ON THE LIMIT POINTS OF THE SEQUENCE $\{\sin n\}$

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Suppose that the points 1, 2, 3, $\cdots$ are "projected" onto the y-axis by the graph of $y=\sin x$. (See Figure 1.) Then it is intuitively evident that no subinterval of $[-1, 1]$ will elude these particles. Or, more precisely, every point in the interval $[-1, 1]$ is a limit point of the sequence $\{\sin n\}$.

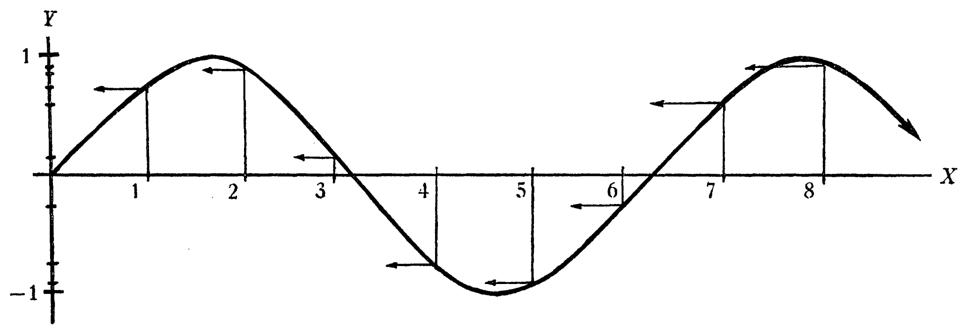

FIG. 1.

This proposition is true. Indeed, it is just one of a family of analogous propositions. And, as is frequently the case with such families, one member is easier to prove than all the others. Our plan here is to first prove this "easy" case and then show how that result can be extended.

We introduce the following notation: $(x)=x-[x]$, where $[x]$ is the greatest integer in x. (Evidently, $0\leqq(x)<1$.) We shall show that every point in $[0, 1]$ is a limit point for the sequence $\{(n\alpha)\}$, provided that α is irrational. But first, we need to know certain properties of the function (x); they arise as corollaries to the following theorem.

THEOREM 1.

$$(x+y)=\begin{cases}(x)+(y), \text{ if } (x)+(y)<1\\(x)+(y)-1, \text{ if } (x)+(y)\geqq 1.\end{cases}$$

Proof. It is immediate from the definition of (x) that $0\leqq(x)+(y)<2$. We suppose first that

$$0\leqq(x)+(y)<1.$$

Equivalently, we have

$$-x-y\leqq-[x]-[y]<1-x-y,$$
$$x+y-1<[x]+[y]\leqq x+y.$$

It follows that $[x]+[y]=[x+y]$. Therefore

$$(x+y)=x+y-[x+y]=x-[x]+y-[y]=(x)+(y).$$

The alternate result is derived in identical fashion, starting with $1\leqq(x)+(y)<2$.

PROPERTY 1. *For* $x \neq$ *integer*, $(-x) = 1 - (x)$.

Proof. Writing $-x$ for y, we have

$$(0) = (x) + (-x) + \{0 \text{ or } -1\}.$$

But (x) and $(-x)$ are both positive.

PROPERTY 2. *If* $(z) > (x)$, *then* $(z-x) = (z) - (x)$.

Proof. Writing $(z-x)$ for y, we have

$$(z) = (x) + (z - x) + \{0 \text{ or } -1\}.$$

But $(z) - (x) = (z-x) - 1 < 0$ denies our hypothesis.

PROPERTY 3. *If* $n(x) < 1$, *where* n *is a natural number, then* $(nx) = n(x)$.

Proof. By induction.

LEMMA. *Given* $\epsilon > 0$ *and an irrational number* α, *there exists a natural number* n *such that* $(n\alpha) < \epsilon$.

Proof. (We may assume that $\epsilon < 1$.) Choose N such that $N > 1/\epsilon$, and consider the set

$$R = \{(\alpha), (2\alpha), (3\alpha), \cdots, (N\alpha)\}.$$

Letting $b = \max R$, we see that R partitions $[0, b]$ into N subintervals. Moreover, the smallest such subinterval must be of length not exceeding b/N. In other words, there exist distinct nonnegative integers k and j such that

$$0 < (k\alpha) - (j\alpha) \leqq b/N < 1/N < \epsilon.$$

(If the smallest subinterval is to the far left, we take $k = 1$ and $j = 0$.) It follows, using Property 2, that

$$0 < (\{k - j\}\alpha) < \epsilon.$$

Now it may happen that $k - j < 0$. (Otherwise, we are finished.) In this event we let $-m = k - j$ and appeal to the following argument: Since $(-m\alpha) < \epsilon$, we may write $-m\alpha = [-m\alpha] + (m\alpha)$, thus,

$$- m\alpha = \text{negative integer} + \epsilon^*,$$

where $0 < \epsilon^* < \epsilon < 1$. Next, multiply both sides by p, where p is the largest natural number such that $p\epsilon^* < 1$. We obtain

$$- pm\alpha = \text{negative integer} + p\epsilon^*.$$

Thus, $(-pm\alpha) = p\epsilon^*$. And also, by our choice of p, we are assured that $0 < 1 - p\epsilon^* < \epsilon^*$. Therefore,

$$0 < 1 - (- pm\alpha) < \epsilon^* < \epsilon.$$

Finally, applying Property 1, we have

$$0 < (pm\alpha) < \epsilon.$$

Thus, we may take $n=pm$.

THEOREM 2. *Let α be irrational. If u is in $[0, 1]$, then u is a limit point for the sequence $\{(n\alpha)\}$.*

Proof. We take u in $(0, 1]$ and immediately apply our lemma: Given $\epsilon>0$, but less than u, choose a natural number k such that $(k\alpha)<\epsilon$. Then take j as the natural number for which

$$j(k\alpha) \leqq u < j(k\alpha) + (k\alpha).$$

It follows that

$$0 \leqq u - j(k\alpha) < (k\alpha) < \epsilon.$$

Since $u<1$, these inequalities imply that $j(k\alpha)<1$. Thus Property 3 is applicable; we may write

$$0 \leqq u - (jk\alpha) < \epsilon.$$

Finally, take $n=jk$.

Example. Every point in the interval $[-1, 1]$ is a limit point for the sequence $\{\sin n\}$.

Proof. Let b belong to $[-1, 1]$. Choose c from $[0, 2\pi]$ such that $\sin c=b$. Then, given $\epsilon>0$, choose $\delta>0$ such that

$$|\sin x - b| < \epsilon \quad \text{for } |x - c| < \delta.$$

We now apply Theorem 2, taking $\alpha=1/2\pi$ and $u=c/2\pi$; there exists a natural number n such that

$$0 < \frac{c}{2\pi} - \left(\frac{n}{2\pi}\right) < \frac{\delta}{2\pi} \cdot$$

Or, $0<c-2\pi(n/2\pi)<\delta$. It follows that

$$|\sin\{2\pi(n/2\pi)\} - b| < \epsilon.$$

But

$$2\pi\left(\frac{n}{2\pi}\right) = 2\pi\left\{\frac{n}{2\pi} - \left[\frac{n}{2\pi}\right]\right\} = n - 2k\pi.$$

Thus, we have $|\sin n-b|<\epsilon$.

EXERCISE 1. Determine the radius of convergence of $\sum(\sin n)x^n$.

EXERCISE 2. Prove: Given any real number u, there exists an increasing sequence of natural numbers, say $\{n_k\}$, such that $\{\tan n_k\}\to u$.

EXERCISE 3. Prove: If f is a piecewise continuous, periodic function having an irrational period, then $\{f(n)\}$ is dense in the range of f.

Remark. Theorem 2 states that the sequence $\{(n\alpha)\}$, α irrational, is dense in

[0, 1]. It is of further interest that this sequence is not "more dense" in some parts of the interval than in others. That is, the points of $\{(n\alpha)\}$ are distributed in [0, 1] in such a manner that the following result holds: Let I be any subinterval of [0, 1], $I_n = \{(\alpha), (2\alpha), \cdots, (n\alpha)\}$, and N_n be the number of elements in $I \cap I_n$. Then $\{N_n/n\} \to$ length of I. We say that $\{(m\alpha)\}$ is *uniformly distributed* in $[-1, 1]$. Two proofs of this deeper result can be found in [1]. On the other hand, due to the nonlinearity of the sine function, the sequence $\{\sin n\}$ is not uniformly distributed in $[-1, 1]$.

Reference

1. Ivan Niven, Irrational Numbers, Carus Monograph 11, The Mathematical Association of America, 1956.

ON THE AUTOMORPHISMS OF THE COMPLEX NUMBER FIELD

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P. B. Yale remarks in [2] that he has not seen a proof of the fact that the complex number field has $2^{2^{\aleph_0}}$ automorphisms. We give below a proof of the same.

THEOREM. *The complex number field C has $2^{2^{\aleph_0}}$ automorphisms.*

Proof. It is well known that the complex number field has $c(=2^{\aleph_0})$ elements. If B is any transcendence base for C over the field Q of rationals, then B must also have cardinal c since the cardinal of C is the same as the cardinal of $Q(B)$ [1, p. 143]. We first show that the number of automorphisms of C is $\geqq 2^c$ by associating with each subset of B an automorphism. Let S be any subset of B. Consider the field $Q(B)$ over the field $Q(S)$. The set $\{-x \mid x \in B \sim S\}$ is also a set of algebraically independent elements for $Q(B)$ over $Q(S)$ and so the map $x \in B \sim S \to -x$ yields an automorphism of $Q(B)$ leaving S fixed. This automorphism of $Q(B)$ which leaves S and all x^2, $x \in B \sim S$ fixed can be extended to an automorphism of the field C. (See Theorem 7 of [2].) Let us denote this automorphism by S_σ. Thus for each subset $S \subset B$ we get an automorphism S_σ. Obviously if $S \neq S'$, $S_\sigma \neq S'_\sigma$, there are at least as many automorphisms as there are subsets of B. Hence the number of automorphisms is $\geqq 2^c$. To show the other inequality we note first that the set of all automorphisms is a subset of the set of all mappings of C into C, i.e., c^c. But this has cardinality c^c which is equal to 2^c. Thus our result follows.

Regarding his comment 1, we may note that Jacobson [1, p. 157] remarks that if B is any transcendence base of C over Q then any 1-1 surjective mapping of B can be extended to an automorphism of C.

References

1. Nathan Jacobson, Lectures in Abstract Algebra III, Theory of Fields and Galois Theory, Van Nostrand, Princeton, 1964.
2. P. B. Yale, Automorphisms of the complex numbers, this MAGAZINE, 39 (1966) 135–141.