Technical report

Overview of parallel architectures for gearing robot

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1 2 DoF planar parallel robot

The designed device is non-redundant planar parallel robot with two translational degrees of freedom (movement in the x axis - shifting up/down and movement in the y axis - shifting left/right), see Fig. 1. The robot is driven by two rotational drives (motors equipped with gears).

Let $\Theta = \begin{bmatrix} \Theta_A & \Theta_B \end{bmatrix}^T$ is the position (rotation) of the motors **A**, **B** (actuated joint coordinates) and $C = \begin{bmatrix} C_x & C_y \end{bmatrix}^T$ is the position of the end effector (generalized coordinates). We denote robot dimensions[1]

$$\begin{aligned} l_{A_1} &= \|AA_1\|, \quad l_{A_2} = \|AA_2\| \\ l_{B_1} &= \|BB_1\|, \quad l_{B_2} = \|BB_2\| \\ l_0 &= \|AB\| \end{aligned} \tag{1}$$

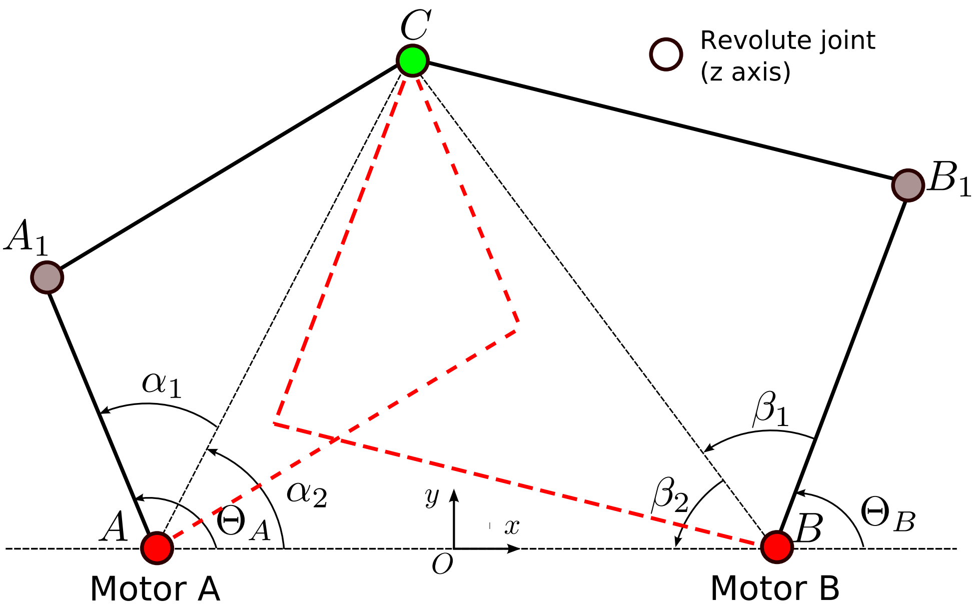


Fig. 1: 2DoF planar parallel robot (red dotted line shows possible variants)

1.1 Inverse kinematics

The relations giving the actuated joint coordinates Θ for given generalized coordinates C are called the *inverse kinematics*.

Let we denote

$$\begin{aligned} l_{AC} &= \|AC\| = \sqrt{(C_x - A_x)^2 + (C_y - A_y)^2} \\ l_{BC} &= \|BC\| = \sqrt{(C_x - B_x)^2 + (C_y - B_y)^2} \end{aligned} \tag{2}$$

[1] vector $AB = \begin{bmatrix} B_x - A_x & B_y - A_y \end{bmatrix}^T$; A_x, A_y denotes components of the point A; AB_x, AB_y denotes components of the vector AB

The following holds for the angles α_1, α_2, β_1, β_2.

$$\begin{aligned}
l_{A_2}^2 &= l_{A_1}^2 + l_{AC}^2 - 2l_{A_1}l_{AC}\cos\alpha_1 \Rightarrow \alpha_1 = \arccos\frac{-l_{A_2}^2 + l_{A_1}^2 + l_{AC}^2}{2l_{A_1}l_{AC}} \\
l_{B_2}^2 &= l_{B_1}^2 + l_{BC}^2 - 2l_{B_1}l_{BC}\cos\beta_1 \Rightarrow \beta_1 = \arccos\frac{-l_{B_2}^2 + l_{B_1}^2 + l_{BC}^2}{2l_{B_1}l_{BC}} \\
l_{BC}^2 &= l_{AC}^2 + l_0^2 - 2l_{AC}l_0\cos\alpha_2 \Rightarrow \alpha_2 = \arccos\frac{-l_{BC}^2 + l_{AC}^2 + l_0^2}{2l_{AC}l_0} \\
l_{AC}^2 &= l_{BC}^2 + l_0^2 - 2l_{BC}l_0\cos\beta_2 \Rightarrow \beta_2 = \arccos\frac{-l_{AC}^2 + l_{BC}^2 + l_0^2}{2l_{BC}l_0}
\end{aligned} \tag{3}$$

Then the inverse kinematic mapping $\Theta = G(C)$ for 4 variants of the 2DoF planar parallel robot is:

- **Variant A**

$$\Theta = \begin{bmatrix}\Theta_A \\ \Theta_B\end{bmatrix} = \begin{bmatrix}\alpha_1 + \alpha_2 \\ \pi - (\beta_1 + \beta_2)\end{bmatrix} \tag{4}$$

- **Variant B**

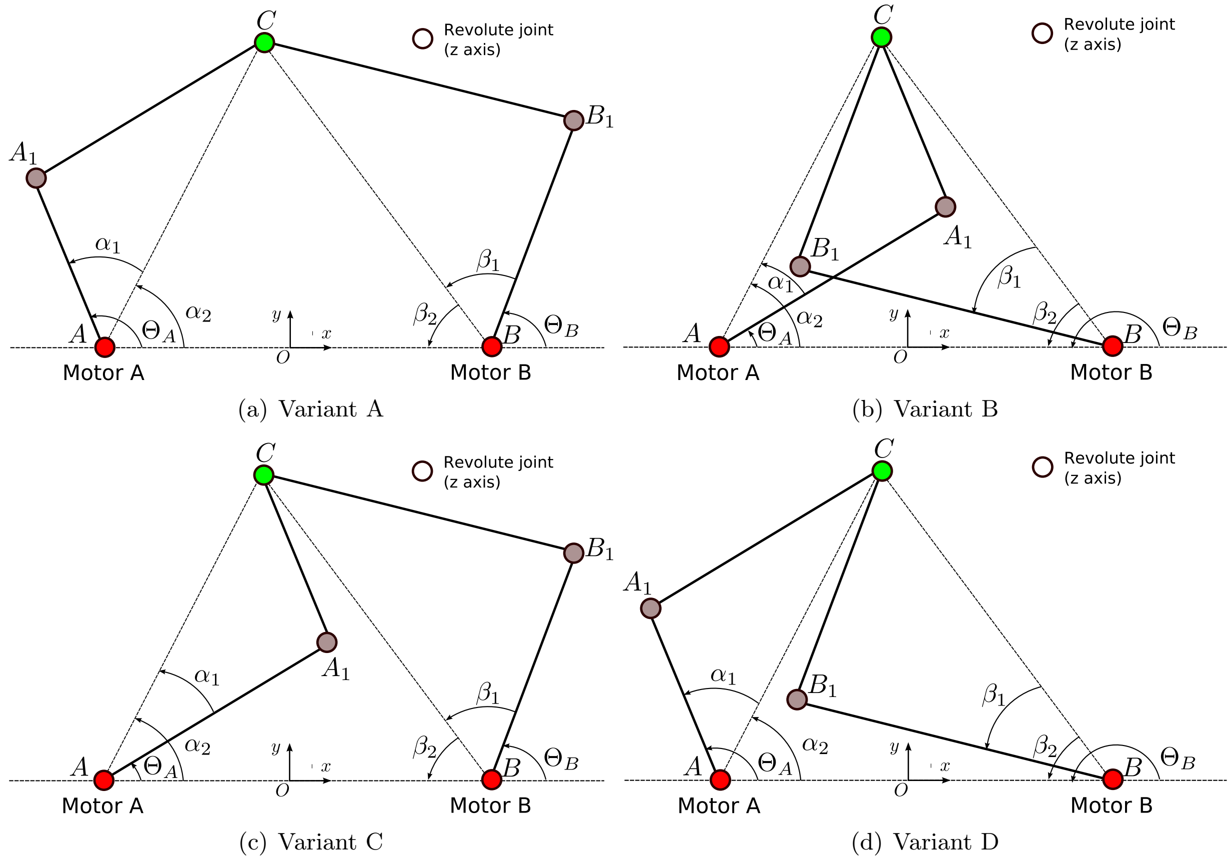
$$\Theta = \begin{bmatrix}\Theta_A \\ \Theta_B\end{bmatrix} = \begin{bmatrix}\alpha_2 - \alpha_1 \\ \pi - (\beta_2 - \beta_1)\end{bmatrix} \tag{5}$$

- **Variant C**

$$\Theta = \begin{bmatrix}\Theta_A \\ \Theta_B\end{bmatrix} = \begin{bmatrix}\alpha_2 - \alpha_1 \\ \pi - (\beta_1 + \beta_2)\end{bmatrix} \tag{6}$$

- **Variant D**

$$\Theta = \begin{bmatrix}\Theta_A \\ \Theta_B\end{bmatrix} = \begin{bmatrix}\alpha_1 + \alpha_2 \\ \pi - (\beta_2 - \beta_1)\end{bmatrix} \tag{7}$$



(a) Variant A (b) Variant B

(c) Variant C (d) Variant D

Fig. 2: Variant of the 2DoF planar parallel robot

1.2 Direct kinematics

The relations giving the position of the end effector C for given actuated joint coordinates Θ is called the *direct kinematics*. The direct kinematic mapping can be solved analytically in a closed form and there are two solutions by virtue of the triangle A_1B_1C, see Fig. 3. Hereafter, we suppose only the solution of the direct kinematics of the 2DoF planar parallel robot for which the point C lies above the line A_1B_1.

Therefore, the end effector position C is given by

$$C = A_1 + l_{A_2} \begin{bmatrix} \cos\Phi & \sin\Phi \end{bmatrix}^T, \tag{8}$$

positions of the points A_1, B_1 are

$$A_1 = l_{A_1} \begin{bmatrix} \cos\Theta_A & \sin\Theta_A \end{bmatrix}^T + \begin{bmatrix} -l_0/2 & 0 \end{bmatrix}^T, \quad B_1 = l_{B_1} \begin{bmatrix} \cos\Theta_B & \sin\Theta_B \end{bmatrix}^T + \begin{bmatrix} l_0/2 & 0 \end{bmatrix}^T \tag{9}$$

and we denote

$$\|A_1B_1\| = l_{A_1B_1}, \quad \|AB_1\| = l_{AB_1} \tag{10}$$

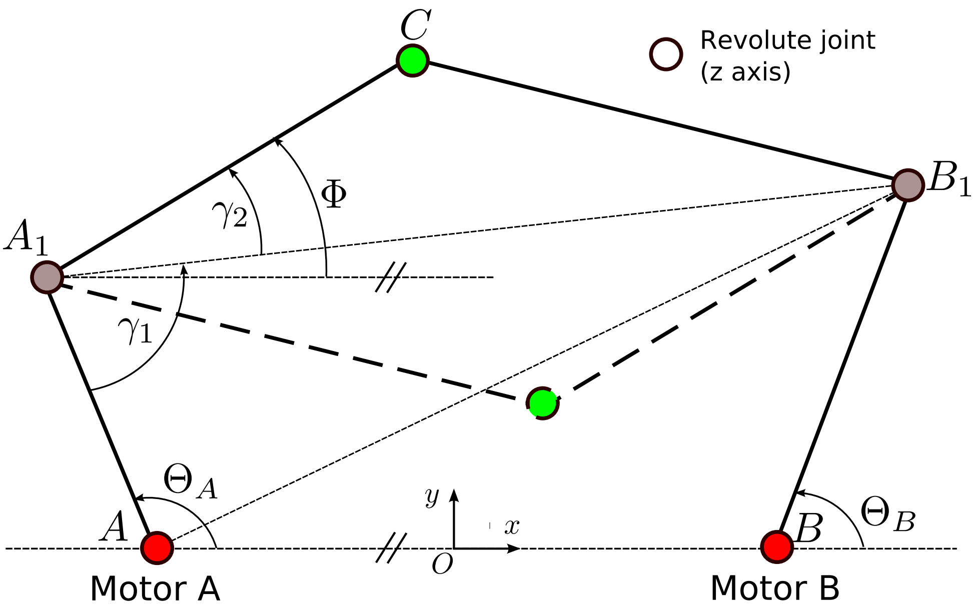
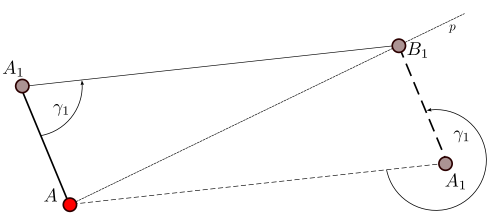
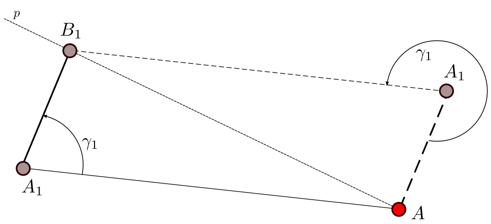


Fig. 3: Scheme for computation of the direct kinematics (dotted line shows the second solution of the direct kinematics)

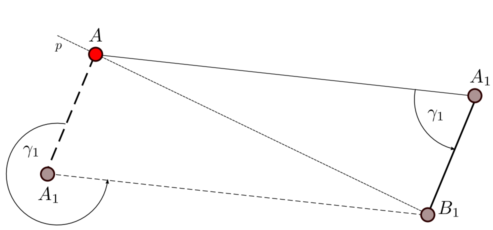
If we define the line p given by the vector AB_1 in slope intercept form $y = Kx$, where $K = \frac{AB_{1y}}{AB_{1x}}$. The angle γ_1 is computed by the Algorithm 1. Possible configurations of γ_1 are illustrated in Fig. 4.



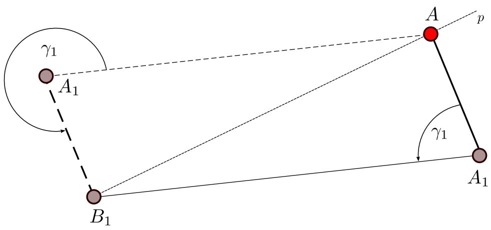
(a) Possibility 1



(b) Possibility 2



(c) Possibility 3



(d) Possibility 4

Fig. 4: Possible configurations of γ_1

♦ **Algorithm 1 (Computation of γ_1)**

1. For $AB_{1x} > 0$ and $AB_{1y} > 0$ (Possibility 1)
   if $[A_{1y} - A_y] - K \cdot [A_{1x} - A_x] > 0$
   then: $\gamma_1 = \arccos \frac{-l^2_{AB_1} + l^2_{A_1} + l^2_{A_1B_1}}{2 l_{A_1} l_{A_1B_1}}$, **else:** $\gamma_1 = 2\pi - \arccos \frac{-l^2_{AB_1} + l^2_{A_1} + l^2_{A_1B_1}}{2 l_{A_1} l_{A_1B_1}}$

2. For $AB_{1x} < 0$ and $AB_{1y} > 0$ (Possibility 2)
   if $[A_{1y} - A_y] - K \cdot [A_{1x} - A_x] > 0$
   then: $\gamma_1 = 2\pi - \arccos \frac{-l^2_{AB_1} + l^2_{A_1} + l^2_{A_1B_1}}{2 l_{A_1} l_{A_1B_1}}$, **else:** $\gamma_1 = \arccos \frac{-l^2_{AB_1} + l^2_{A_1} + l^2_{A_1B_1}}{2 l_{A_1} l_{A_1B_1}}$

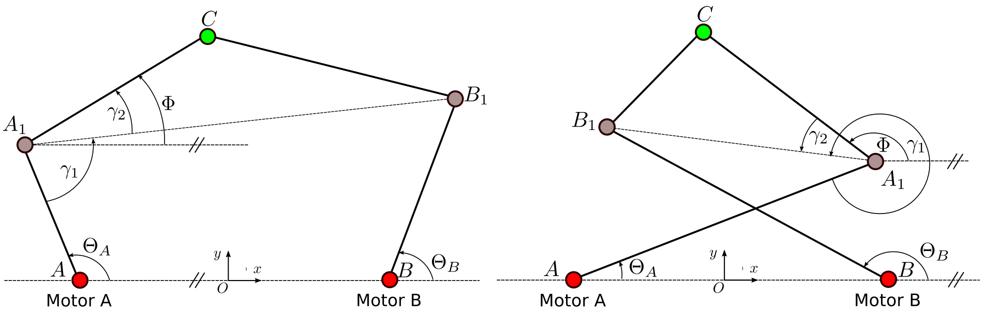
3. For $AB_{1x} > 0$ and $AB_{1y} < 0$ (Possibility 3)
   if $[A_{1y} - A_y] - K \cdot [A_{1x} - A_x] > 0$
   then: $\gamma_1 = \arccos \frac{-l^2_{AB_1}+l^2_{A_1}+l^2_{A_1B_1}}{2l_{A_1}l_{A_1B_1}}$, **else:** $\gamma_1 = 2\pi - \arccos \frac{-l^2_{AB_1}+l^2_{A_1}+l^2_{A_1B_1}}{2l_{A_1}l_{A_1B_1}}$
4. For $AB_{1x} < 0$ and $AB_{1y} < 0$ (Possibility 4)
   if $[A_{1y} - A_y] - K \cdot [A_{1x} - A_x] > 0$
   then: $\gamma_1 = 2\pi - \arccos \frac{-l^2_{AB_1}+l^2_{A_1}+l^2_{A_1B_1}}{2l_{A_1}l_{A_1B_1}}$, **else:** $\gamma_1 = \arccos \frac{-l^2_{AB_1}+l^2_{A_1}+l^2_{A_1B_1}}{2l_{A_1}l_{A_1B_1}}$

♦

The angle γ_2 is given by

$$\gamma_2 = \arccos \frac{-l^2_{B_2} + l^2_{A_2} + l^2_{A_1B_1}}{2l_{A_2}l_{A_1B_1}} \tag{11}$$

The angle Φ is computed by the Algorithm 2. Possible configurations of Φ are illustrated in Fig. 5.



(a) Possibility 1 (b) Possibility 2

Fig. 5: Possible configurations of Φ

♦ **Algorithm 2 (Computation of Φ)**

1. **if** $A_{1x} < B_{1x}$ (Possibility 1)
   then: $\Phi = \Theta_A - \pi + (\gamma_1 + \gamma_2)$
   else: $\Phi = \Theta_A - \pi + (\gamma_1 - \gamma_2)$

♦

The final direct kinematic mapping consist of the following steps:

Compute γ_1 (Algorithm 1) → compute γ_2 (11) → compute Φ (Algorithm 2) → compute position C of the end effector (8).

1.3 Differential kinematics

The differential kinematic mapping gives the relationship between the actuated joint velocities $\dot{\Theta} = \begin{bmatrix} \dot{\Theta}_A & \dot{\Theta}_B \end{bmatrix}^T$ and the generalized end effector velocities $\dot{C} = \begin{bmatrix} \dot{C}_x & \dot{C}_y \end{bmatrix}^T$.

The velocity vectors $\dot{AA_1}$, $\dot{BB_1}$ of the points A_1, B_1 respectively can be considered as

$$\dot{AA_1} = E \cdot AA_1 \cdot \dot{\Theta}_A, \quad \dot{BB_1} = E \cdot BB_1 \cdot \dot{\Theta}_B, \tag{12}$$

where $E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

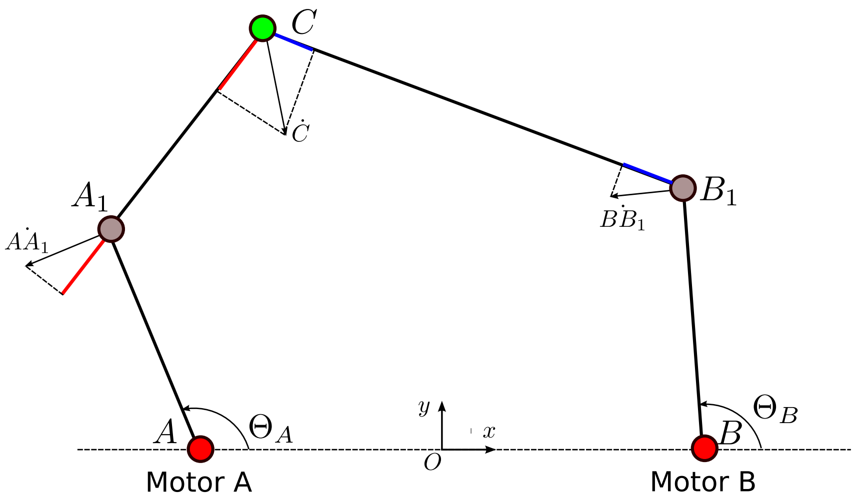


Fig. 6: Differential kinematics

Then, for given position C of the end effector, the projection[2] of the velocity vectors $A\dot{A}_1$ and $\dot{C}$ on the direction A_1C have to be equal. Analogously it holds for the vectors $B\dot{B}_1$, $\dot{C}$ and the direction B_1C, see Fig. 6.

$$\begin{aligned} \left(A\dot{A}_1\,,\,A_1C\right) &= \left(\dot{C}\,,\,A_1C\right) \\ \left(B\dot{B}_1\,,\,B_1C\right) &= \left(\dot{C}\,,\,B_1C\right) \end{aligned} \tag{13}$$

Substituting (12) into (13) we get the differential kinematic mapping

$$\begin{aligned} A(C)\cdot\dot{\Theta} &= B(C)\cdot\dot{C} \\ \dot{\Theta} &= \underbrace{A^{-1}(C)\cdot B(C)}_{J^{-1}(C)}\cdot\dot{C}, \end{aligned} \tag{14}$$

where

$$A(C) = \begin{bmatrix} (E\cdot AA_1\,,\,A_1C) & 0 \\ 0 & (E\cdot BB_1\,,\,B_1C) \end{bmatrix}, \quad B(C) = \begin{bmatrix} A_1C^T \\ B_1C^T \end{bmatrix} \tag{15}$$

The matrix $J^{-1}(C)$ is so called *inverse jacobian*. By applying of the *principle of virtual work* it can be proven that the relationship between the motors torque $M = \begin{bmatrix} M_A & M_B \end{bmatrix}^T$ and the end effector force $F = \begin{bmatrix} F_x & F_y \end{bmatrix}^T$ is

$$M = J^T(C)\cdot F \tag{16}$$

1.4 Optimization of the robot dimensions

The requirements on the robot's end effector (point C) are listed in Tab 1. Three variants of the robot are discussed further.

[2] $(x\,,\,y)$ denotes a *dot product* of the vector x and y.

Maximal force in the x axis	400	N
Maximal force in the y axis	150	N
Maximal speed in the x axis	2	m/s
Maximal speed in the y axis	1	m/s
Maximal movement range x	300+50	mm
Maximal movement range y	150+20	mm

Tab. 1: Requirements on the robot end effector

Required reachable area W (rectangle 350×170 mm, see tab. 1) is placed in the robot workspace[3] W_{robot} so that for the criterion $K(C)$

$$K(C) = 1/\mathrm{cond}\{J(C)\} = 1/\mathrm{cond}\left\{J(C)^{-T}\right\} \in \langle 0,\, 1 \rangle$$

holds

$$W_{opt} = \arg\max_{W \in W_{robot}} \left[\min_{C \in W} [K(C)]\right]$$

The optimization was always performed for one set of robot dimensions ($l_{A_1} = l_{B_1}$, $l_{A_2} = l_{B_2}$, l_0) from a bounded set of admissible dimensions and the robot variant with the workspace maximizing $\min_{C \in W_{opt}} [K(C)]$ was chosen.

The results of the optimization process for three variants (A, B, C) of the 2DoF planar parallel robot is shown below.

[3]The area where the robot end effector can move.

Variant A

Robot dimensions and requirements on the motors reflecting the optimal force-speed ratios between motors and robot's end effector:

Robot dimensions		
l_0	203	mm
$l_{A_1} = l_{B_1}$	185	mm
$l_{A_2} = l_{B_2}$	300	mm

Requirements on the motors		
Max. motor torque **A**	83	Nm
Max. motor torque **B**	83	Nm
Max. motor speed **A**	191	$ot./min.$
Max. motor speed **B**	191	$ot./min.$

Tab. 2: Robot dimensions and requirements on the motors (variant A)

The robot workspace and the optimal placement of the required reachable area (see Tab. 1) and the physical robot workspace for the end effector movement restricted to the required reachable area.

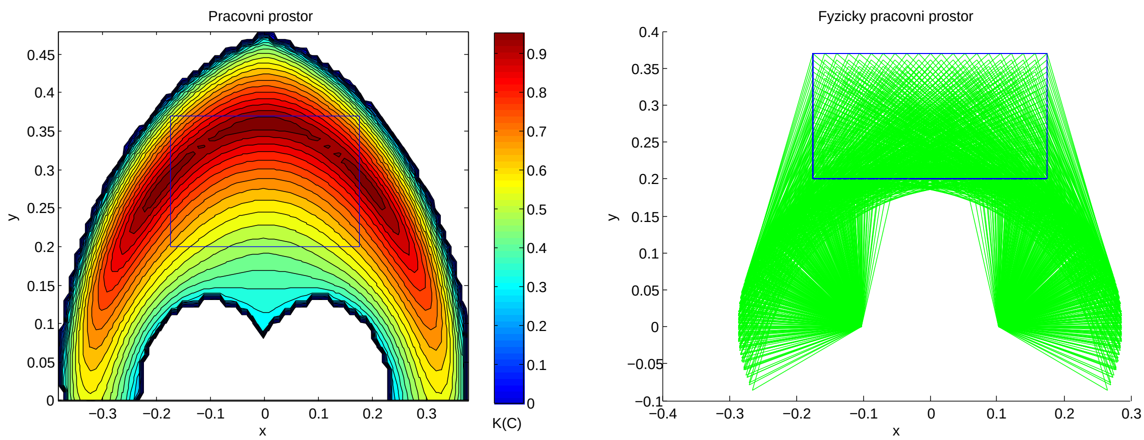


Fig. 7: Robot workspace (left) and physical robot workspace. Blue rectangle - required reachable area. (variant A)

Variant B

Robot dimensions and requirements on the motors reflecting the optimal force-speed ratios between motors and robot's end effector:

Robot dimensions		
l_0	208	mm
$l_{A_1} = l_{B_1}$	395	mm
$l_{A_2} = l_{B_2}$	276	mm

Requirements on the motors		
Max. motor torque **A**	183	Nm
Max. motor torque **B**	183	Nm
Max. motor speed **A**	67	$ot./min.$
Max. motor speed **B**	67	$ot./min.$

Tab. 3: Robot dimensions and requirements on the motors (variant B)

The robot workspace and the optimal placement of the required reachable area (see Tab. 1) and the physical robot workspace for the end effector movement restricted to the required reachable area.

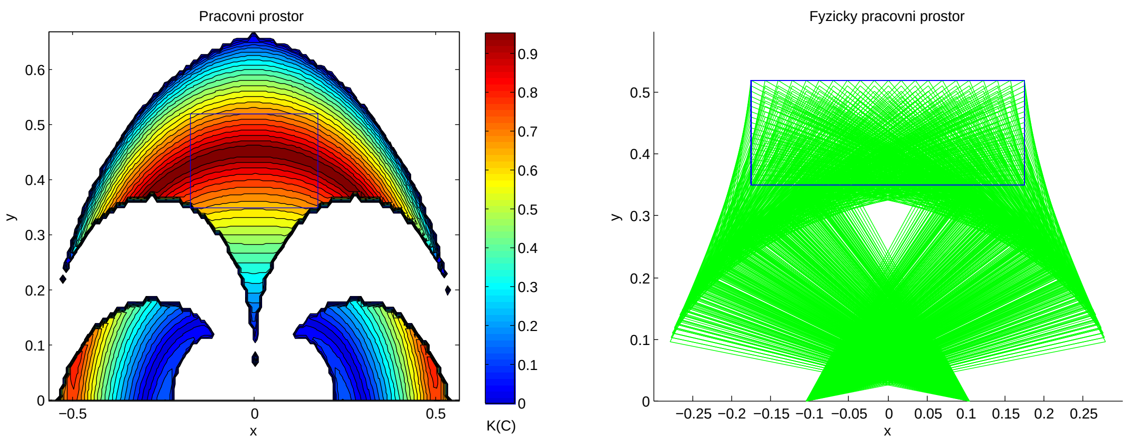


Fig. 8: Robot workspace (left) and physical robot workspace. Blue rectangle - required reachable area. (variant B)

Variant C

Robot dimensions and requirements on the motors reflecting the optimal force-speed ratios between motors and robot's end effector:

Robot dimensions		
l_0	379	mm
$l_{A_1} = l_{B_1}$	216	mm
$l_{A_2} = l_{B_2}$	290	mm

Requirements on the motors		
Max. motor torque **A**	87	Nm
Max. motor torque **B**	120	Nm
Max. motor speed **A**	277	$ot./min.$
Max. motor speed **B**	258	$ot./min.$

Tab. 4: Robot dimensions and requirements on the motors (variant C)

The robot workspace and the optimal placement of the required reachable area (see Tab. 1) and the physical robot workspace for the end effector movement restricted to the required reachable area.

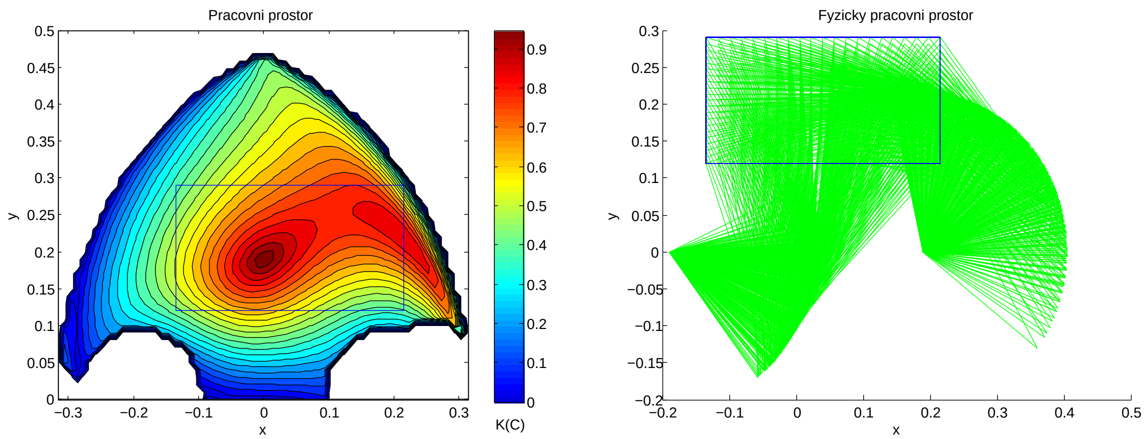


Fig. 9: Robot workspace (left) and physical robot workspace. Blue rectangle - required reachable area. (variant C)

2 2 DoF spatial parallel robot

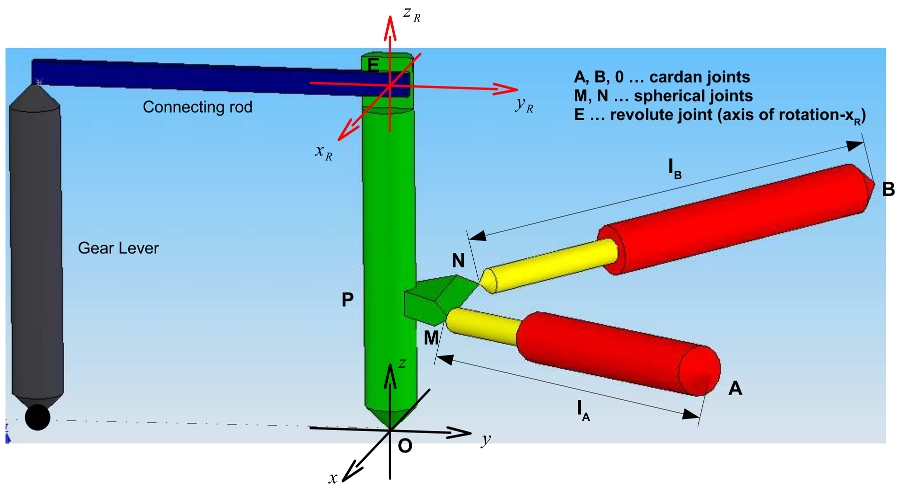


Fig. 10: Gear robot layout

The *Gear robot*, see Fig. 10 consists of two active legs, which are attached to the base at the points O, A, B. From a kinematic viewpoint, the legs are represented by the prismatic joints (linear actuators).

We denote the length of the active legs as the actuated joint coordinates

$$\Theta = [l_A \; l_B]^T \tag{17}$$

And the position of the end effector as the generalized coordinates

$$E = [E_x \; E_y \; E_z]^T \tag{18}$$

We suppose robot dimensions, see Fig. 11.

$$l_1 = 0.05\,m,\; l_2 = 0.35\,m,\; l_3 = 0.5\,m,\; l_4 = 0.4\,m,\; l_5 = 1\,m \tag{19}$$

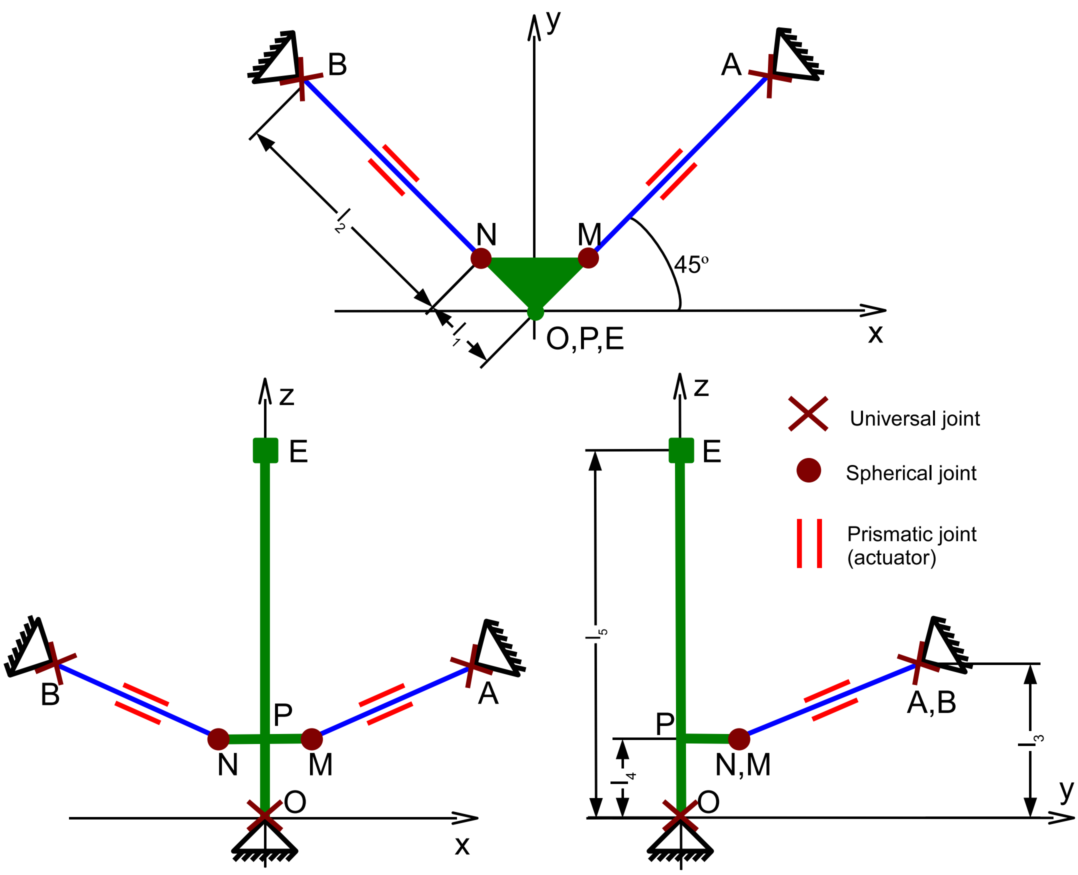


Fig. 11: Gear robot (dimensions)

2.1 Inverse kinematics

The relations giving the actuated joint coordinates for given generalized coordinates are called the *inverse kinematics.* Note that the end effector can never leave sphere surface with radius $L = |OE|$.

Therefore

$$E_z = \sqrt{L^2 - E_x^2 - E_y^2} \quad \text{and} \quad L^2 \geq E_x^2 + E_y^2 \tag{20}$$

It holds for actuated joint coordinates

$$\begin{aligned} l_A &= \|AO + OE + R \cdot EM_R\| \\ l_B &= \|BO + OE + R \cdot EN_R\| \end{aligned} \tag{21}$$

The vectors EM_R, EN_R are (from the dimensions of the gear robot) the known coordinate vectors with respect to the moving frame $E - x_R y_R z_R$ and R is the rotation matrix from the moving frame to the reference frame $O - xyz$. The rotation matrix R (22) consists of the elementary rotations (23), (24), which depend on the given generalized coordinates E. The rotation of the moving frame with respect to the reference frame is described by the XY Euler angles α, β,

$$R(E) = R_x(E) \cdot R_{y'}(E), \tag{22}$$

where

$$R_x(E) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}; \quad \alpha = \arctan\frac{E_y}{-E_z} \tag{23}$$

$$R_{y'}(E) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}; \quad \beta = \arctan\frac{E'_x}{E'_z} \tag{24}$$

(25)

$$E' = \begin{bmatrix} E'_x & E'_y & E'_z \end{bmatrix}^T = R_x^T(E) \cdot \begin{bmatrix} E_x & E_y & E_z \end{bmatrix}^T \tag{26}$$

The inverse kinematics[4] may be written as

$$\Theta = G(E) \tag{27}$$

2.2 Differential kinematics

The differential kinematic mapping gives the relationship between the actuated joint velocities $\dot{\Theta} = \begin{bmatrix} \dot{l}_A & \dot{l}_B \end{bmatrix}^T$ and the generalized end effector velocities $\dot{E} = \begin{bmatrix} \dot{E}_x & \dot{E}_y \end{bmatrix}^T$. Note that by differentiating (20) with respect to time we get

$$\dot{E}_z = \begin{bmatrix} \frac{-E_x}{\sqrt{L^2 - E_x^2 - E_y^2}} & \frac{-E_y}{\sqrt{L^2 - E_x^2 - E_y^2}} \end{bmatrix} \cdot \begin{bmatrix} \dot{E}_x \\ \dot{E}_y \end{bmatrix} \tag{28}$$

The differential kinematic mapping may be formulated as

$$\dot{\Theta} = \underbrace{\frac{\partial G(E)}{\partial E}}_{J^{-1}(E)} \cdot \dot{E}, \tag{29}$$

where the matrix $J^{-1}(E)$ is an *inverse jacobian.*

Due to complexity of the kinematic constraints (27), the partial derivatives in (29) have to be solved by the help of a computation software and lead to a very long terms which make this approach practically unusable. Therefore we suppose only the estimation of the inverse jacobian $\hat{J}^{-1}(E)$ as

$$\hat{J}^{-1}(E) = \begin{bmatrix} \mathbb{C}_1 & \mathbb{C}_2 \end{bmatrix}, \tag{30}$$

[4]The mapping $E \mapsto \Theta$ is often called *kinematic constraints.*

where $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$ are columns of the inverse jacobian matrix

$$\begin{aligned} \mathbb{C}_1 &= \Big(G(E+\left[\begin{array}{ccc} h & 0 & 0 \end{array}\right]^T) - G(E)\Big)/h \\ \mathbb{C}_2 &= \Big(G(E+\left[\begin{array}{ccc} 0 & h & 0 \end{array}\right]^T) - G(E)\Big)/h \end{aligned}$$

Parameter h is chosen according to required accuracy of the estimation.

2.3 Direct kinematics

The relations giving the position of the end effector for given actuated joint coordinates is called the *direct kinematics*. Note that the kinematic constraints given by (27) is complicated non-linear function and in principle it is hard to find inverse function $G^{-1}(\Theta)$ analytically.

Otherwise, we can use the kinematic constraints from (27). Suppose the difference e between the measured actuated joint coordinates Θ_m and recomputed actuated joint coordinates Θ from computed (estimated) generalized coordinates E.

Let

$$e = \Theta_m - \Theta = \Theta_m - G(E) \tag{31}$$

be the expression of such difference. The time derivative of error (31) is:

$$\dot{e} = \dot{\Theta}_m - J^{-1}(E) \cdot \dot{E} \tag{32}$$

Where $J^{-1}(E) = \frac{\partial G(E)}{\partial E}|_E$ denote as a inverse Jacobian in the terminology of the parallel robots.

The relation (32) between the generalized velocities $\dot{E}$ and the actuated joint velocities $\dot{\Theta}$ gives a differential equation, which describes difference evolution over time. Now, it is necessary to choose a relation between $\dot{e}$ and E that ensures convergence of the difference to zero. Assume a choice:

$$\dot{e} = \dot{\Theta}_m - J^{-1}(E) \cdot \dot{E} \stackrel{!}{=} -K\left[\Theta_m - G(E))\right] = -K \cdot e \tag{33}$$

that leads to linear system

$$\dot{e} + Ke = 0 \tag{34}$$

If K is a positive definite matrix, the linear system (34) is asymptotically stable. Consequently, the difference e converges to zero, the recomputed actuated joint coordinates Θ converge to the measurement joint coordinates Θ_m and the computed generalized coordinates E converge to the actual position of the end effector. Suppose that the inverse jacobian $J^{-1}(E)$ is nonsingular for all positions E of the end effector through the whole workspace. It means that the robot has not any parallel singularities in the workspace. Consequently, final dynamic system solving the direct kinematics can be written from (33):

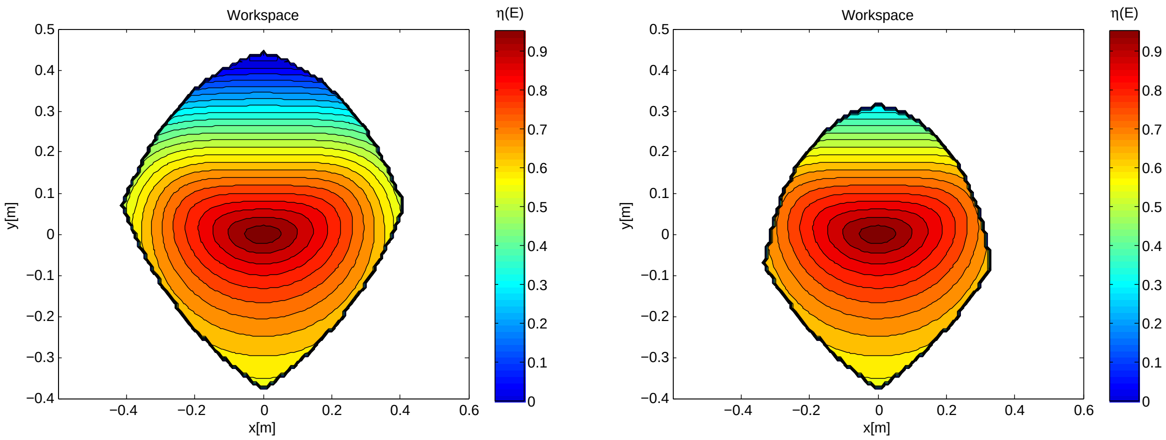
$$\dot{E} = J(E)\left[\dot{\Theta}_m + K\left[\Theta_m - G(E)\right]\right] \tag{35}$$

2.4 Workspace

We can say that the end effector position E of the robot belongs to the workspace if

- the extension of the actuators l_A, l_B lie within a given interval $\langle l_{min}\, l_{max}\rangle$
- the slope angle at the universal joints (O, A, B,) and spherical joints (M, N) is smaller or equal γ

Fig. 12 shows workspace of the 2DoF spatial parallel robot for the dimensions (19). The local dexterity index is suppose to be $\eta(E) = 1/\texttt{cond}[\hat{J}^{-1}(E)]$.



(a) $\gamma = \pi/4,\ l_{min} = l_2 - l_2/3,\ l_{max} = l_2 + l_2/3$ (b) $\gamma = \pi/6,\ l_{min} = l_2 - l2/3,\ l_{max} + l2/3$

Fig. 12: Workspace of the 2DoF spatial parallel robot