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Semestral Work in Mathematical Modeling

Optimizing in Outdated Goods Sales

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Contents

1 Introduction

We deal with the problematic which probably every shop manager has to face. The question is how a seller should deal with the goods which should be discounted for some reason. For illustration we will point on sales of bread.

In developed countries there appears the following phenomenon. Bread which is only one day old is already considered not to be fresh. Therefore it is not acceptable to be sold unless being discounted. The question is if the seller should offer the "old" bread in a discount.

On one hand, the seller receives money for the old bread which he sells and would give away anyway. On the other hand, by doing this he risks that some of the customers who would buy fresh bread will buy old bread instead. The customers would do so because they can safe money by buying one day old bread which has not much less value then the fresh one from their point of view. This might cause a domino effect that every evening there remains much of the fresh bread just because a significant part of the customers bought the old bread. Hence the following day there is again much old bread which is bought instead of the fresh one.

In order to analyze the problematic we will consider a shop in a monopoly position. We will use a model with assumptions simplifying the situation.

2 Problematic

Consider a small village in which there is a single food store. The population of the village is 500 and all of the citizens buy bread only in this store. The demand for bread varies from day to day. In order to simplify the model, we consider that there are two kinds of a day: On any *profuse day* the demand for bread is $D_p = 400$ loafs, while on any *hungry day* the demand is only $D_h = 200$ loafs. The seller can not predict which kind of a day is to come, the only thing he knows is that every day is *profuse* with probability $\alpha \in [0,1]$.

The loaf of fresh bread is sold for $p_f = 4\,\$$. The seller has three options how to deal with old bread:

a) give it away;

b) offer a loaf for $p_o = 3.2\,\$$;

c) offer a loaf for $p_o = 2.8\,\$$.

The unit cost of a loaf for the seller is $c = 3\,\$$.

The value of a loaf of fresh bread for any customer is $v_f = 5\,\$$ while the value of an old one is $v_o = 4\,\$$.

Question: How many loafs should the seller order every day and how should he deal with old bread? Consider that the seller has to order the same number N of loafs every day.

We will determine the seller's utility in particular case of his decision. In the further text we will omit using the dollar symbol "\$".

Notion	Value	Meaning
D_h	200	Demand on a hungry day
D_p	400	Demand on a profuse day
α	0.5	Probability that a profuse day comes
c	3	Cost of a loaf of bread
p_f	4	Price of fresh bread
$p_{o,b}$	3.2	Price of a loaf of old bread in the case b)
$p_{o,c}$	2.8	Price of a loaf of old bread in the case c)
N	1. D_h 2. D_p 3. variable	Number of loafs of bread that the seller orders each day depending on the seller's choice
v_f	5	Value of a loaf of fresh bread to a consumer
v_o	4	Value of a loaf of old bread to a consumer
R		Seller's revenge
E		Seller's expense
U	$R - E$	Seller's utility
u_f	$v_f - p_f$	A customer's utility from buying a loaf of fresh bread
$u_{o,b}$	$v_o - p_{o,b}$	A customer's utility from buying a loaf of old bread in the case b)
$u_{o,b}$	$v_o - p_{o,b}$	A customer's utility from buying a loaf of old bread in the case c)

Table 1: Notion used in the model.

3 Cases leading to a deterministic model

3.1 Minimal order (case 1.)

First, we look at the case in which the seller orders $N = D_h = 200$ loafs of bread. Then all the bread the seller receives on a day is sold, no matter what day it is. Hence there is no bread left for the following day and the problem with old bread does not arise. The seller's revenue is $R = D_h p_f$ and his expense is $E = D_h c$. Consequently the seller's utility in mean of profit is

$$U_1 = R - E = D_h(p_f - c) = 200 \cdot (4 - 3) = 200.$$

3.2 Maximal order (case 2.)

Next, we solve the case in which the seller orders $N = D_p = 400$ loafs of bread. In this case there is no reason for the seller to offer old bread because he has always enough fresh bread to offer. The seller would only lose money by selling old bread because he could sell fresh bread (for a higher price) to the same customer instead. The seller's revenue is $R_p = D_p p_f$ on a profuse day and $R_h = D_h p_f$ on a hungry day. The seller's expense is $E = D_p c$. Consequently the seller's utility in mean of average profit is

$$\begin{aligned} U_2 &= \alpha R_p + (1 - \alpha) R_h - E \\ &= (R_h - E) + \alpha(R_p - R_h) \\ &= (D_h p_f - D_p c) + \alpha(D_p - D_h) p_f \\ &= -400 + 800\alpha. \end{aligned}$$

3.3 Medium order - old bread given away (case 3.a)

Finally, we will solve the case $D_h < N < D_p$. The expense does not depend on the seller's choice and is $E = Nc$.

If the seller gives old bread away then the situation is simple. The seller sells D_h loafs on a hungry day and N loaf on a profuse day. Consequently, $R_h = D_h p_f$, $R_p = N p_f$. The seller's utility then is

$$\begin{aligned} U_{3a} &= \alpha R_p + (1 - \alpha) R_h - E \\ &= (R_h - E) + \alpha(R_p - R_h) \\ &= D_h p_f - Nc + \alpha(N - D_h) p_f. \end{aligned}$$

Namely, for $N = 300$, we get

$$U_{3a} = -100 + 400\alpha.$$

3.4 Medium order - fresh bread preferred (case 3.b)

The seller does better by offering old bread at the price $p_{o,b} = 3.2\$$. Then each customer prefers to buy fresh bread rather than old bread. It is so because a loaf of fresh bread gives a customer utility $u_f = v_f - p_f = 5 - 4 = 1$ while a loaf of old bread gives him utility of only $u_{o,b} = v_o - p_{o,b} = 4 - 3.2 = 0.8$. In such a situation each customer will buy old bread only in case there is no fresh bread offered. Therefore there is not a risk for the seller that offering old bread would reduce the demand for the fresh one.

On a hungry day only fresh bread is sold. Hence, $R_h = D_h p_f$. There remains $N - D_h$ loafs of bread for the next day. (The bread which is old on a day can not be sold the following day, e.i. only one day old bread can be sold.)

On a profuse day which follows after a hungry day all fresh bread is sold and moreover

$$M = \min\{N - D_h, D_p - N\}$$

old bread is sold (since $N - D_h$ is the supply and $D_p - N$ is the demand for old bread). Hence $R_{ph} = Np_f + Mp_{o,b}$. Then no bread is left to the following day.

On a profuse day which follows after a profuse day only fresh bread is sold because no bread from the preceding day remains. Then, $R_{pp} = Np_f$.

The seller's utility then is

$$\begin{aligned} U_{3b} &= \alpha[\alpha R_{pp} + (1-\alpha)R_{ph}] + (1-\alpha)R_h - E \\ &= (R_h - E) + \alpha(R_{ph} - R_h) + \alpha^2(R_{pp} - R_{ph}) \\ &= (D_h p_f - Nc) + \alpha[(N - D_h)p_f + Mp_{o,b}] - \alpha^2 M p_{o,b}. \end{aligned}$$

Namely, for $N = 300$, we get

$$U_{3b} = -100 + 720\alpha - 320\alpha^2.$$

4 Stochastic model: Medium order - old bread preferred (case 3.c)

4.1 Consumption and old bread in stock

The situation becomes more complicated if the seller offers bread for a lower price $p_o = 2.8$. Then each customer prefers to buy old bread rather than fresh one. It is so because a loaf of fresh bread gives a customer utility $u_f = v_f - p_f = (5 - 4) = 1$ while a loaf of old bread gives him a higher utility $u_{o,c} = v_o - p_{o,c} = (4 - 2.8) = 1.2$. In such a situation old bread is being sold first and fresh bread is being sold as late as after the old bread is sold out.

After some break (e.g. holiday) there is no old bread in a stock. We will say that a day is n-th if it is the n-th day after a break. Any time we will speak about state of stock on the n-th day we will mean the state in the early morning (before fresh bread is delivered). Denote x_n the number of loafs of old bread which is in stock the n-th day. Obviously $x_1 = 0$. Further, we will determine the state x_{n+1} of stock when knowing the state x_n on the preceding day and kind of the day.

For some n, consider the n-th day. There is x_n old bread in stock in the morning. Notice that $x_n \leq N$ because there can not remain more bread than what is ordered on that day. If the day is profuse then $\max\{N + x_n - D_p, 0\}$ loafs of fresh bread remain (old bread is consumed at first, because $x_n \leq N < D_p$). If the day is hungry then $\min\{N + x_n - D_h, N\}$ loafs of fresh bread remain.

4.2 Stochastic description

We have described the way how the state of the old bread stock change from one day to the next one. In order to determine the average consumption of old and fresh bread we need to use some theoretical background from the theory of stochastic processes.

Denote

$$d := \gcd(N - D_p, N - D_h, N) = \gcd(D_h, N, D_p)$$

and $K = \frac{N}{d}$. Denote $k \in \{0, \ldots, K\}$ the state at which there is $s_k = kd$ loafs of old bread in stock. Then $S := \{0, \ldots, K\}$ is obviously the set of all reachable states, so-called *state space*. Denote X_n the random variable determining the state of the stock on the n-th day. Then $\{X_n, n \in \mathbb{N}_0\}$ is a stochastic process ($\mathbb{N}_0$ denotes natural numbers with zero). More over the process

$$\{X_n, n \in \mathbb{N}_0\}$$

is a time-homogeneous Markov chain. The fact that the chain is Markov means that it has a Markov property. Markov property says that, given the present state, future states are independent of the past states. Formaly,

$$PX_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0 = PX_{n+1} = j | X_n = i,$$

for all $n \in \mathbb{N}_0$ and all $i, j, j_{n-1}, \ldots, j_0 \in S$. The time-homogenity of the process means that $PX_{n+1} = j | X_n = i$ is does not depend on n, for all $i, j \in S$.

Denote $p_i(n) := PX^(n) = i$ the probability that the stock of old bread is s_i on the beginning of the n-th day. Then

$$\begin{aligned} p_j(n+1) &= PX_{n+1} = j && (1)\\ &= \sum_{i \in S} PX_{n+1} = j, X_n = i && (2)\\ &= \sum_{i \in S} PX_{n+1} = j | X_n = i PX_n = i && (3)\\ &= \sum_{i \in S} p_{i,j} p_i(n), && (4) \end{aligned}$$

in which $p_{i,j} := PX_{n+1} = j | X_n = i$ is the *transition probability* from the state i to the state j (in one day time), for all $i, j \in S$ and $n \in \mathbb{N}_0$.

The probabilities of particular states at a day n form vector $\mathbf{p}(n) = (p_i(n+1))_{i=0}^{K}$. The transition probabilities form the probability matrix $\mathbf{P} = (p_{i,j})_{i,j=0}^{K}$. Then the equation (1) can be written in a vector form

$$\mathbf{p}^T(n+1) = \mathbf{p}^T(n)\, \mathbf{P}.$$

Consequently, using induction, we conclude

$$\mathbf{p}^T(n) = \mathbf{p}^T(0)\, \mathbf{P}^n. \qquad (5)$$

4.3 Stationary distribution in the case $N = 300$

The transitions from a state to another one were described in the section 4.1 based on the fact that the day is profuse or that it is hungry. Recall that a profuse day comes with the probability α and a hungry day with the probability $1 - \alpha$.

We will first show the solution in the case of $N = 300$ (while $D_h = 200, D_p = 400$). Then the transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} \alpha & 1-\alpha & 0 & 0 \\ \alpha & 0 & 1-\alpha & 0 \\ 0 & \alpha & 0 & 1-\alpha \\ 0 & 0 & \alpha & 1-\alpha \end{bmatrix}.$$

The characteristic polynomial of the matrix $\mathbf{P}$ is ($\mathbf{I}$ is the identity matrix)

$$Det(\lambda\mathbf{I} - \mathbf{P}) = \lambda(\lambda - 1)(\lambda^2 - 2\alpha + 2\alpha^2).$$

Consequently the eigenvalues of the matrix $\mathbf{P}$ are $\lambda_1 = 1, \lambda_2 = 0$ and $\lambda_{3,4} = \pm\sqrt{2(\alpha - \alpha^2)}$. Denote $\xi_1, \ldots, \xi_4$ the associated eigenvectors and write the initial distribution as $\mathbf{p}(n) = \sum_{i=1}^{4} k_i\xi_i$. For $i = 1, \ldots, 4$ we have $\xi_i^T\,\mathbf{P} = \lambda_i\xi_i$, consequently induction yields us

$$\xi_i^T\,\mathbf{P}^n = \lambda_i^n\xi_i^T.$$

Hence we can write the equation 5 as

$$\mathbf{p}^T(n) = (\sum_{i=1}^{4} k_i\xi_i)^T\,\mathbf{P}^n = \sum_{i=1}^{4} k_i\lambda_i^n\xi_i^T.$$

Notice that

$$|\lambda_{3,4}| = \sqrt{2(\alpha - \alpha^2)} = \sqrt{1/2 - 2(\alpha - 1/2)^2} < \sqrt{2}/2 < 1.$$

Hence

$$\lim_{n\to\infty} \lambda_i^n = \begin{cases} 1, & i = 1, \\ 0, & i = 2, 3, 4, \end{cases}$$

and so the stationary distribution is

$$\pi = \lim_{n\to\infty} \mathbf{p}(n) = k_1\xi_1. \tag{6}$$

Notice, that the sum $\|\mathbf{p}(n)\|_1$ of elements of any vector $\mathbf{p}(n)$ must be 1, consequently it must be so even for π. Hence $k_1 = 1/\|\xi_1\|_1$. After calculating the eigenvector ξ_1 we conclude that

$$\pi = \frac{\xi_1}{\|\xi_1\|_1} = \frac{1}{2\alpha^2 - 2\alpha + 1}(\alpha^3, (1-\alpha)\alpha^2, (1-\alpha)^2\alpha, (1-\alpha)^3)^T.$$

Namely, for $\alpha = 0.5$, we get

$$\pi = \frac{1}{4}(1, 1, 1, 1)^T.$$

4.4 Stationary distribution for general N

Further we will consider general $N(D_h < N < D_p)$.

Denote

$$k_h := \frac{D_h - N}{d} < 0, \qquad k_p := \frac{D_p - N}{d} > 0$$

Then the transition probabilities are

$$p_{i,j} = \begin{cases} \alpha & i - j = k_p, \\ \alpha & j = 0 \wedge i - j < k_p, \\ 1-\alpha & i - j = k_h, \\ 1-\alpha & j = K \wedge i - j > k_h, \\ 0 & \textit{otherwise}. \end{cases}$$

Then the transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 1-\alpha & & & & & \\ \vdots & & & & & 1-\alpha & & & & \\ \alpha & & & & & & \ddots & & & \\ \alpha & & & & & & & 1-\alpha & & \\ & \alpha & & & & & & & & 1-\alpha \\ & & \ddots & & & & & & & \vdots \\ & & & & \alpha & & 0 & \cdots & 0 & 1-\alpha \end{bmatrix}$$

and the stationary distribution of the process is again

$$\pi = \frac{\xi_1}{\|\xi_1\|_1},$$

in which ξ_1 is the eigenvector associated to the eigenvalue 1.

4.5 Average profits

Consider a day at which the stock of old bread is in the state $k \in S$ (so there is $s_k = kd$ loafs of old bread in the stock). If the day is hungry, then $\min\{s_k, D_h\}$ loafs of old bread are sold (we consider minimum of supply and demand of the old bread) and $\max\{D_h - s_k, 0\}$ loafs of fresh bread are sold. If the day is profuse, then all s_k loafs of old bread are sold and $\min\{N, D_p - s_k\}$ loafs of fresh bread are sold.

Recall that a day is profuse with the probability α and hungry with the probability $1 - \alpha$. Hence the Seller's average revenue on a day such that the old bread stock is in state k is

$$R_k = \alpha(s_k p_{o,c} + \min\{N, D_p - s_k\}p_f) + (1-\alpha)(\min\{s_k, D_h\}p_{o,c} + \max\{D_h - s_k, 0\}p_f)$$

Considering the stationary distribution over the states we conclude that the seller's average revenue on a day is

$$R = \sum_{k=0}^{K} \pi_k R_k.$$

Finally the seller's average profit is $U = R - E$ (recall that the expense is $E = Nc$).

5 Comparison

5.1 Cases in which the old bread is not being sold

We will compare the cases of minimal order (1.), maximal order (2.) and medium order and old bread given away (3.a). Recall that

$$U_1 = D_h(p_f - c) = 200,$$

$$U_2 = (D_h p_f - D_p c) + \alpha(D_p - D_h)p_f = -400 + 800\alpha,$$

$$U_{3a} = D_h p_f - Nc + \alpha(N - D_h)p_f,$$

namely, for $N = 300$,

$$U_{3a} = -100 + 400\alpha.$$

One can simply observe that, for the given parameters D_p, D_h, p_f, the utility $U_2 > U_1$ only if the probability $\alpha > \frac{3}{4}$. Consequently there is advantage in maximal order over the minimal order only if the probability of profuse day is higher then 75%. Generally,

$$U_2 > U_1 \quad \Longleftrightarrow \quad \alpha > \frac{c}{p_f}.$$

Remark, that we did not consider the fact that the customers are more glad in the case 2 in which their supply is always satisfied compare to the case 1. The situation would change as soon as we would consider a model with reputation factors of considered a duopoly competition model.

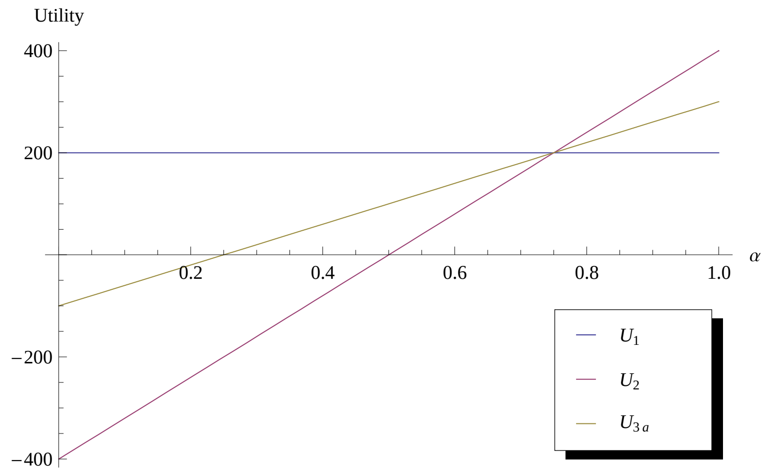

Figure 1: Comparison of the minimal order (1.), maximal order (2.) and medium order with old bread given away (3a.). For the medium order we consider $N = \frac{1}{2}(D_h + D_p) = 300$.

Notice that, for $N = 300$, we have

$$U_{3a} = -100 + 400\alpha = \frac{1}{2}(200 - 400 + 800\alpha) = \frac{1}{2}(U_1 + U_2).$$

Generally we can verify that

$$U_{3a} = \frac{(D_p - N)D_h(p_f - c) + (N - D_h)[(D_h p_f - D_p c) + \alpha(D_p - D_h)p_f]}{D_p - D_h}$$
$$= \frac{D_p - N}{D_p - D_h}U_1 + \frac{N - D_h}{D_p - D_h}U_2,$$

in which $\frac{D_p-N}{D_p-D_h}$ and $\frac{N-D_h}{D_p-D_h}$ are positive coefficients which add up to 1. Consequently U_{3a} is weighted arithmetic average of U_1 and U_2. Hence either $U_{3a} \le U_1$ (if $\alpha < \frac{c}{p_f}$) or $U_{3a} \le U_2$ (if $\alpha \ge \frac{c}{p_f}$). Consequently if the seller knows all the parameters of the model and he does not to deal with old bread then he should always decide for either minimal or maximal order (never medium order unless the case $\alpha = \frac{c}{p_f}$).

5.2 Medium order with fresh bread preferred

Consider the case that the fresh bread is preferred over the old bread by each customer. Recall that

$$U_{3b} = (D_h p_f - Nc) + \alpha[(N - D_h)p_f + Mp_{o,b}] - \alpha^2 M p_{o,b}. \tag{7}$$

in which $M = \min\{N - D_h, D_p - N\}$. Namely, for $N = 300$, we get

$$U_{3b} = -100 + 720\alpha - 320\alpha^2.$$

Logical deduction yields us that under the circumstances mentioned above there is an advantage in offering old bread because it is sold only after all fresh bread is sold out. This can be also verified by calculation

$$U_{3b} = -100 + 400\alpha + 320(\alpha - \alpha^2) = U_{3a} + 320\alpha(1 - \alpha) > U_{3a},$$

and generally

$$U_{3b} = D_h p_f - Nc + \alpha[(N - D_h)p_f + Mp_{o,b}] - \alpha^2 M p_{o,b}$$
$$= D_h p_f - Nc + \alpha(N - D_h)p_f + Mp_{o,b}(\alpha - \alpha^2)$$
$$= U_{3a} + Mp_{o,b}\alpha(1 - \alpha)$$
$$> U_{3a}.$$

The question is which N is optimizing U_{3b}. First, consider $N \le \frac{1}{2}(D_h + D_p)$. Then $M = N - D_h$ and by substituting to (7) and deriving with respect to N we get

$$\frac{\partial U_{3b}}{\partial N} = -p_{o,b}\alpha^2 + (p_f + p_{o,b})\alpha - c =: g_1(\alpha).$$

Since $g_1(1) = p_f - c > 0$ and $g_1(0) = -c < 0$, we have $g_1(\alpha) > 0$ for $\alpha \in (\alpha_1, 1)$, in which

$$a_1 = \frac{p_f + p_{o,b} - \sqrt{D_1}}{2p_{o,b}}, \quad D_1 = (p_f + p_{o,b})^2 - 4p_{o,b}c.$$

Further, consider $N > \frac{1}{2}(D_h + D_p)$. Then $M = D_p - N$ and by substituting to (7) and deriving with respect to N we get

$$\frac{\partial U_{3b}}{\partial N} = p_{o,b}\alpha^2 + (p_f - p_{o,b})\alpha - c =: g_2(\alpha).$$

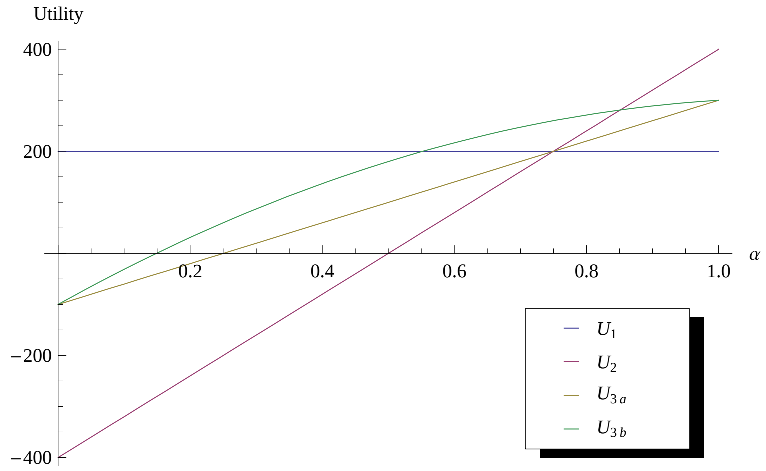

Figure 2: Comparison of the minimal order (1.), maximal order (2.), medium order with old bread given away (3a.) and medium order with fresh bread preferred (3b.). For the medium order we consider $N = \frac{1}{2}(D_h + D_p) = 300$.

Since $g_2(1) = p_f - c > 0$ and $g_2(0) = -c < 0$, we have $g_2(\alpha) > 0$ for $\alpha \in (a_2, 1)$, in which

$$a_2 = \frac{-p_f + p_{o,b} - \sqrt{D_2}}{2p_{o,b}}, \quad D_2 = (p_f - p_{o,b})^2 + 4p_{o,b}c.$$

We can observe that

$$g_1(\alpha) - g_2(\alpha) = -2p_{o,b}\alpha^2 + 2p_{o,b}\alpha = 2p_{o,b}\alpha(1-\alpha) > 0,$$

and hence $g_1(\alpha) > g_2(\alpha)$. Consequently $a_1 < a_2$ and there can appear the following 3 cases:

- $\alpha \in (0, a_1]$, then $0 \geq g_1(\alpha) > g_2(\alpha)$ and the best choice for the seller is the minimal order $N = D_h$;
- $\alpha \in [a_2, 1)$, then $g_1(\alpha) > g_2(\alpha) \geq 0$ and the best choice for the seller is the maximal order $N = D_p$;
- $\alpha \in (a_1, a_2)$, then $g_1(\alpha) > 0 > g_2(\alpha)$ and the best choice for the seller is the medium order $N = \frac{1}{2}(D_h + D_p)$;

Notice, that for given parameters p_f and $p_{o,b}$ we have $\alpha_1 \cong 0.552$ and $\alpha_2 \cong 0.851$.

5.3 Medium order: fresh vs. old bread preferred

As we can see from the figure 3 (see the Appendix), there is not an advantage in selling the bread for such a price that the customers prefer to buy old bread rather than the fresh one. It is caused by two factors: First, the seller receives more money when he sells a loaf of old bread for the price $p_{o,b} = 3.2$ rather than for the smaller price $p_{o,c} = 2.8$. Second, the customers will often buy old bread instead of fresh bread and consequently the fresh bread is sold as late as next day for the lowered price $p_{o,c} = 2.8$.

On the other hand, under some circumstances, there can be an advantage in offering old bread for such a price, that the customers prefer buying old bread instead of the fresh one. For the example we can consider $v_o = 4.5, p_{o,b} = 3.7$ and $p_{o,c} = 3.3$. Then the situation is depicted in the figure 4 (see the Appendix). We can see that for the example for $\alpha = 0.5$ or $\alpha = 0.8$ the seller should offer the bread for the lower price $p_{o,c}$.

6 Conclusion

We have determined the seller's utility in particular cases. We considered variable daily order of bread N (number of loafs). We also considered that the seller can decide whether to offer the old bread (one day old bread) and eventually he can choose the price at which the old bread is sold.

The maim issue was if there is any advantage in offering old bread for such a price (further we will simply say "low price") that the customers will prefer old bread to the fresh one. The case with old bread sold in a low price leads to a stochastic model of old bread deposit. There are special cases in which the seller does better by offering the old bread at a low price. However, the most efficient strategy for the seller appeared the following: Offer the old bread for the price $p_{o,b}$ warning us that the customers will always prefer fresh bread to the old one. Set the number N of loafs of bread ordered each day to

$$N = \begin{cases} D_h & \alpha \le a_1, \\ \frac{1}{2}(D_h + D_p) & a_1 < \alpha < a_2, \\ D_p & a_2 < \alpha, \end{cases}$$

in which

$$a_1 = \frac{p_f + p_{o,b} - \sqrt{D_1}}{2p_{o,b}}, \quad D_1 = (p_f + p_{o,b})^2 - 4p_{o,b}c.$$

$$a_2 = \frac{-p_f + p_{o,b} - \sqrt{D_2}}{2p_{o,b}}, \quad D_2 = (p_f - p_{o,b})^2 + 4p_{o,b}c.$$

The results we got hold only under the assumption that the seller is not interested in his reputation and his shop is at a monopoly position. If this assumption was omitted then the seller would be pushed to order a higher number N of loafs of bread and he would be also pushed to lower the price at which he offers the old bread.

7 Future research

First of all, I would recommend to involve correction in the seller's utility so that we consider his reputation (or generally customers' satisfaction). The customer's utility is $u_f = p_f - v_f$ if he buys a loaf of fresh bread, $u_o = p_o - v_o$ if he buys a loaf of old bread (where $p_o \in p_{o,b}, p_{o,c}$ is the price of the old bread) and 0 if he buys nothing. Consider that there are sold N_f loafs of fresh bread and N_o loafs of bread sold on a day. Then the utility that all the customers receive together is $U_c = N_f u_f + N_o u_o$. We can assume, that the seller's utility corrected by the reputation factor is

$$U^* = U + \rho U_c,$$

in which ρ is the coefficient determining how much the seller is interested in his reputation. Such a slightly modified model can be analyzed similarly as we have done (for the case $\rho = 0$).

Next, the model will be more reliable after we consider the cost of storing (and exhibiting) old bread. I suggest to consider the cost to be linear to the maximal storage which can be provided. On the other hand, the seller should have a possibility to decide how much old bread he wants to store at maximum. By limiting the volume of his storage the seller can avoid situations in which old bread is sold instead of the fresh one.

Obviously there are not only two kinds of days (hungry and profuse), but the demand for the bread is a random variable. Moreover, every consumer has different values v_f and v_o of a loaf of fresh and old bread. The difference $v = v_f - v_o$ can be considered to be a random variable v with some distribution F_v. We can use the normal distribution $v \sim N(\mu_v, \sigma_v^2)$. Denote d the random variable determining the number of loafs of bread a customer wants to buy on a day. Consider that d has the Poisson distribution, e.i. $d \sim Po(\alpha)$. Assume that the random variables v and d are independent. Denote ω a customer. Then the customer ω prefers to buy old bread to the fresh one if and only if $v(\omega) < p_f - p_o$. Hence $QF_v(p_f - p_o)$, in which Q is the population of the village, is the average number of customers who prefer old bread and $Q(1 - F_v(p_f - p_o))$ is one of those who prefer fresh bread. Notice that a sum of random variables with Poisson distribution is again a random variable with a Poisson distribution. Consequently the random variables D_o and D_f representing the demand for old and fresh bread respectively are

$$D_o \sim Po(\alpha Q F_v(p_f - p_o)),$$

$$D_f \sim Po(\alpha Q(1 - F_v(p_f - p_o))).$$

Considering such random demands we can get a more realistic model. Mainly, the seller can determine the rate of those who prefer old bread to the fresh one by choosing p_o.

References

[Jones (2001)] P. W. Jones, P. Smith *Stochastic processes: An introduction*. Hodder Arnold

[Fudenberg (1991)] Fudenberg, Drew a Tirole, Jean. *Game Theory*. MIT Press, 1991.

8 Appendix

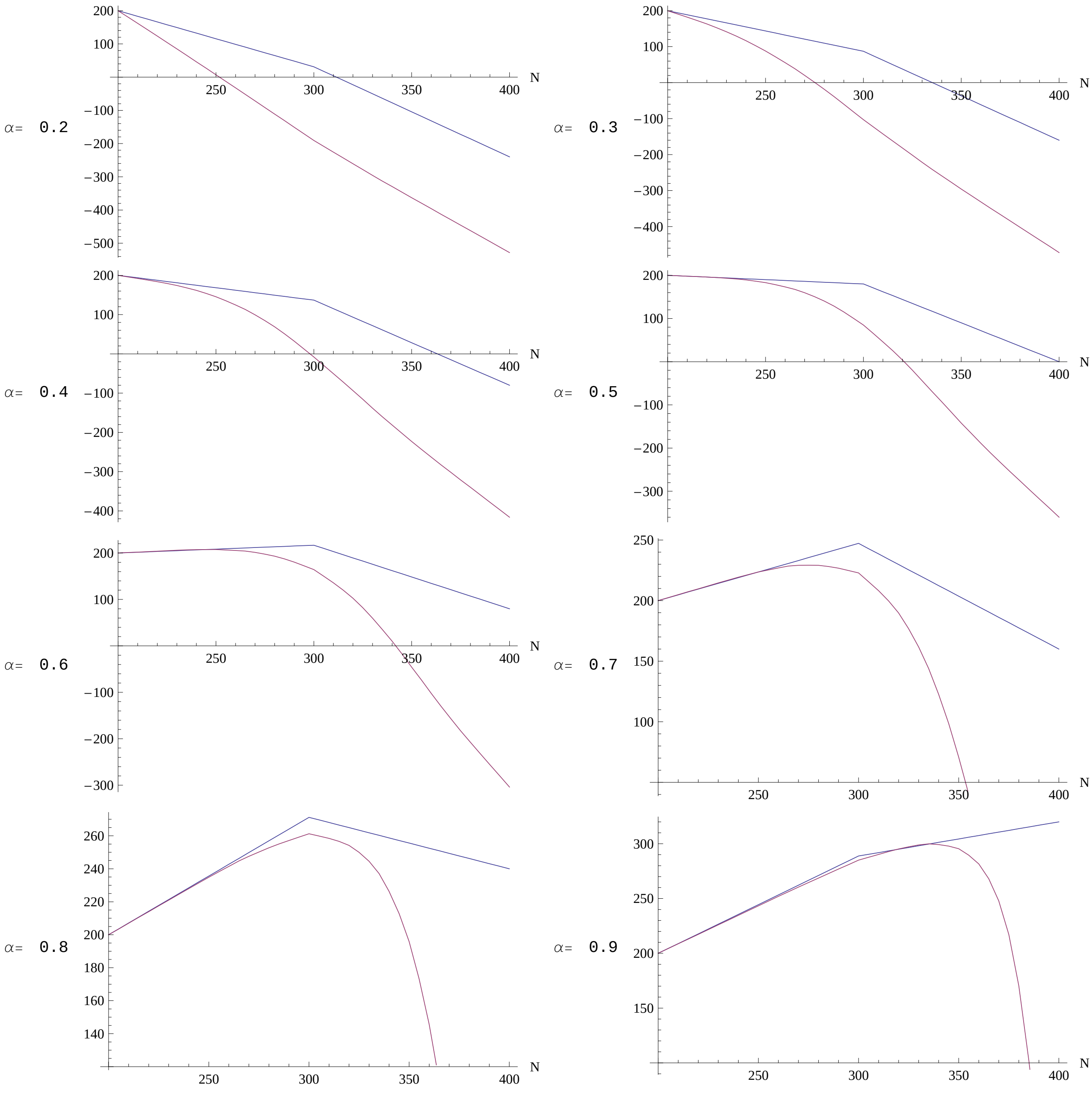

Figure 3: Comparison of the medium order with fresh bread preferred (3b.) and the medium order with old bread preferred (3c.). The graph of U_{3b} is blue and the graph of U_{3c} is pink. We consider particular values of α.

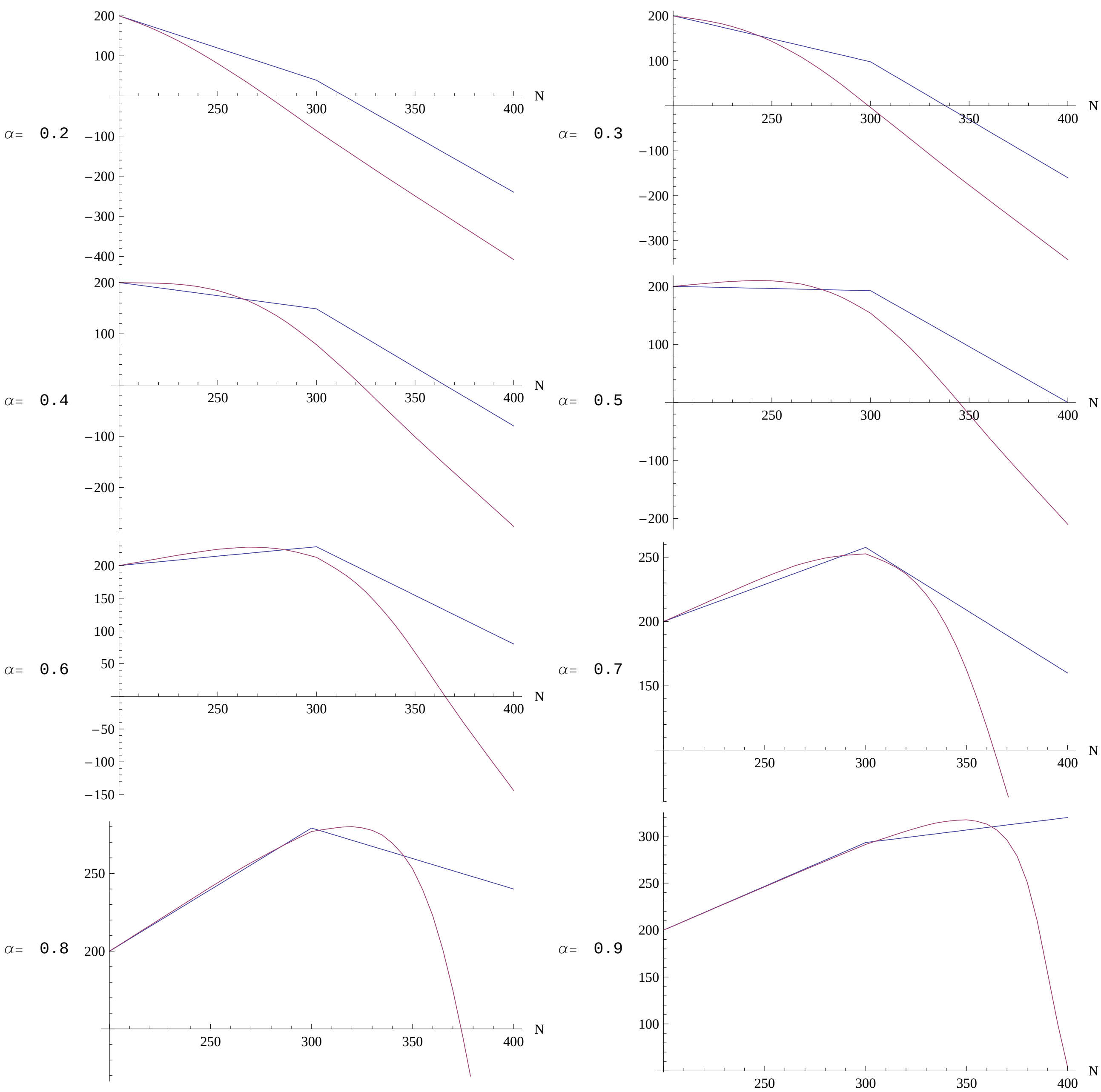

Figure 4: Results for slightly modified parameters. We consider $v_o = 4.5, p_{o,b} = 3.7$ and $p_{o,c} = 3.3$. Comparison of the medium order with fresh bread preferred (3b.) and the medium order with old bread preferred (3c.). The graph of U_{3b} is blue and the graph of U_{3c} is pink. We consider particular values of α.