3

SOLUTION

$Q = 21\ m^3/s$ $\qquad u_o = 3\ m/s$ $\qquad A = Q/u = 7\ m^2$

For minimum friction the optimal value of B is $B = 2h_b\left(\frac{1}{\sin\theta} - \frac{1}{\tan\theta}\right)$

$$B = 2h_b\left(\frac{1}{\sin 45} - \frac{1}{\tan 45}\right) = 0.8284h_b \quad b = h_b$$

$A = (B + b)h_b$ $\qquad 7 = (0.8284h_b + h_b)h_b = 1.8284h_b^2$

$h_b = \sqrt{(7/1.8284)} = 1.957$ m

B = 1.621 m $\qquad$ b = 1.957 m

P = B + 2b/sin 45 = 1.621 + 2 x 1.957/sin 45 = 7.155

A = 7

R_h = 7/7.155 = 0.978 m (Note that for 45^o $R_h = 0.5\ h_b$)

Manning formula $\qquad u = \frac{R^{2/3}S^{1/2}}{n}$ $\qquad 3 = \frac{0.978^{2/3}S^{1/2}}{0.012}$

S = 0.001333

The specific energy head is $h_s = 1.957 + 3^2/2g = 2.416$ m

The critical depth is $\qquad h_c = \frac{(2B + 4b)}{(3B + 5b)}h_s = \frac{2 \text{ x } 1.621 + 4 \text{ x } 1.957}{3 \text{ x } 1.621 + 5 \text{ x } 1.957} = \frac{11.07}{14.648} = 0.756 \text{ m}$

Since the actual depth is larger the flow is sub critical.

SELF ASSESSMENT EXERCISE No. 3

These are exam standard questions.

1. An open channel has a trapezoidal section with sides inclined at 45^o to the vertical. The channel must carry 20 m^3/s of water with a mean velocity of 2.5 m/s. Determine the smallest slope of the bed possible and the corresponding depth and dimensions of the channel. The constant n in the Manning formula is 0.012. Show that this is a sub critical flow.

   $u = (R^{2/3}S^{1/2})/n$

   (Answer S = 0.000845, h = 2.1, B = 2.1 m and b = 2.1 m.)

2. A channel has a trapezoidal section 5 m wide at the bottom. The sides slope at 1 metre up for each 2 horizontal. The bed has a slope of 1/3600 and n in the manning formula is 0.024.

   Calculate the flow rates corresponding to mean velocities of 0.3 and 0.6 m/s.

   (Ans. 0.549 m^3/s and 4.81 m^3/s)

HYDROLOGY - TUTORIAL 4

UNSTEADY FLOW IN CHANNELS

In this tutorial you will

- Derive equations associated with a rise in the level of the bed.
- Define a hydraulic jump and derive the equations for it.
- Define a narrow weir and derive equations for it.
- Define a broad weir and derive equations for it.
- Derive equations for the flow rate through a Venturi Flume.
- Solve questions from past papers.

This tutorial is a continuation of tutorial 1 and 2 and these should be studied first.

UNSTEADY FLOW

If the depth of the water is not constant, we have unsteady flow. This might occur when the frictional losses do not match the change in potential energy. In this case the hydraulic gradient 'i' is not the same as the slope 'S'. The height of the bed relative to the datum level is z.

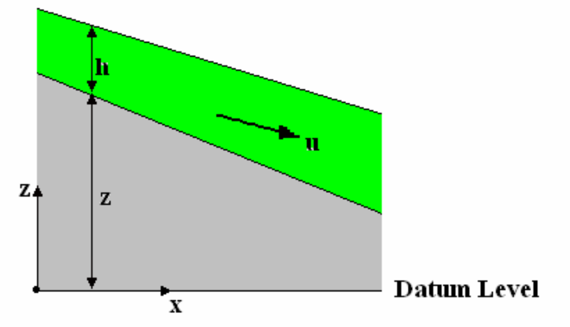


Figure 1

The total head is defined as h_s plus the additional potential head z

$$h_T = h_S + z = h + z + u^2/2g$$

Start with $\quad h_S = h_T - z$

Differentiate with respect to x (the distance along the channel from a given datum point).

$$dh_S/dx = dh_T/dx - dz/dx$$

dz/dx is the gradient of the bed and this is clearly negative so dz/dx = - S

$$dh_S/dx = dh_T/dx + S$$

The change in the total head can only be due to frictional losses and this will be a reduction so we can define this as the hydraulic gradient $i = dh_f/dx = - dh_T/dx$

$$dh_S/dx = S - i$$

We had $\quad h_S = h + u^2/2g$

Differentiate this with respect to h $\quad dh_S/dh = 1 + (u/g)du/dh$

u = Q/A.

Differentiate with respect to A $\quad du/dA = -Q/A^2$

If A is a function of depth, then this is difficult. For a rectangular channel A = B h

Differentiate with respect to h $\quad dA/dh = B$

Substitute $\quad du/dh = (-Q/A^2)\,B$

Substitute into $dh_S/dh = 1 + (u/g)du/dh$

$$dh_S/dh = 1 - (u/g)\,(Q/A^2)\,B$$

$$dh_S/dh = 1 - (u/g)\,(Q/A^2)\,B$$

Substitute Q/A = u $\quad dh_S/dh = 1 - (u^2/g)\,B/A$

Substitute B/A = 1/h $\quad dh_S/dh = 1 - (u^2/gh) = 1 - F_r^2$

For maximum or minimum specific head $dh_S/dh = 0$ and this can only occur if $F_r = 1$

Flow is at the critical depth when $\mathbf{F_r = 1}$

The velocity that produces the critical depth is $\mathbf{u_c = \sqrt{(g\,h)}}$

Note that it has been assumed that h is constant at all widths so the Froude number is only 1 when the channel is rectangular in section.

HYDRAULIC JUMP

It has already been shown that for a given flow rate in an open channel, there are two possible depths. One is when the flow is slow and deep called TRANQUIL FLOW and the other when it is shallow and fast called RAPID FLOW or SHOOTING FLOW.

It is possible rapid flow to change to tranquil flow quite suddenly and spontaneously and when it does we get a phenomenon called a hydraulic jump. It is not possible for the reverse to happen.

For a hydraulic jump to occur, the $F_r > 1$, i.e. the flow must be supercritical.

The jump might occur because the slope of the bed is insufficient for friction to balance the loss of potential energy. Since the losses are smaller for the tranquil flow, the balance can be restored.

A jump can be made to occur if there is an obstacle on the bed higher than the critical depth.

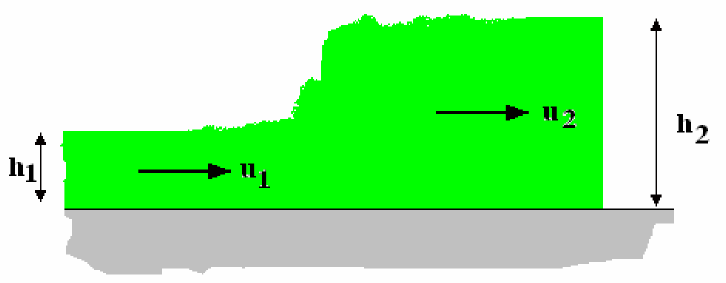


Figure 2

When the change occurs there is a reduction in momentum and an increase in the hydrostatic force. The solution is based on equating them.

u_o = mean velocity. The mean depth is h/2

Pressure force on a cross section is $F_p = \rho gAh/2$ Momentum force at a section $=F_m = \rho Au^2$

The cross sectional area is A = B h where B is the width.

$$\text{Change in pressure force} = \frac{\rho g A_1 h_1}{2} - \frac{\rho g A_2 h_2}{2} = \frac{\rho g B h_1^2}{2} - \frac{\rho g B_2 h_2^2}{2}$$

$$\text{Change in pressure force} = \frac{\rho g B}{2}\left(h_1^2 - h_2^2\right) = \frac{\rho g B}{2}(h_1 - h_2)(h_1 + h_2)$$

Change in momentum force = $\rho B h_1 u_1^2 - \rho B h_2 u_2^2 = \rho B(h_1 u_1^2 - h_2 u_2^2)$

For continuity of flow $u_2^2 = (u_1 h_1/h_2)^2$

$$\text{Change in momentum force} = \rho B\left(h_1 u_1^2 - \frac{h_2 h_1^2}{h_2^2}u_1^2\right) = \rho B u_1^2 \frac{h_1}{h_2}(h_2 - h_1)$$

The change in pressure and momentum forces may be equated.

$$\frac{\rho g B}{2}(h_1 - h_2)(h_1 + h_2) = \rho B u_1^2 \frac{h_1}{h_2}(h_2 - h_1)$$

$$\frac{g}{2}(h_1 + h_2) = u_1^2 \frac{h_1}{h_2}$$

$$u_1^2 = (h_1 + h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2} \quad \text{................(1)}$$

In terms of the flow rate

$$Q^2 = B^2 h_1^2 (h_1 + h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2} = B^2 h_1 h_2 (h_1 + h_2)\frac{g}{2}$$

The flow per unit width is usually given as $q^2 = g h_1 h_2 \frac{(h_1 + h_2)}{2}$ ....(2)

From (1) $\frac{2h_1u_1^2}{g} = \left(h_1h_2 + h_2^2\right)$ $\qquad h_2^2 + h_1h_2 - h_1\frac{2u_1^2}{g} = 0$

h_2 may be solved with the quadratic equation giving:

$2h_2 = -h_1 \pm h_1\sqrt{\left\{1+\frac{8u_1^2}{gh_1}\right\}}$ and since h_2 cannot be negative $2h_2 = -h_1 + h_1\sqrt{\left\{1+\frac{8u_1^2}{gh_1}\right\}}$

Substitute the Froude Number $F_r^2 = \frac{u_1^2}{gh_1}$ $\qquad 2h_2 = -h_1 + h_1\sqrt{\left\{1+8F_r^2\right\}}$

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1+8F_r^2\right\}} - 1\right]$$

Next consider the energy balance before and after the jump.

Energy Head before the jump = $h_1 + u_1^2/2g$ Energy Head after the jump = $h_2 + u_2^2/2g$

Head loss = $h_L = h_1 + u_1^2/2g - h_2 - u_2^2/2g$ $\qquad h_L = h_1 - h_2 + (u_1^2 - u_2^2)/2g$

Continuity of flow $Q = u_1\ B\ h_1 = u_2\ B\ h_2$ $\qquad u_2 = u_1\frac{h_1}{h_2}$

Hence $u_1^2 - u_2^2 = u_1^2 - u_1^2\left(\frac{h_1}{h_2}\right)^2 = u_1^2\left\{1-\left(\frac{h_1}{h_2}\right)^2\right\}$

We already found equation (1) was $u_1^2 = (h_1+h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2}$

Substitute

$$u_1^2 - u_2^2 = (h_1+h_2)\left(\frac{h_2}{h_1}\right)\frac{g}{2}\left\{1-\left(\frac{h_1}{h_2}\right)^2\right\}$$

$$u_1^2 - u_2^2 = \frac{g}{2}(h_1+h_2)\left(\frac{h_2}{h_1}\right)\left\{\frac{h_2^2-h_1^2}{h_2^2}\right\}$$

$$\left(u_1^2 - u_2^2\right) = \frac{g\left\{(h_1+h_2)\left(h_2^2-h_1^2\right)\right\}}{2h_1h_2}$$

Substitute into the formula for h_L

$h_L = h_1 - h_2 + (u_1^2 - u_2^2)/2g$

$$h_L = (h_1-h_2) + \frac{g(h_1+h_2)\left(h_2^2-h_1^2\right)}{2 \text{ x } 2gh_1h_2} = (h_1-h_2) + \frac{(h_1+h_2)\left(h_2^2-h_1^2\right)}{4h_1h_2}$$

$$h_L = \frac{4h_1h_2(h_1-h_2) + \left\{(h_1+h_2)\left(h_2^2-h_1^2\right)\right\}}{4h_1h_2}$$

$$h_L = \frac{4h_1^2h_2 - 4h_1h_2^2 + h_2^3 - h_1^3 + h_1h_2^2 - h_1^2h_2}{4h_1h_2}$$

$$h_L = \frac{3h_1^2h_2 - 3h_1h_2^2 + h_2^3 - h_1^3}{4h_1h_2}$$

$$h_L = \frac{(h_2 - h_1)^3}{4h_1h_2}$$

Useful website http://www.lmnoeng.com/Channels/HydraulicJump.htm

WORKED EXAMPLE No.1

50 m^3/s of water flows in a rectangular channel 8 m wide with a depth of 0.5 m. Show that a hydraulic jump is likely to occur.

Calculate the depth after the jump and the energy loss per second.

SOLUTION

$A_1 = 8 \times 0.5 = 4\ m^2$ $u_1 = Q/A_1 = 50/4 = 12.5$ m/s
Froude Number $F_{r1} = u/\sqrt{gh} = = 12.5/\sqrt{(9.81 \times 0.5)} = 31.855$ This is supercritical so a jump is possible.

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1+8F_r^2\right\}}-1\right] = \frac{0.5}{2}\sqrt{1+8 \times 31.855^2} = 3.748\ \text{m}$$

$A_2 = 8 \times 3.748 = 30\ m^2$ $u_2 = Q/A_2 = 50/30 = 1.667$ m/s

$$h_L = \frac{(h_2 - h_1)^3}{4h_1h_2} = \frac{(3.748 - 0.5)^3}{4 \times 0.5 \times 3.748} = 4.573\ \text{m}$$

Energy loss $= mgh_L = 50\,000 \times 9.81 \times 4.573 = 2.243$ MJ/s
m = 50 000 kg/s

SELF ASSESSMENT EXERCISE No. 1

1. Show by applying Newton's Laws that when a hydraulic jump occurs in a rectangular channel the depth after the jump is

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1+8F_r^2\right\}}-1\right]$$

Go on to show that the head loss is $h_L = \dfrac{(h_2 - h_1)^3}{4h_1h_2}$

2. 40 m^3/s of water flows in a rectangular channel 10 m wide with a depth of 1.0 m. Show that a hydraulic jump is likely to occur. Calculate the depth after the jump and the energy loss per second.
(Answers 1.374 m and 3.735 kJ/s)

3. Water has a depth H = 1.5 m behind a sluice gate and emerges from the gate with a depth of 0.4 m. Downstream a hydraulic jump occurs. Calculate depth after the jump and the mean velocity before and after the jump. (Note use Bernoulli to find u_1)

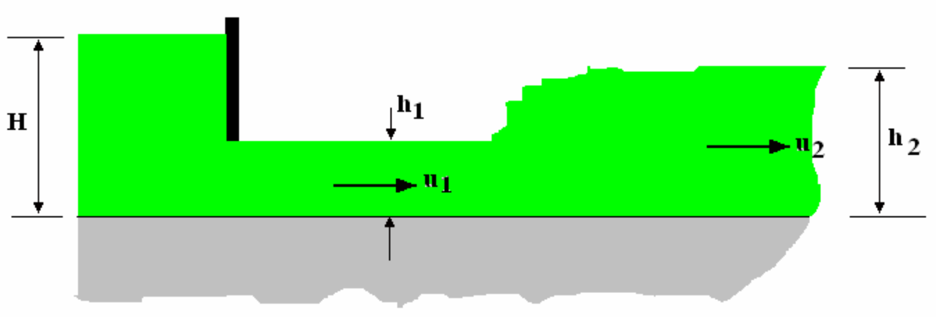


Figure 3

(Answers 1.416 and 1.627 m/s)

RISE IN LEVEL OF BED

In this section we will examine what happens to the level of water flowing in a channel when there is a sudden ride in the level of the bed.

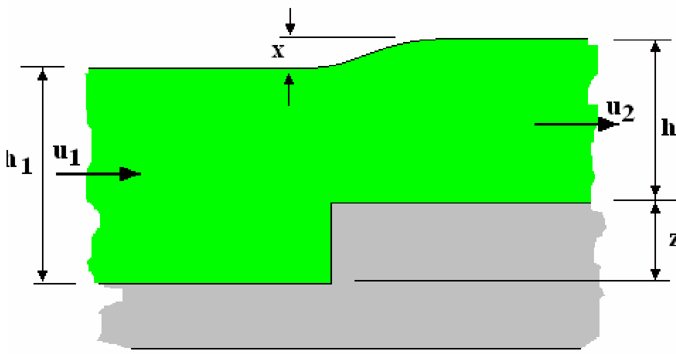


Figure 4

First apply Bernoulli between points (1) and (2)

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g} + z$$

From the continuity equation substitute $u_2 = \frac{u_1 h_1}{h_2}$

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{\left(\frac{u_1 h_1}{h_2}\right)^2}{2g} + z$$

Rearrange $$h_1 + \frac{u_1^2}{2g} - h_2 - \frac{u_1^2 h_1^2}{2g h_2^2} - z = 0$$

$$\left(h_1 + \frac{u_1^2}{2g} - h_2 - z\right) h_2^2 - \frac{u_1^2 h_1^2}{2g} = 0$$

Substitute $h_2 = x + h_1 - z$

$$\left\{h_1 + \frac{u_1^2}{2g} - (x + h_1 - z) - z\right\}(x + h_1 - z)^2 - \frac{u_1^2\{(x + h_1 - z)^2\}}{2g} = 0$$

$$\left\{\frac{u_1^2}{2g} - x\right\}(x + h_1 - z)^2 - \frac{u_1^2\{(x + h_1 - z)^2\}}{2g} = 0$$

$$\{u_1^2 - 2gx\}(x + h_1 - z)^2 - u_1^2(x + h_1 - z)^2 = 0$$

$$\{u_1^2 - 2gx\}(x + h_1 - z)^2 - u_1^2(x + h_1 - z)^2 = 0$$

There follows a long bout of more algebra to produce a cubic equation for x :

$$x^3 + 2x^2\left(h_1 - z - \frac{u_1^2}{4g}\right) + x(h_1 - z)\left(h_1 - z - \frac{u_1^2}{g}\right) + z\frac{u_1^2}{2g}(2h_1 - z) = 0$$

This equation may be used to solve the change in height of the water. If the values of x and z are small, we may neglect products and higher powers of small numbers so the equation simplifies to:

$$x\left(h_1 - \frac{u_1^2}{g}\right) + \frac{z u_1^2}{g} = 0$$

$$x\left(1 - \frac{u_1^2}{g h_1}\right) + \frac{z u_1^2}{g h_1} = 0$$

The Froude number approaching the change is $F_{r1} = \frac{u_1}{\sqrt{gh_1}}$ hence

$$x\left(1 - F_{r1}^{\ 2}\right) + zF_{r1}^{\ 2} = 0$$

$$x = \frac{zF_{r1}^{\ 2}}{\left(F_{r1}^{\ 2} - h_1\right)}$$

The equation indicates that if flow is supercritical ($F_{r1}>1$) then x is positive and the surface rises. If the flow is sub critical (tranquil $F_{r1}<1$) then x is negative and the surface is depressed.

WORKED EXAMPLE No. 2

Water flows in a rectangular channel with a depth of 0.55 m and a mean velocity of 4.5 m/s. Downstream there is a rise in the level of the bed of 0.075 m. Determine the depth and mean velocity after the rise. Is Bernoulli's equation is satisfied?

SOLUTION

$z = 0.075$ m $h_1 = 0.55$ m $\quad u_1 = 4.5$ m/s $\quad F_{r1} = \frac{u_1}{\sqrt{gh_1}} = \frac{4.5}{\sqrt{9.81 \mathrm{x} 0.55}} = 1.937$

The flow is supercritical (Rapid) $x = \frac{0.075\,(1.937)^2}{\left(1.937^2 - 0.55\right)} = 0.102\mathrm{m}$

Depth = 0.55 + 0.102 = 0.577 m

$$u_2 = \frac{u_1 h_1}{h_2} = \frac{4.5 \mathrm{\ x\ } 0.55}{0.577} = 4.289\mathrm{m/s}$$

Energy head before rise is $h_1 + \frac{u_1^2}{2g} = 0.55 + \frac{4.5^2}{2g} = 1.582$

Energy Head after the rise is $h_2 + \frac{u_2^2}{2g} + z = 0.577 + \frac{4.289^2}{2g} + 0.102 = 1.616$

There is a small discrepancy.

SELF ASSESSMENT EXERCISE No. 2

1. Water flows in a rectangular channel with a depth of 1.0 m. Downstream there is a rise in the level of the bed of 0.1 m. Determine the mean velocity after the rise and the critical depth upstream if the depth after the rise is :

(a) 1.1 m (Answers 3.13 and 1 m)
(b) 0.8 m (Answer 2.215 m/s and 0.833 m)

WEIRS

We shall consider two forms of weirs, a narrow one with a sharp edge and a broad one with a rounded edge.

NARROW WEIRS

The flow over a sharp edge weir is in essence the same as flow through a rectangular notch. Consider the flow from section (1) to section (2).

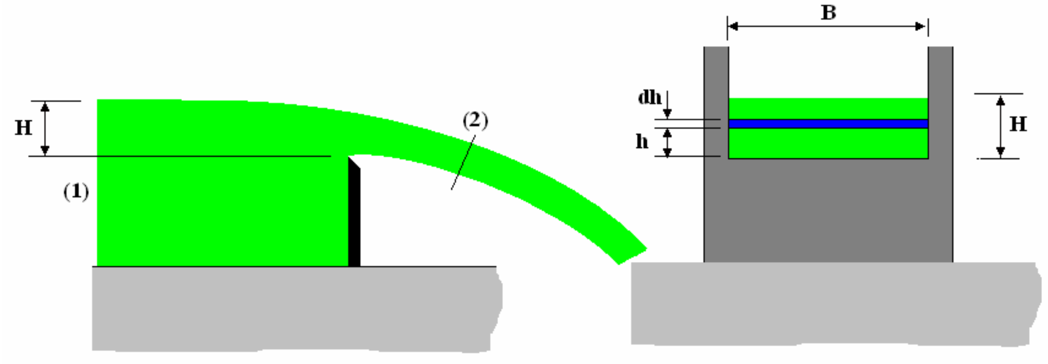


Figure 5

We shall only consider the depth upstream relative to the top of the weir. The mean velocity at a given point is u. The pressure head at (2) is atmospheric Applying Bernoulli between (1) and (2) we have:

$$h_1 + \frac{u_1^2}{2g} = h_2 + \frac{u_2^2}{2g}$$

The velocity upstream is usually small so we can neglect u_1 and if we use gauge pressure then $h_2 = 0$

$$h_1 = \frac{u_2^2}{2g} \text{ and } u_2 = \sqrt{2gh_1}$$

Next consider a thin horizontal strip at distance h from the bottom of the weir and height dh. The volume flow through it is $dQ = u_2\, Bdh$

$$dQ = \sqrt{2gh_1}\,Bdh$$

$$Q = \sqrt{2g}B\int_0^H h_1^{1/2}dh$$

$$Q = \frac{2}{3}\sqrt{2g}BH^{2/3}$$

It is normal to introduce a coefficient of discharge to correct for losses.

$$Q = \frac{2}{3}C_d\sqrt{2g}BH^{2/3}$$

BROAD CRESTED WEIR

Earlier we examined what happens when the bed of the channel suddenly rises. In the case of the broad crested weir, the level falls as it passes from the weir and it can be proved that at some point on the weir, the flow becomes critical. At this point $h_2 = h_c$.

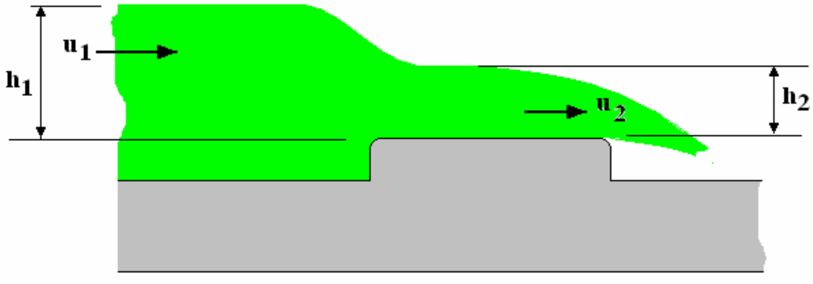


Figure 6

It was shown earlier that $h_c = \frac{2h_s}{3}$ where h_s is the specific total energy head.

It was also shown that the critical flow rate is

$$Q_c = (B+b)g^{1/2}\left(\frac{8h_s^3}{27}\right)^{1/2}$$

For a rectangular section this becomes

$$Q_c = B\left(\frac{8gh_s^3}{27}\right)^{1/2} = 1.705h_s^{3/2}$$

This gives the flow rate over the weir. Usually a coefficient of discharge is used to correct for losses.

$$Q_c = 1.705\,C_d\,h_s^{3/2}$$

If the approach velocity u_1 is negligible then $h_s = h_1$ and makes it easy to solve Q.

WORKED EXAMPLE No. 3

A rectangular channel takes the flow from the foot of a steep spillway with a flow of 10 m^3/s per metre of width. The flow in the channel approaches a broad crested weir with a Froude number of 3. Calculate the following.

i) The mean velocity in the channel.
ii) The minimum height of the weir which will cause a hydraulic jump to occur in the channel.

SOLUTION

The diagram illustrates the problem

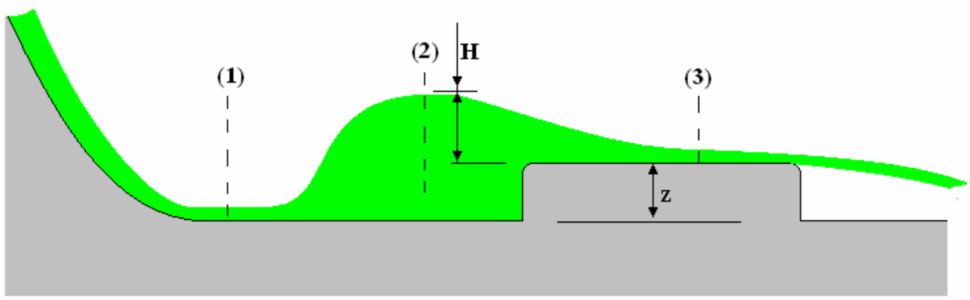


Figure 7

CHANNEL – At section (1) $F_r^2 = 9 = u_1^2/gh_1$ $h_1 = q/u_1$

Combine and $9 = u_1^3/10g$ hence $u_1 = 9.593$ m/s $h_1 = 10/9.593 = 1.042$ m

JUMP

$$h_2 = \frac{h_1}{2}\left[\sqrt{\left\{1 + 8F_r^2\right\}} - 1\right] = 3.932\text{m}$$

WEIR – At section (3) the flow is assumed critical so $F_r = 1$

$$F_r^2 = \frac{u_3^2}{gh_3} = 1 \text{ and } u_3^2 = gh_3$$

$$u_3 = \frac{q}{h_3} \text{ or } h_3 = \frac{q}{u_3} \text{ substitute and } u_3^3 = gq \quad \text{hence } u_3 = 4.612 \text{ m/s}$$

$h_3 = q/u_3 = 2.168$ m

Bernoulli between (2) and (3)

$$\frac{u_2^2}{2g} + h_2 = \frac{u_3^2}{2g} + h_3 + z \quad \text{hence } z = 1.009$$

$$\text{Check } F_r = \frac{u_3}{\sqrt{gh_3}} = 1$$

SELF ASSESSMENT EXERCISE No. 3

1. A rectangular channel takes the flow from the foot of a spillway with a flow of 20 m^3/s per unit width. The flow in the channel approaches a broad crested weir with a Froude number of 2.24.

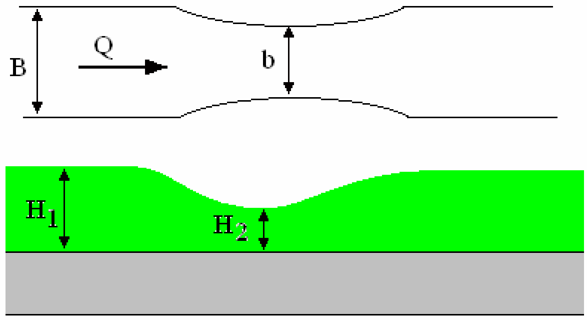
   Calculate the minimum height of the weir to produce a hydraulic jump in the channel.

   (Answer 0.968 m)

2. A wide rectangular spillway has a flow of water of 12 m^3/s per unit width. A broad weir in the path causes a hydraulic jump to occur. The Froude number approaching the jump is 2.5.

   Calculate the minimum height of the weir assuming the flow is critical at some point over it. (Answer 0.85 m)

2. VENTURI FLUME



A venturi flume is a flume that narrows to the throat and then widens back out again. The reduction in width causes a change in velocity and hence height. In a Venturi meter the change is reflected as a change in static pressure but in a flume it is height.

If energy is conserved the total energy head at inlet and at the throat are the same so from Bernoulli we have:

$$H_1 + \frac{u_1^2}{2g} = H_2 + \frac{u_2^2}{2g}$$

The flow rate is $Q = Au = B_1 H_1 u_1 = B_2 H_2 u_2$ Hence $u_1 = u_2 \frac{B_2 H_2}{B_1 H_1}$

$$H_1 + \left(u_2 \frac{B_2 H_2}{B_1 H_1}\right)^2 \frac{1}{2g} = H_2 + \frac{u_2^2}{2g}$$

$$H_1 - H_2 = + \frac{u_2^2}{2g}\left\{1 - \left(\frac{B_2 H_2}{B_1 H_1}\right)^2\right\} \qquad u_2 = \sqrt{\frac{2g(H_1 - H_2)}{1 - \left(\frac{B_2 H_2}{B_1 H_1}\right)^2}}$$

$$Q = B_2\, H_2 \sqrt{\frac{2g(H_1 - H_2)}{\left[1 - \frac{b^2\, H_2^2}{B^2\, H_1^2}\right]}}$$ Allowing for energy losses $$Q = C_d B_2\, H_2 \sqrt{\frac{2g(H_1 - H_2)}{\left[1 - \frac{b^2\, H_2^2}{B^2\, H_1^2}\right]}}$$

If the flow rate is a maximum, the depth at the throat will be the critical depth and a hydraulic jump will form downstream of the throat. In this case

$$H_2 = \frac{2}{3} h_s \text{ and } h_s = H_2 + \frac{u_2^2}{2g} = \frac{2h_s}{3} + \frac{u_2^2}{2g} \quad \text{Hence } u_2 = \sqrt{\frac{2gh_s}{3}}$$

$$Q = B_2 H_2 u_2 = B_2 H_2 \sqrt{\frac{2gh_s}{3}} = \frac{2B_2 hs}{3}\sqrt{\frac{2gh_s}{3}}$$

Introducing the coefficient of discharge $Q = 1.705 C_d\, h_s^{2/3}$

This is the same as for Broad Crested weir. Solving Q with this formula is not straight forward because h_s contains the velocity term. The examination often asks for the derivation of this formula.

WORKED EXAMPLE No.4

A rectangular channel is 1.2 m wide and narrows to 0.6 m wide in a venturi flume. The depth at the entrance and throat are 0.6 m and 0.55 m respectively. Calculate the flow rate given Cd = 0.88.

SOLUTION

$$Q = C_d B_2\, H_2 \sqrt{\frac{2g(H_1 - H_2)}{\left[1 - \frac{B_2^2\, H_2^2}{B_1^2\, H_1^2}\right]}} = 0.88 \text{ x } 0.6 \text{ x } 0.55 \sqrt{\frac{2g(0.05)}{\left[1 - \frac{0.6 \text{ x } 0.55}{1.2 \text{ x } 0.6}\right]^2}} = 0.324 \text{ m}^3/3$$

SELF ASSESSMENT EXERCISE No.4

Show that for critical flow with a Froude Number of 1 in a rectangular channel, the depth of flow yc is related to the specific energy head H by the expression $H = 3y_c/2$

Describe with sketches a broad crested weir and a venturi flume.

Show that for both structures the flow rate is related to the critical depth by the relationship

$$Q = 1.705C_d\ H^{3/2}$$

Note the symbols are not the same as in the notes and are the ones used typically in the EC exam.

ENGINEERING COUNCIL EXAM

HYDROLOGY D204

TUTORIAL 13 - PIPE NETWORKS

In this tutorial you will

- Revise pipe friction equations.
- Derive iterative balance equations for nodes and loops.
- Solve problems about networks with a common junction.
- Solve problems about networks with connected loops.

Students are advised to complete tutorial 1 and study the flow in pipes before doing this tutorial.

BRIEF REVISION OF PIPE FLOW

PIPE FRICTION

When a fluid flows in a pipe the friction head is defined by the Darcy equation $h_f = \dfrac{4C_f L u_m^2}{2gD}$

h_f = friction head
u_m = mean velocity = Q/A
D = pipe bore
L = pipe length
R_e = Reynolds number = $\rho u_m D/\mu$
R = Hydraulic resistance

The friction coefficient C_f is dependent on the surface roughness and degree of turbulence and there are many theories about its relationship. The most common method for finding C_f is from the Moody Chart or from one of many developed formulae such as :

BLASIUS $\quad C_f = 0.0791\ R_e^{0.25}$

LEE $\quad C_f = 0.0018 + 0.152\ R_e^{0.35}$.

HAALAND $\quad \dfrac{1}{\sqrt{C_f}} = -3.6\log_{10}\left\{\dfrac{6.9}{R_e} + \left(\dfrac{\varepsilon}{3.71}\right)^{1.11}\right\}$

For laminar flow n = 1 and $C_f = 16/R_e$

Basically we are saying $h_f \propto u_m^n$ and n is only 2 for complete turbulent flow.

We must remember that C_f depends on the Reynolds number. In this case we would need to think about recalculating C_f every time we change the flow.

In terms of flow rate Q $\quad h_f = R\,Q^n$

R (or often K) is the fluid resistance. When n = 2 $\quad R = \dfrac{32C_f L}{g\pi^2 D^5}$ and when n is not 2, C_f and hence R changes with the flow rate. The units of R are s^2/m^5.

MINOR LOSSES

Minor losses occur at Entry and exit from a reservoir, at sudden changes in sections and sharp bends. In general these are small compared to pipe friction when the pipes are long and are neglected for short connections they are important. In terms of pressure head the losses are usually formulated as:

$h_L = RQ^2$ where the resistance is given by $R = \dfrac{8k}{\pi^2 D^4}$ where k is a factor that depends on the pipe sizes and can be found in literature.

The Moody chart and other details concerning pipe losses can be found in tutorial 1.

GUESSING and CORRECTING

Starting with $h_f = R\,Q^n$ differentiate to get $dh_f = nRQ^{n-1}dQ = \dfrac{nRQ^n dQ}{Q}$ and since $R\,Q^n = h_f$

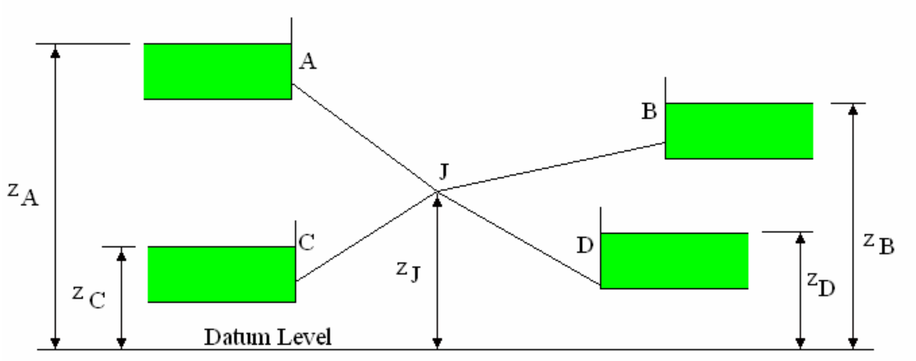
$$dh_f = \frac{nh_f dQ}{Q} \text{ or } dQ = \frac{Qdh_f}{nh_f}$$

If this relationship holds approximately true for finite changes then $\delta h_f = \dfrac{nh_f\delta Q}{Q}$ or $\delta Q = \dfrac{Q\,\delta h_f}{nh_f}$

These equations are used to make corrections in the guessing game that follows.

NETWORKS WITH A COMMON JUNCTION

The diagram shows a typical example with four reservoirs A, B, C and D connected to a common junction J. The problem is to find the pressure head h at the junction.



For a system such as that illustrated, suppose that we need to find the flow in each pipe but we don't know the pressure at the junction.

We could do four simultaneous equations in order to find the flow in each pipe but these days with fast computational methods it is relatively easier to guess at values and make corrections.

Applying Bernoulli between the free surface of any reservoir and the junction gives:

$$h + z + u^2/2g = h_J + z_J + u_J^2/2g + h_f$$

At the free surface $h = 0$ and $u = 0$ (This is an assumption normally made).

$$z = h_J + z_j + u_J^2/2g + h_f \qquad h_f = z - h_J - z_j - u_J^2/2g$$

Many sources of information ignore the velocity term and state $h_f = z - z_j - h_J$ and this will be so here. For a given pipe we calculate the hydraulic resistance R (or K) and get the form $h_f = R\,Q^n$

Hence for any pipe $h_f = R\,Q^n = z - z_J - h_J \qquad Q = \{(z - z_J - h_J)^{1/n}/R\}$

Suppose we guess at the value of h_J (or more likely $h_J + z_J$). With a suitable programme such as Excel™ it is easy to guess the head and keep changing it until $\Sigma Q = 0$.

The point is that at the junction the total or net flow rate must be zero so each time we guess we add up the total flow until $\Sigma Q = 0$. If it is not very close to zero then we guess again.

A more systematic method of arriving at the correct answer is to make a correction after each guess based on the error formula derived previously.

Using the guessed value of head we work out Q for each pipe. Any error in the guess will produce an error in the flow of $\delta Q = \dfrac{Q\,\delta h_f}{nh_f}$ and the correction to the head must be $\delta h_f = \dfrac{nh_f\delta Q}{Q}$ for each pipe.

For four pipes A, B, C and D the total error is:

$$\sum \delta h_f = \left(\frac{nh_f\ \delta Q}{Q}\right)_A + \left(\frac{nh_f\ \delta Q}{Q}\right)_B + \left(\frac{nh_f\ \delta Q}{Q}\right)_C + \left(\frac{nh_f\ \delta Q}{Q}\right)_D \qquad \sum \delta h_f = \frac{n\ \delta Q}{\sum Q/h_f}$$

This method is called the nodal balance method.

WORKED EXAMPLE No.1

Based on the diagram previous, the following data applies.
$z_A = 143$ m $z_B = 134$ m $z_C = 120$ m $z_D = 100$ m $z_J = 80$

$D_A = 0.4$ m $D_B = 0.3$ m $D_C = 0.5$ m $D_D = 0.4$ m

$L_A = 4200$ m $L_B = 1200$ m $L_C = 1250$ m $L_D = 1200$ m

$C_f = 0.005$ for all pipes. The constant n is 2.

Find the pressure head at the junction by guessing and hence the flow rate in or out of each reservoir.

SOLUTION

Calculate R for each pipe. $R = \frac{32C_fL}{g\pi^2D^5} = 0.001653\frac{L}{D^5}$

$R_A = 677.8$
$R_B = 816$
$R_C = 66.1$
$R_D = 193.7$

The following solution was done on a spread sheet. Remember that on the spread sheet the values in the table will automatically change when you change h_J in the programme. Adjusting h_J until $\Sigma Q \approx 0$ gives the following.

PIPE	R	$z - z_J$	$h_f = z - z_J - h_J$	$Q=\sqrt{h_f/R}$	$\Delta Q/\Delta h_f$	Guess $h_J = 39.998$
A	677.8	63	23.002	0.18422	0.00801	
B	816	54	14.002	0.13099	0.00936	
C	66.1	40	0.002	0.0055	2.75033	
D	193.7	20	-19.998	-0.3213	0.01607	
				ΣQ =-0.0006	Σ =2.78376	

Note that if Q is minus it is implied that h_f is minus but this will cause a problem in the calculations so use $Q = \sqrt{\text{Modulus}(h_f/R)}$

In Excel the formula would be entered =SQRT(ABS(cell1/cell2))*(cell3/ABS(cell3)) where cell 1 and cell2 are the cells containing h_f and R and cell3 contains h_f.

The final values of flow rate are
$Q_A = 0.18422\ m^3/s$
$Q_B = 0.13099\ m^3/s$
$Q_C = 0.13099\ m^3/s$
$Q_D = -0.3213\ m^3/s$

In an examination you may not have access to a suitable programmable calculator so you would have to keep repeating all the calculations with an adjustment to the value of h_J each time.

WORKED EXAMPLE No.2

Repeat the last worked example but use iteration to arrive at the answer.

The correction to be made after each iteration is $\sum h_f = \dfrac{n\,\delta Q}{\sum Q/h_f}$

With this formula you can calculate the adjustment each time. With no other data, a good idea for the starting value is the mean height of the reservoirs. Also note that the height of the junction is not normally given but if we guess at $h_J + z_J$ we get the same result but do not know the static head at J. Also note that K is commonly used instead of R. A good starting guess might be $h_J = (63 + 54 + 40 + 20)/4 = 44$ m

PIPE	R	$z - z_J$	h_f	Q	Q/h_f
A	677.8	63	19	0.16743	0.00881
B	816	54	10	0.1107	0.01107
C	66.1	40	-4	-0.246	0.0615
D	193.7	20	-24	-0.352	0.01467
				Σ=-0.3199	Σ=0.09605

Now find the correction

$$\sum \delta h_f = \frac{2\,\delta Q}{\sum Q/h_f} = \frac{2 \text{ x } (-0.3199)}{0.09605} = -6.66$$ Change the values of h_J to 37.34 and repeat.

PIPE	R	$z - z_J$	h_f	Q	Q/h_f
A	677.8	63	25.66	0.19457	0.00758
B	816	54	16.66	0.14289	0.00858
C	66.1	40	2.66	0.2006	0.07542
D	193.7	20	-17.34	-0.2992	0.01725
				Σ=0.23886	Σ=0.10883

Now find the correction

$$\sum \delta h_f = \frac{2\,\delta Q}{\sum Q/h_f} = \frac{2 \text{ x } (0.23886)}{0.10883} = 4.39$$ Change the values of h_J to 41.73 and repeat.

PIPE	R	$z - z_J$	h_f	Q	Q/h_f
A	677.8	63	21.27	0.17715	0.00833
B	816	54	12.27	0.12262	0.00999
C	66.1	40	-1.73	-0.1618	0.09351
D	193.7	20	-21.73	-0.3349	0.01541
				Σ=-0.1969	Σ=0.12725

Now find the correction

$$\sum \delta h_f = \frac{2\,\delta Q}{\sum Q/h_f} = \frac{2 \text{ x } (-0.1969)}{0.1273} = -3.09$$ Change the values of h_J to 38.6 and repeat.

PIPE	R	$z - z_J$	h_f	Q	Q/hf
A	677.8	63	24.64	0.19066	0.00774
B	816	54	15.64	0.13844	0.00885
C	66.1	40	1.64	0.15751	0.09605
D	193.7	20	-18.36	-0.3079	0.01677
				Σ=0.17875	Σ=0.1294

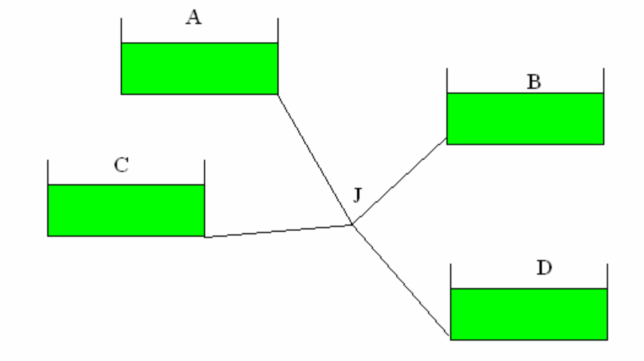
$$\sum \delta h_f = \frac{2\,\delta Q}{\sum Q/h_f} = \frac{2 \text{ x } (0.17875)}{0.1294} = 2.76$$ Change the values of h_J to 41.4 and repeat.

The number of iterations depends on how accurate you want the answer to be but you can see the answer is converging on 40 m.

WORKED EXAMPLE No.3 (EC Exam Standard)

The table shows the data for the network of pipes shown connecting four reservoirs to a common junction.

Reservoir	Water Level (m) above a datum	K for Pipe connecting to J (s^2/m^5)
A	50	4.0
B	45	3.0
C	40	2.0
D	30	2.0



Calculate the flow in each pipe using iteration until the final head correction at the junction is less than 0.1 m.

SOLUTION

The height of the datum is not given so we can only calculate the combined head and height. The best guess is usually the mean height of the reservoirs which is (50 + 45 + 40 + 30)/4 = 41.25

1st ITERATION

PIPE	K	z	Δh_f	Q	Q/h_f	Guess h_J+z_J
A	4	50	8.75	1.47902	0.16903	41.25
B	3	45	3.75	1.11803	0.29814	
C	2	40	-1.25	-0.7906	0.63246	
D	2	30	-11.25	-2.3717	0.21082	
				-0.5652	1.31045	

$$\sum \delta h_f = \frac{2\,\delta Q}{\sum Q/h_f} = \frac{2 \text{ x } (-0.5652)}{1.310} = -0.863 = \text{Correct } h_J + z_J = 40.4$$

2nd ITERATION

PIPE	R	z	Δh_f	Q	Q/h_f	Guess h_J+z_J
A	4	50	9.6	1.54919	0.16137	41.25
B	3	45	4.6	1.23828	0.26919	
C	2	40	-0.4	-0.4472	1.11803	
D	2	30	-10.4	-2.2804	0.21926	
				0.05991	1.76786	

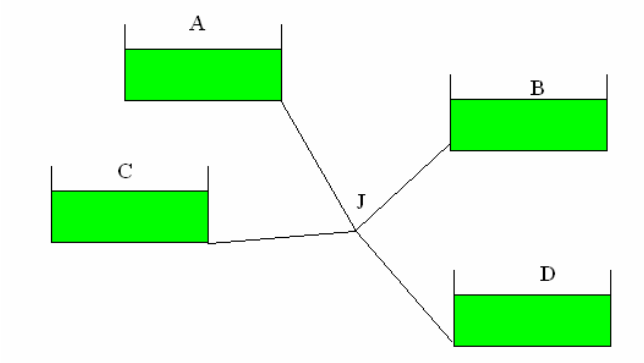
$$\sum \delta h_f = \frac{2\,\delta Q}{\sum Q/h_f} = \frac{2 \text{ x } (0.5991)}{1.76786} = 0.0678 \text{ This is less than 0.1 so meets the answer}$$

$Q_A = 1.55\ m^3/s$ $Q_B = 1.24\ m^3/s$ $Q_C = -0.45\ m^3/s$ $Q_D = -2.28\ m^3/s$

SELF ASSESSMENT EXERCISE No.1

The table shows the data for the network of pipes shown connecting four reservoirs to a common junction.

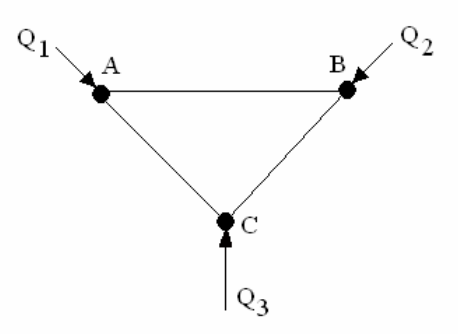
Reservoir	Water Level (m) above a datum	K for Pipe connecting to J (s^2/m^5)
A	120	420
B	90	316
C	60	250
D	40	120



Calculate the flow in each pipe using iteration until the final head correction at the junction is less than 0.1 m.

(0.36, 0.27, -0.16 and-0.47 m^3/s)

NETWORKS



Consider a small network laying in the horizontal plane as shown in the diagram. There are three nodes A, B and C and three pipes AB, BC and AC. The purpose is to find the flow in each pipe.

Suppose the pressure head at node A is h_A. Assume the flow is clockwise around the loop.

The pressure head at node B must be $h_A + h_f(AB)$.
The pressure head at node C must be $h_A + h_f(AB) + h_f(BC)$.
The pressure head at node A must be $h_A + h_f(AB) + h_f(BC) + h_f(CB)$.
We are back to where we started so $h_A = h_A + h_f(AB) + h_f(BC) + h_f(CB)$.
It follows that $h_f(AB) + h_f(BC) + h_f(CB) = 0$

If the flow in any pipe is the opposite way, then h_f will be negative and all is taken care of.

If we went anti-clockwise around the loop, the same would be true.

Notation is clockwise is positive (opposite to maths convention, typical engineering)
Anti- clockwise is negative.

The solution is based on calculating the total h_f for either the clockwise or anti-clockwise flow and adjusting the Q values until Σh_f is zero.

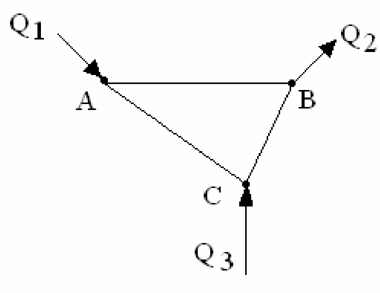
The correction to the flow in each pipe is $\delta Q = \dfrac{\sum h_{f1}}{\sum n h_{f1}/Q}$

We start by guessing the flow in each pipe (ensuring balance at each node) and calculating the friction head for each. We add up the friction heads and if it is not close to zero we correct our guess and do it again.

The method is known as the Hardy Cross method or Loop Balance.

IMPORTANT NOTE – the correction must be SUBTRACTED

WORKED EXAMPLE No.4



In the simple network shown $Q_1 = 0.8\ m^3/s$, $Q_2 = -1.2\ m^3/s$.
The resistance of each pipe is as follows.

Pipe AB $R = 50\ s^2/m^5$
Pipe BC $R = 30\ s^2/m^5$
Pipe AC $R = 60\ s^2/m^5$

Determine the flow in the three pipes. Take n = 2

SOLUTION

By conservation of flow, $Q_3 = 0.4\ m^3/s$

Guess the flow in each pipe bearing in mind the total flow at a node is zero. Clockwise is positive. The starting guess is:
Q (AB) = 0.6   Q (BC) = -0.6   Q (BC) = -0.4

First iteration

PIPE	R	Q	h_f	h_f/Q
AB	50	0.6	18	30
BC	30	-0.6	-10.8	18
AC	60	-0.2	-2.4	12
		-0.2	4.8	60

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{4.8}{2 \times 60} = 0.04$$ Correct the Q values by subtracting

Second iteration

PIPE	R	Q	h_f	h_f/Q
AB	50	0.56	15.68	28
BC	30	-0.64	-12.288	19.2
AC	60	-0.24	-3.456	14.4
		-0.32	-0.064	61.6

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{-0.32}{2 \times 61.6} = -0.000524$$ Correct the Q values by subtracting

Third iteration

PIPE	R	Q	h_f	h_f/Q
AB	50	0.56052	15.708	28.025
BC	30	-0.6395	-12.2688	19.185
AC	60	-0.2395	-3.44162	14.37
		-0.3185	-0.00005	61.58

This is one iteration more than we need. The head loss is so close to zero that this is the correct answer.

Q (AB) = $0.56052\ m^3/s$, Q (BC) = $-0.6395\ m^3/s$ and Q (AC) = $-0.2395\ m^3/s$

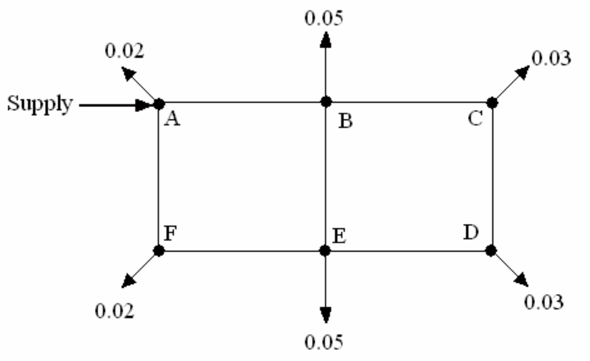
If we check the flow into each node we will see that the original figures have been maintained.

MULTIPLE LOOPS

When a network contains multiple loops, there will be pipes common to adjoining loops with a clockwise flow in one loop appearing as anti-clockwise in the other. Each loop must be identified and the corrections made systematically to each loop in turn. The correction to the flows must be made each time before moving on to the next loop. For more than two loops in a network, the process becomes very elaborate and computer methods need to be used.

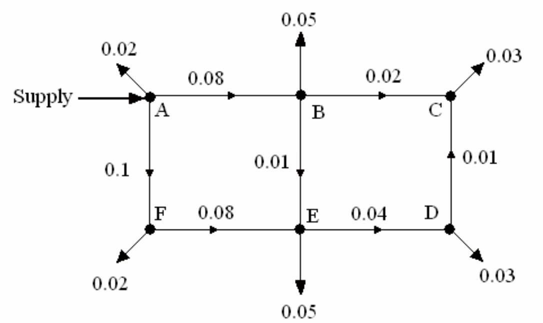
WORKED EXAMPLE No. 5

The diagram shows a water supply network with the demands indicated at the nodes. The value of K for each pipe is 1000 s^2/m^5 except for BE which is 7500 s^2/m^5. The supply pressure head at A is 50 m above the ground elevation for the area served which is flat and level. Calculate the pressure head at each node.



SOLUTION

The problem must be solved as two loops with a common pipe BE.
First make a guess at the flow rates.
The supply must be
$0.02 + 0.05 + 0.03 + 0.03 + 0.05 + 0.02 = 0.2$ m^3/s.
Bear in mind that the net flow is zero at all nodes.



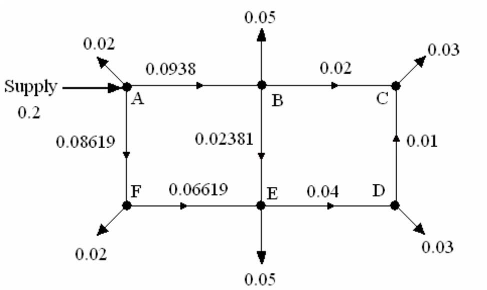
Data shown for initial guess

Start with loop ABEFA

PIPE	R	Q	h_f	h_f/Q
AB	1000	0.08	6.4	80
BE	7500	0.01	0.75	75
EF	1000	-0.08	-6.4	80
FA	1000	-0.1	-10	100
			-9.25	335

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{-9.25}{2 \times 335} = -0.01386$$

Correct all flows in this loop by adding 0.01386.



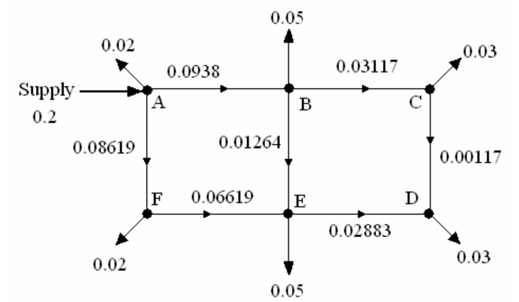
Data after first correction to left loop

Now do loop BCDEB

PIPE	R	Q	h_f	h_f/Q
BC	1000	0.02	0.4	20
CD	1000	-0.01	-0.1	10
DE	1000	-0.04	-1.6	40
BE	7500	-0.02381	-4.2504	178.5
			-5.550	248.5

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{-5.55}{2 \text{ x } 248.5} = -0.01117$$

Correct all flows in this loop by adding 0.01117.



Data after first correction to the right loop

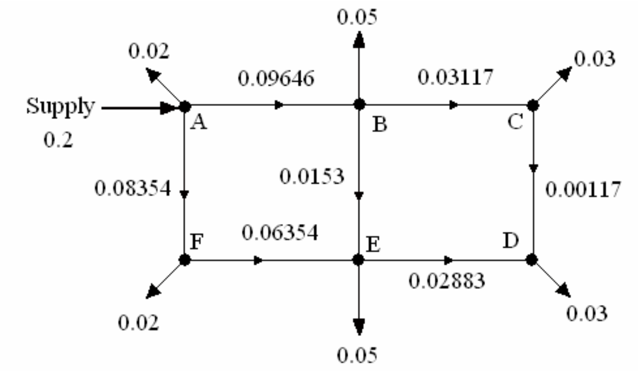
This completes the first iteration so now do loop ABEFA again.

PIPE	R	Q	hf	hf/Q
AB	1000	0.09381	8.79956	93.80597
BE	7500	0.01264	1.19829	94.80084
EF	1000	-0.0662	-4.38165	66.19403
FA	1000	-0.0862	-7.42941	86.19403
			-1.81321	340.9949

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{-1.8132}{2 \text{ x } 341} = -0.00266$$

Correct all flows by adding 0.00266

Data after second correction to the left loop



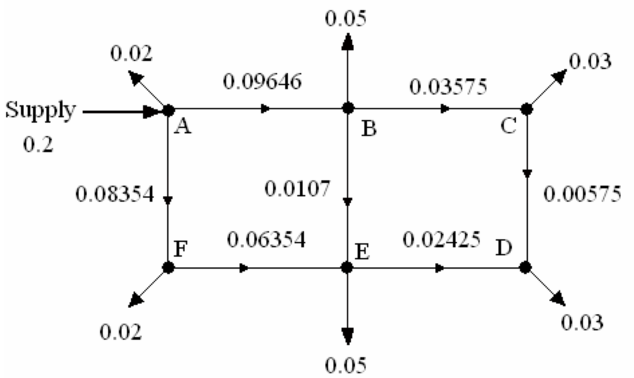
Now do loop BCDEB again.

PIPE	R	Q	h_f	h_f/Q
BC	1000	0.03117	0.971311	31.16586
CD	1000	0.00117	0.001359	1.165859
DE	1000	-0.0288	-0.83141	28.83414
BE	7500	-0.0153	-1.7554	114.7411
			-1.61414	175.907

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{1.6141}{2 \text{ x } 175.9} = -0.00459$$

Correct all flows by adding 0.00459

Data after second correction to right loop



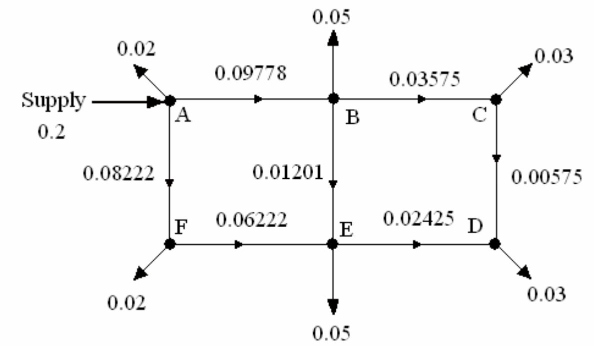
We need to keep going until h_f is very small. Our initial guess was not very good.
Do ABEFA Again

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{-0.849}{2 \times 323.866} = -0.00131$$

PIPE	R	Q	h_f	h_f/Q
AB	1000	0.09646	9.30543	96.46467
BE	7500	0.01071	0.8604	80.33071
EF	1000	-0.0635	-4.03674	63.53533
FA	1000	-0.0835	-6.97815	83.53533
			-0.84905	323.866

Correct all flows by adding 0.00131

Data after third correction to the left loop

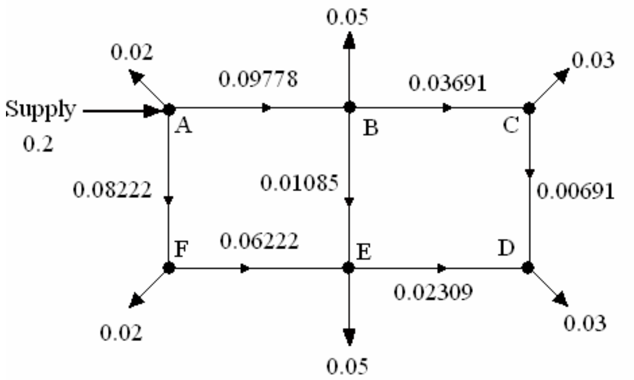


Do BCDEB again

PIPE	R	Q	h_f	h_f/Q
BC	1000	0.03575	1.278342	35.75391
CD	1000	0.00575	0.033107	5.75391
DE	1000	-0.0242	-0.58787	24.24609
BE	7500	-0.012	-1.08389	90.16178
			-0.36031	155.9157

$$\delta Q = \frac{\sum h_{f1}}{2\sum h_{f1}/Q} = \frac{-0.3603}{2 \times 155.916} = -0.00116$$

Correct all flows by adding 0.00116



Final flow rates

We have a total friction head of less than 1 metre in both loops so we will end here.
To find the pressure head at each node we must evaluate the friction heads with these flows.

PIPE	R	Q	h_f
AB	1000	0.09778	9.56004
BE	7500	0.01087	0.88554
EF	1000	-0.0622	-3.87189
FA	1000	-0.0822	-6.76087
BC	1000	0.03691	1.362302
CD	1000	0.00691	0.047739
DE	1000	-0.0231	-0.53318
BE	7500	-0.0109	-0.88554

Pressure at B = 50 – 9.6 = 40.4 m

Pressure at F = 50 – 6.8 = 43.2 m

Pressure at C = 40.4 – 1.4 = 39 m

Pressure at D = 39.4 – 0.5 = 38.9 m (check)

Pressure at E = 40.4 – 0.9 = 39.5 m

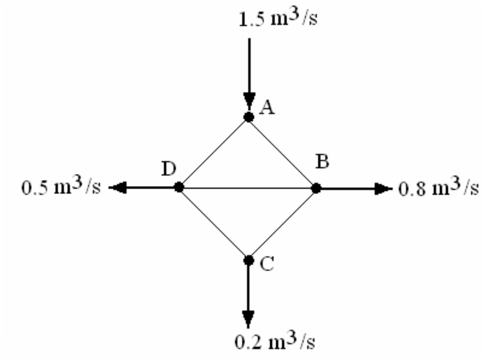
Pressure at E = 43.2 – 3.9 = 39.3 m (check)

Pressure at D = 39 – 0.05 = 39 m

SELF ASSESSMENT EXERCISE No.2

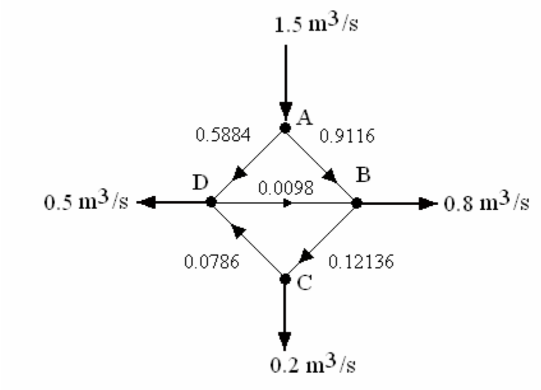
1. The diagram shows a simple pipe network in the horizontal plane with nodes at A, B, C and D with flow rates as indicated. The hydraulic resistance of each pipe is as follows:

AB	500 s^2/m^5
BC	500 s^2/m^5
CD	1200 s^2/m^5
DA	1200 s^2/m^5
BD	600 s^2/m^5

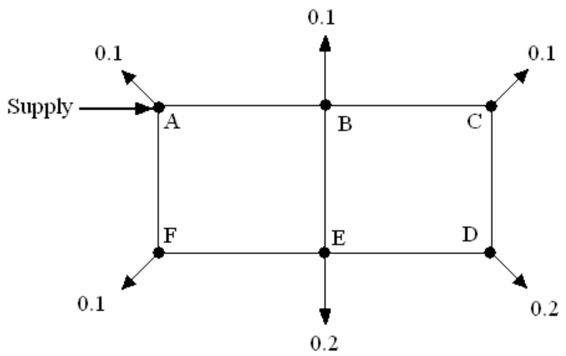


Determine the flow in each pipe

The answers are shown next.



2. The diagram shows a water supply network with the demands indicated at the nodes. The value of K for each pipe is 500 s^2/m^5 except for BE which is 600 s^2/m^5. The supply pressure head at A is 120 m above the ground elevation for the area served which is flat and level. Calculate the flow rate in each pipe and the pressure head at each node.



ANSWERS

	Q m^3/s	h_f m	Node	h m
AB	0.373	69.6	A	120
BE	0.126	9.6	B	50.4
EF	-0.227	-25.7	C	35.4
FA	-0.327	-53.4	D	32.7
BC	0.173	15.0	E	40.7
CD	0.0732	2.7	F	66.6
DE	-0.127	-8.0		
BE	-0.126	-9.6		