FAKULTA APLIKOVANÝCH VĚD ZÁPADOČESKÉ UNIVERZITY V PLZNI

www.KMA.zcu.cz SINCE 1954

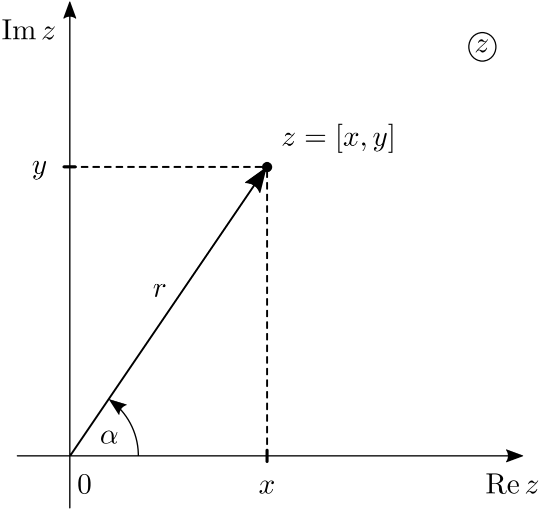
Komplexní čísla, komplexní funkce a Riemannova hypotéza

Petr Nečesal

Katedra matematiky FAV ZČU

pnecesal@kma.zcu.cz

Komplexní čísla a jejich reprezentace



- algebraický tvar

$$z = x + \mathrm{i} \cdot y$$

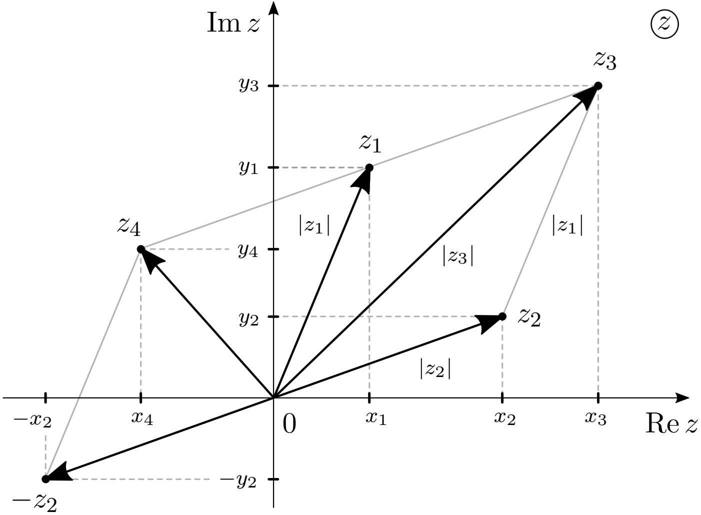
- goniometrický tvar ($z \neq 0$)

$$z = r\,(\cos\alpha + \mathrm{i} \cdot \sin\alpha)$$

- exponenciální tvar ($z \neq 0$)

$$z = r\,\mathrm{e}^{\mathrm{i} \cdot \alpha}$$

Sčítání a odečítání komplexních čísel



$$z_1 = x_1 + \mathrm{i} \cdot y_1$$
$$z_2 = x_2 + \mathrm{i} \cdot y_2$$

- součet $z_3 = z_1 + z_2$

  $$z_3 = (x_1+x_2)+\mathrm{i} \cdot (y_1+y_2)$$

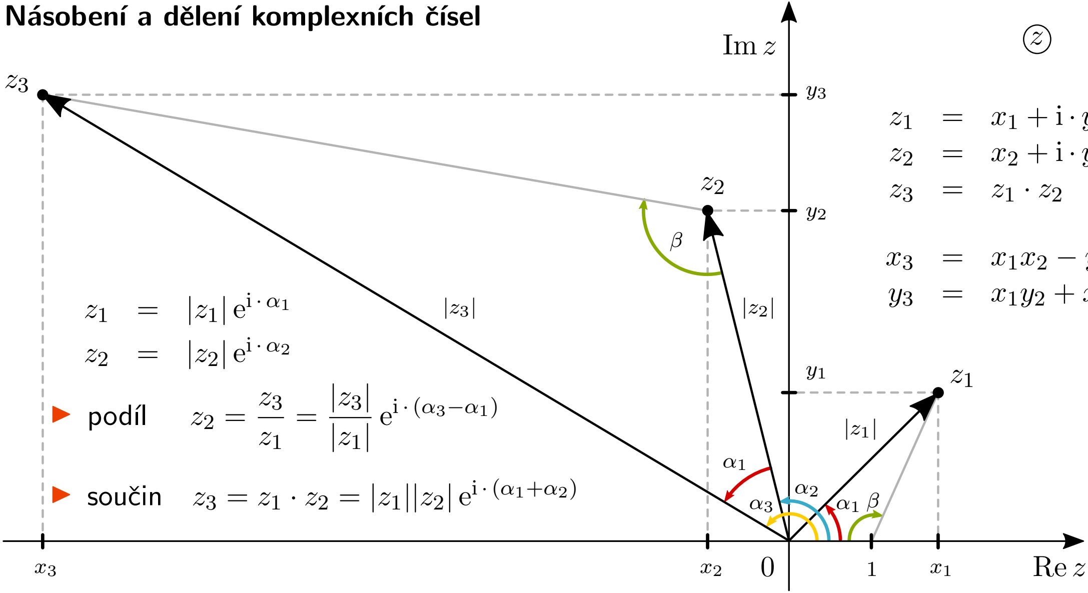
- rozdíl $z_4 = z_1 - z_2$

  $$z_4 = (x_1-x_2)+\mathrm{i} \cdot (y_1-y_2)$$

- trojúhelníková nerovnost

  $$|z_3| = |z_1+z_2| \leq |z_1|+|z_2|$$

Násobení a dělení komplexních čísel



$$\begin{aligned} z_1 &= x_1 + \mathrm{i} \cdot y_1 \\ z_2 &= x_2 + \mathrm{i} \cdot y_2 \\ z_3 &= z_1 \cdot z_2 \end{aligned}$$

$$\begin{aligned} x_3 &= x_1 x_2 - y_1 y_2 \\ y_3 &= x_1 y_2 + x_2 y_1 \end{aligned}$$

$$\begin{aligned} z_1 &= |z_1|\, \mathrm{e}^{\mathrm{i} \cdot \alpha_1} \\ z_2 &= |z_2|\, \mathrm{e}^{\mathrm{i} \cdot \alpha_2} \end{aligned}$$

- podíl $z_2 = \dfrac{z_3}{z_1} = \dfrac{|z_3|}{|z_1|}\, \mathrm{e}^{\mathrm{i} \cdot (\alpha_3 - \alpha_1)}$
- součin $z_3 = z_1 \cdot z_2 = |z_1||z_2|\, \mathrm{e}^{\mathrm{i} \cdot (\alpha_1 + \alpha_2)}$

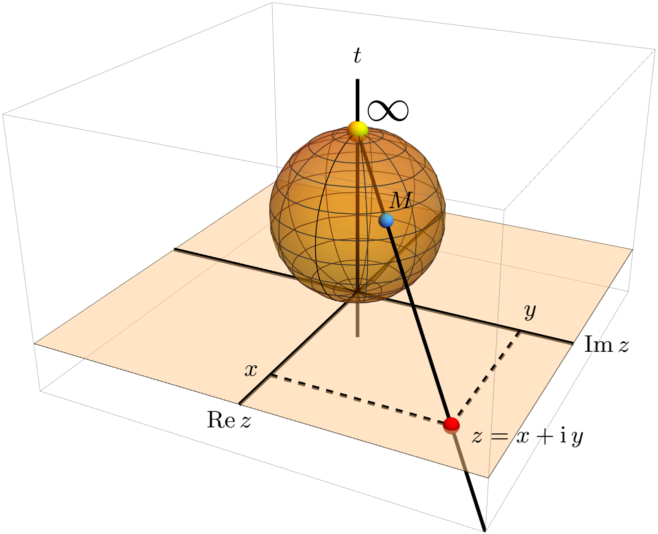
Komplexní čísla nelze rozumně uspořádat

- Pro libovolná reálná čísla a, b, c mimo jiné platí:

  1. $a < b \quad \vee \quad a = b \quad \vee \quad b < a$ (nastává právě jedna z možností),
  2. $a < b \quad \wedge \quad b < c \quad \Rightarrow \quad a < c,$
  3. $a < b \quad \Rightarrow \quad a + c < b + c,$
  4. $a < b \quad \wedge \quad 0 < c \quad \Rightarrow \quad a \cdot c < b \cdot c.$

- Předpokládejme, že lze komplexní čísla uspořádat tak, že body 1 – 4 platí pro všechna komplexní čísla a, b, c.
  - Pokud $0 < \mathrm{i}$, potom $0 < (-1)$, $0 < (-\mathrm{i})$, $\mathrm{i} < 0$, což je spor ↯.
  - Pokud $\mathrm{i} < 0$, potom $0 < (-\mathrm{i})$, $0 < (-1)$, $0 < \mathrm{i}$, což je spor ↯.

Rozšířený obor komplexních čísel $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$



- stereografická projekce

$$M = \left(\frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}, \frac{1}{2}\left(1 + \frac{x^2+y^2-1}{x^2+y^2+1}\right)\right)$$

- pravidla pro komplexní nekonečno

$$z \pm \infty = \infty, \qquad z \in \mathbb{C},$$

$$z \cdot \infty = \infty, \qquad z \in \mathbb{C}^* \setminus \{0\},$$

$$\frac{z}{\infty} = 0, \quad \frac{\infty}{z} = \infty, \qquad z \in \mathbb{C},$$

$$\frac{z}{0} = \infty, \qquad z \in \mathbb{C}^* \setminus \{0\}.$$

- neurčité výrazy

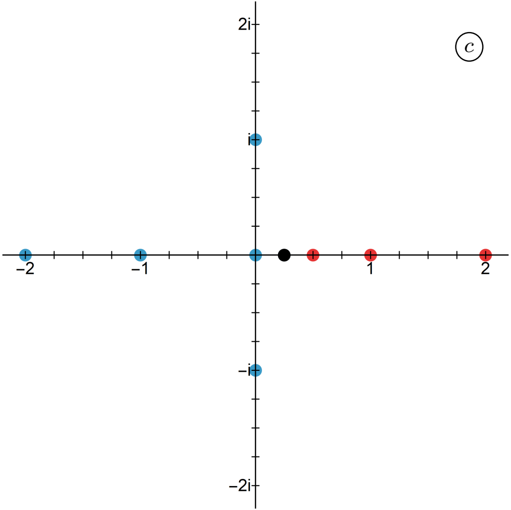
$$\infty \pm \infty, \quad 0 \cdot \infty, \quad \frac{\infty}{\infty}, \quad \frac{0}{0},$$

$$1^\infty, \quad 0^0, \quad \infty^0.$$

Mandelbrotova množina $M = \{c \in \mathbb{C} : \ (z_n) \text{ je omezená posloupnost}\}$

$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$

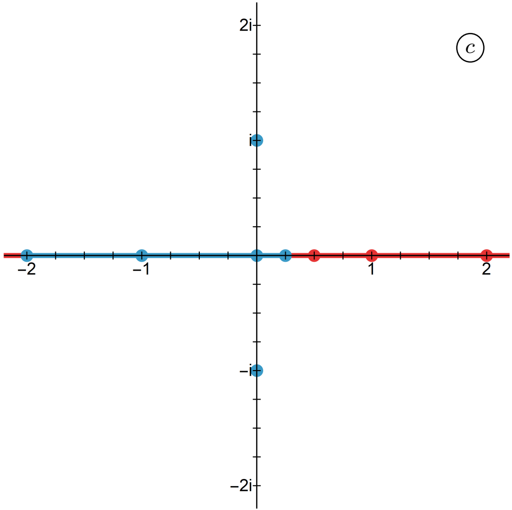
- $c = 0$: $(z_n) = (0,\ 0,\ 0,\ 0,\ \dots)$
- $c = -2$: $(z_n) = (-2,\ 2,\ 2,\ 2,\ \dots)$
- $c = -1$: $(z_n) = (-1,\ 0,\ -1,\ 0,\ -1,\ 0,\ \dots)$
- $c = +\mathrm{i}$: $(z_n) = (+\mathrm{i},\ -1+\mathrm{i},\ -\mathrm{i},\ -1+\mathrm{i},\ -\mathrm{i},\ \dots)$
- $c = -\mathrm{i}$: $(z_n) = (-\mathrm{i},\ -1-\mathrm{i},\ +\mathrm{i},\ -1-\mathrm{i},\ +\mathrm{i},\ \dots)$
- $c = +\frac{1}{4}$: $(z_n) = (0.25,\ 0.3125,\ 0.34765625,\ \dots)$
- $c = +\frac{1}{2}$: $(z_n) = (0.5,\ 0.75,\ 1.0625,\ 1.62890625,\ \dots)$
- $c = +1$: $(z_n) = (1,\ 2,\ 5,\ 26,\ 677,\ 458\,330,\ \dots)$
- $c = +2$: $(z_n) = (2,\ 6,\ 38,\ 1446,\ 2\,090\,918,\ \dots)$



Mandelbrotova množina $M = \{c \in \mathbb{C} : \ (z_n)$ je omezená posloupnost$\}$

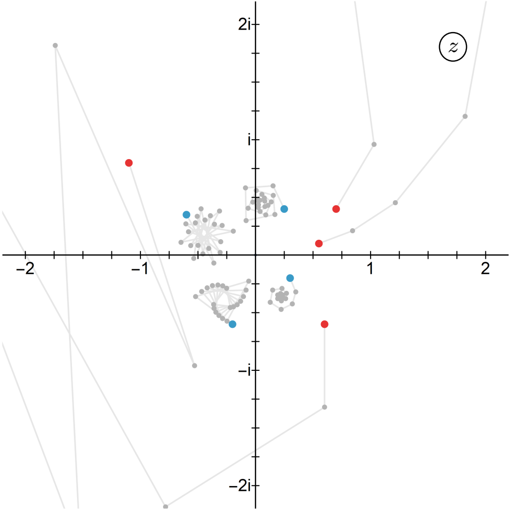
$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$

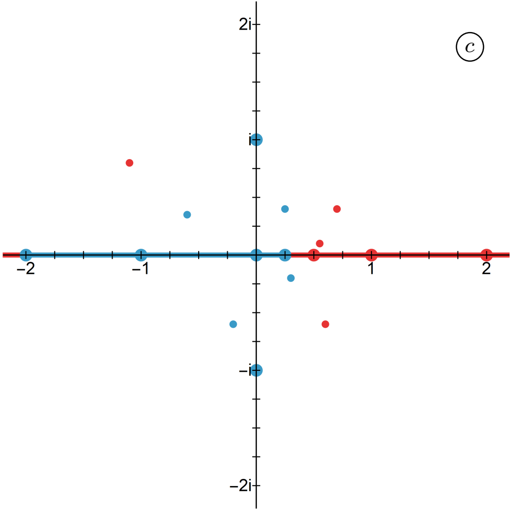
- $c = 0 \quad : \quad (z_n) = (0,\ 0,\ 0,\ 0,\ \ldots)$
- $c = -2 \quad : \quad (z_n) = (-2,\ 2,\ 2,\ 2,\ \ldots)$
- $c = -1 \quad : \quad (z_n) = (-1,\ 0,\ -1,\ 0,\ -1,\ 0,\ \ldots)$
- $c = +\mathrm{i} \quad : \quad (z_n) = (+\mathrm{i},\ -1+\mathrm{i},\ -\mathrm{i},\ -1+\mathrm{i},\ -\mathrm{i},\ \ldots)$
- $c = -\mathrm{i} \quad : \quad (z_n) = (-\mathrm{i},\ -1-\mathrm{i},\ +\mathrm{i},\ -1-\mathrm{i},\ +\mathrm{i},\ \ldots)$
- $c = +\frac{1}{4} \quad : \quad (z_n) = (0.25,\ 0.3125,\ 0.34765625,\ \ldots)$
- $c = +\frac{1}{2} \quad : \quad (z_n) = (0.5,\ 0.75,\ 1.0625,\ 1.62890625,\ \ldots)$
- $c = +1 \quad : \quad (z_n) = (1,\ 2,\ 5,\ 26,\ 677,\ 458\,330,\ \ldots)$
- $c = +2 \quad : \quad (z_n) = (2,\ 6,\ 38,\ 1446,\ 2\,090\,918,\ \ldots)$



Mandelbrotova množina $M = \{c \in \mathbb{C} : (z_n) \text{ je omezená posloupnost}\}$

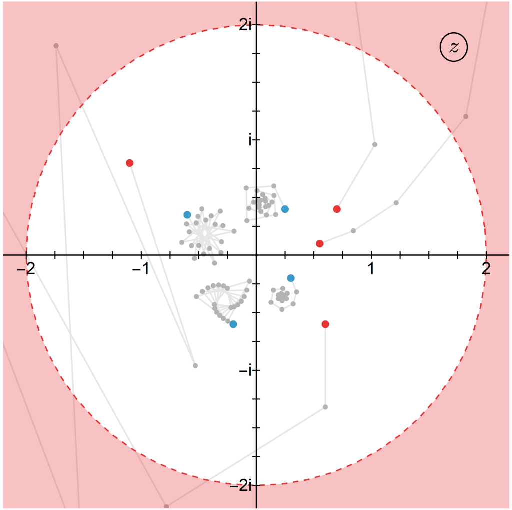
$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$

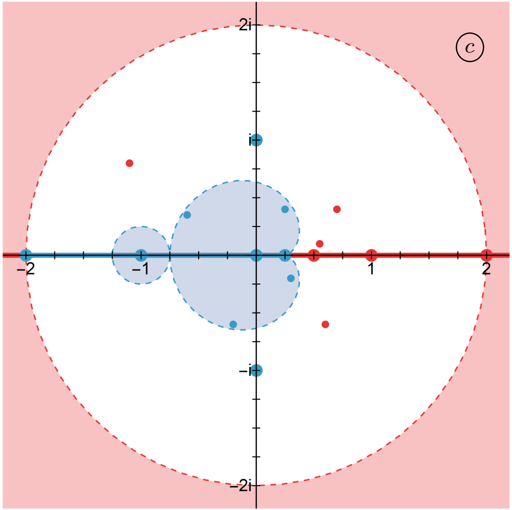




Mandelbrotova množina $M = \{c \in \mathbb{C} : (z_n)$ je omezená posloupnost$\}$

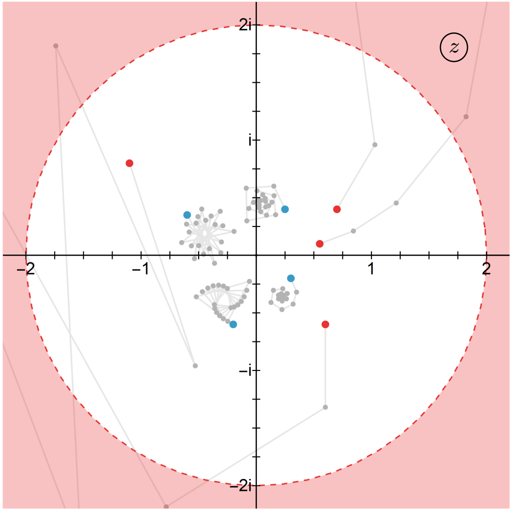
$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$

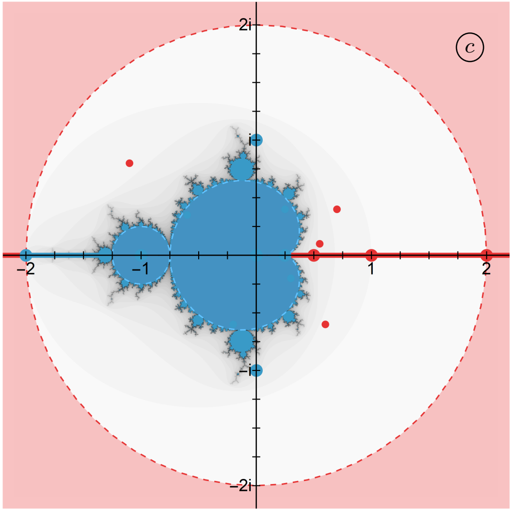




Mandelbrotova množina $M = \{c \in \mathbb{C} : (z_n)$ je omezená posloupnost$\}$

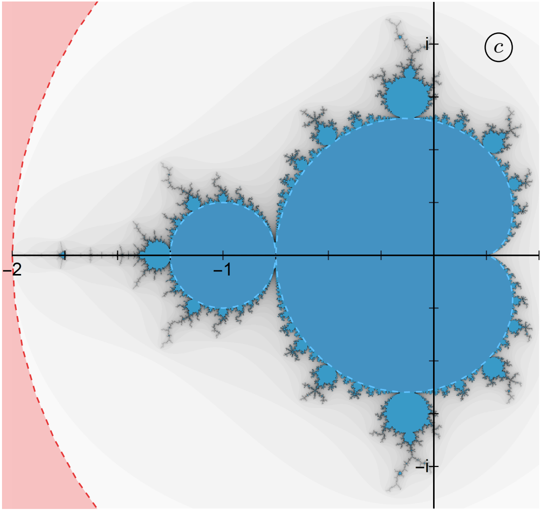
$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$

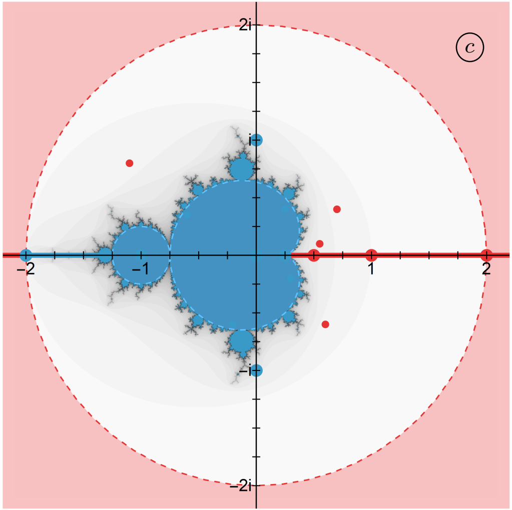




Mandelbrotova množina $\quad M = \{c \in \mathbb{C} : \ (z_n)$ je omezená posloupnost$\}$

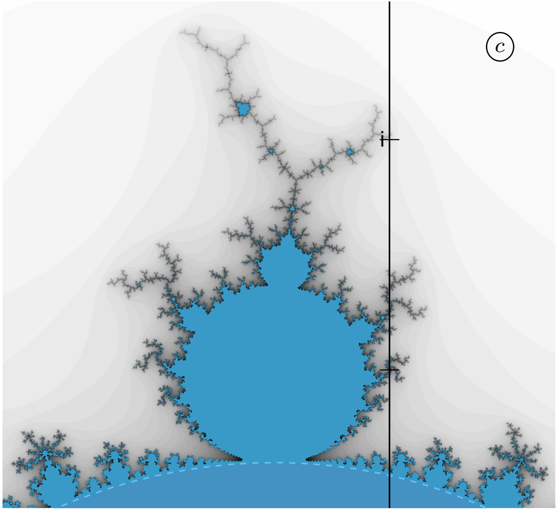
$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$

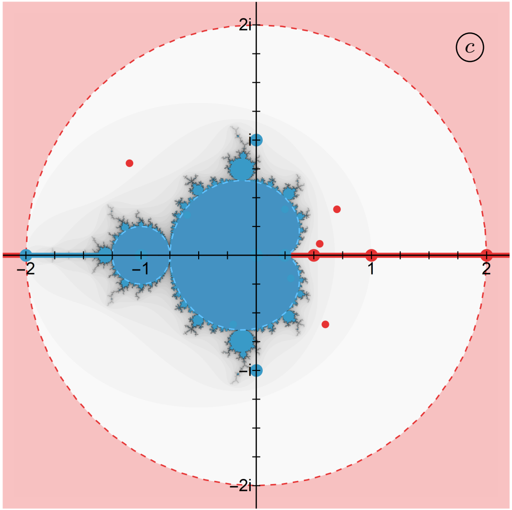




Mandelbrotova množina $M = \{c \in \mathbb{C} : \ (z_n) \text{ je omezená posloupnost}\}$

$$z_1 = c, \quad z_{n+1} = z_n \cdot z_n + c$$





Komplexní funkce komplexní proměnné $f : \mathbb{C}^* \to \mathbb{C}^*$

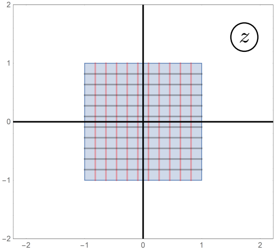
příklady jednoznačných funkcí

- $f : w = \overline{z}$
- $f : w = az + b$ (lineární fce)
- $f : w = \dfrac{1}{\overline{z}}$ (kruhová inverze)
- $f : w = \dfrac{1}{z}$ (převrácená hodnota)
- $f : w = z^2$ (druhá mocnina)
- $f : w = z^3$ (třetí mocnina)
- $f : w = \arg z$ (h.h. argumentu)
- $f : w = \exp z$ (exponenciální funkce)

příklady víceznačných funkcí

- $f : w = \operatorname{Arcsin} z$ (arkus sinus)
- $f : w = \operatorname{Arccos} z$ (arkus kosinus)
- $f : w = \operatorname{Arctg} z$ (arkus tangens)
- $f : w = \operatorname{Arccotg} z$ (arkus kotangens)
- $f : w = \sqrt{z}$ (druhá odmocnina)
- $f : w = \sqrt[3]{z}$ (třetí odmocnina)
- $f : w = \operatorname{Arg} z$ (argument)
- $f : w = \operatorname{Ln} z$ (logaritmická funkce)

Lineární funkce $f: w = az + b, \quad a, b \in \mathbb{C}, \ a \neq 0, \quad D(f) = \mathbb{C}^*, \ H(f) = \mathbb{C}^*$

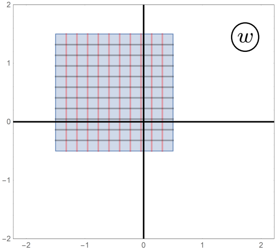


$$w = a \cdot z + b = |a|\,\mathrm{e}^{\mathrm{i}\arg a} \cdot |z|\,\mathrm{e}^{\mathrm{i}\arg z} + b$$

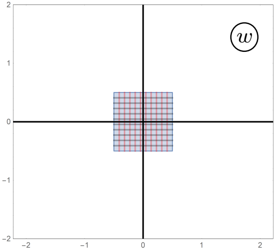
$$= \underbrace{|a| \cdot |z|}_{=|w|}\,\mathrm{e}^{\mathrm{i}(\overbrace{\arg a + \arg z}^{=\arg w + 2k\pi,\ k\in\mathbb{Z}})} + b$$

- posunutí o b
- stejnolehlost s $|a|$
- otočení o $\arg a$

$a = 1, \ b = -1/2 + 1/2\,\mathrm{i}$

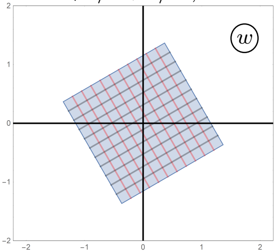


$a = 1/2, \ b = 0$

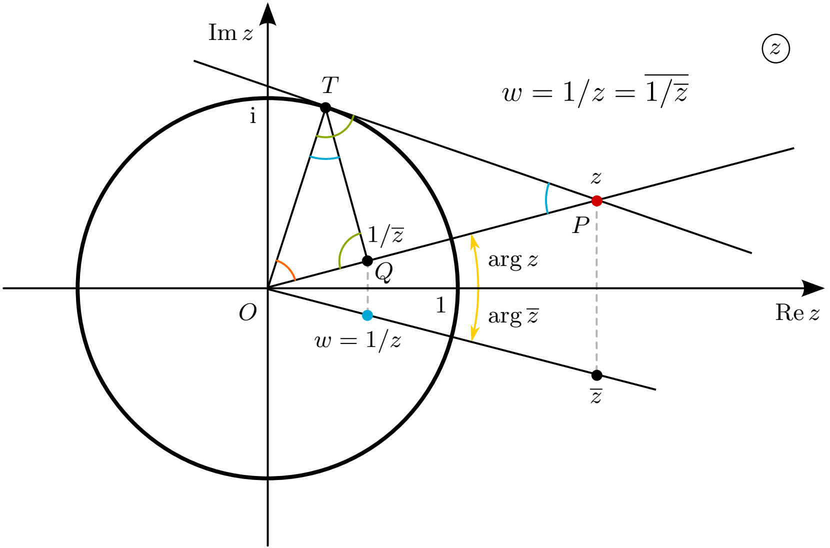


$|a| = 1, \ \arg a = \pi/6$

$a = \sqrt{3}/2 + 1/2\,\mathrm{i}, \ b = 0$



Základní lineární lomená funkce $f : w = 1/z$, $D(f) = \mathbb{C}^*$, $H(f) = \mathbb{C}^*$



$$z = |z|\,e^{i \arg z}$$

$$\frac{1}{z} = \frac{1}{|z|}\,e^{-i \arg z}$$

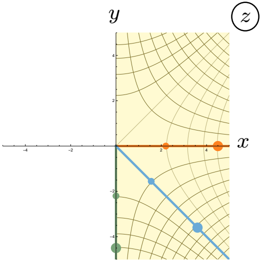
$$|z| = |\overline{z}|$$

$$\arg z = -\arg \overline{z}$$

$$\overline{z} = |\overline{z}|\,e^{i \arg \overline{z}}$$

$$\frac{1}{\overline{z}} = \frac{1}{|\overline{z}|}\,e^{-i \arg \overline{z}}$$

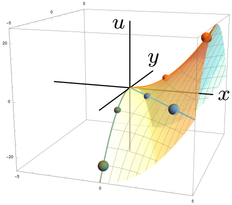
Komplexní druhá mocnina a odmocnina



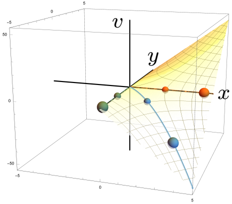
$$z = x + \mathrm{i}\, y$$

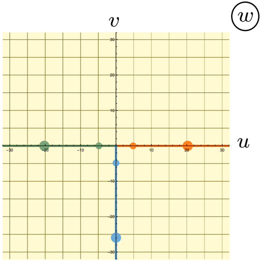
$$w = z^2$$

$u = x^2 - y^2$



$v = 2xy$

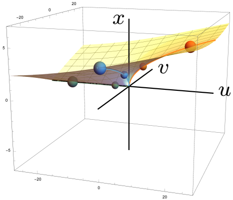




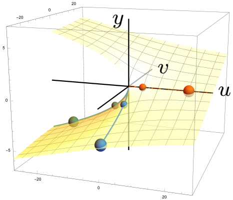
$$w = u + \mathrm{i}\, v$$

$$z = \sqrt{w}$$

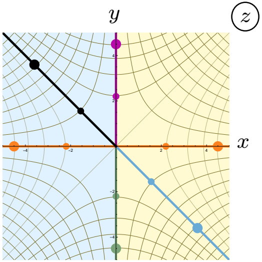
$x = \sqrt{\frac{1}{2}u + \frac{1}{2}\sqrt{u^2 + v^2}}$



$y = \dfrac{v}{\sqrt{2u + 2\sqrt{u^2 + v^2}}}$

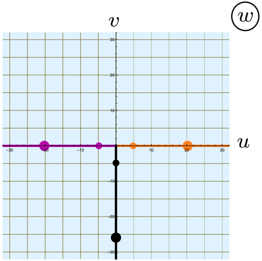


Komplexní druhá mocnina a odmocnina



$$z = x + \mathrm{i}\, y$$

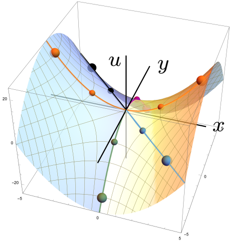
$$w = z^2$$



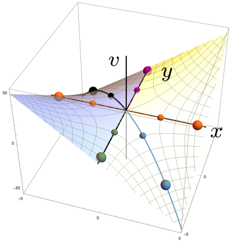
$$w = u + \mathrm{i}\, v$$

$$z = \sqrt{w}$$

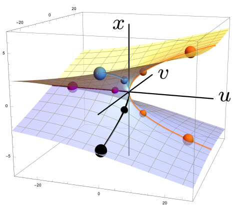
$u = x^2 - y^2$



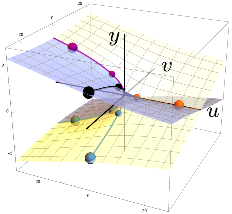
$v = 2xy$



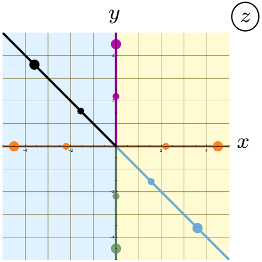
$x = \pm\sqrt{\frac{1}{2}u + \frac{1}{2}\sqrt{u^2 + v^2}}$



$y = \dfrac{\pm v}{\sqrt{2u + 2\sqrt{u^2 + v^2}}}$

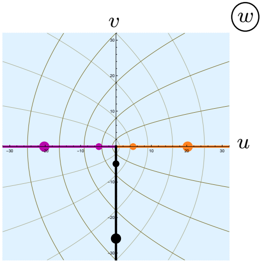


Komplexní druhá mocnina a odmocnina



$$z = x + \mathrm{i}\, y$$

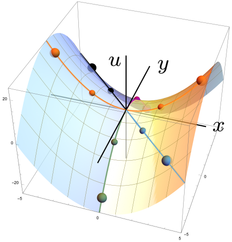
$$w = z^2$$



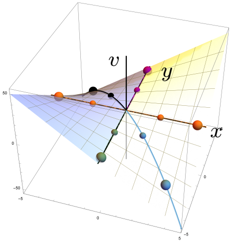
$$w = u + \mathrm{i}\, v$$

$$z = \sqrt{w}$$

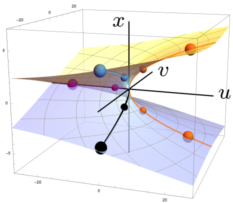
$$u = x^2 - y^2$$



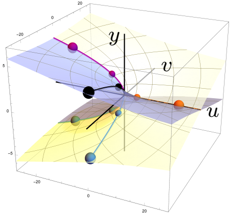
$$v = 2xy$$



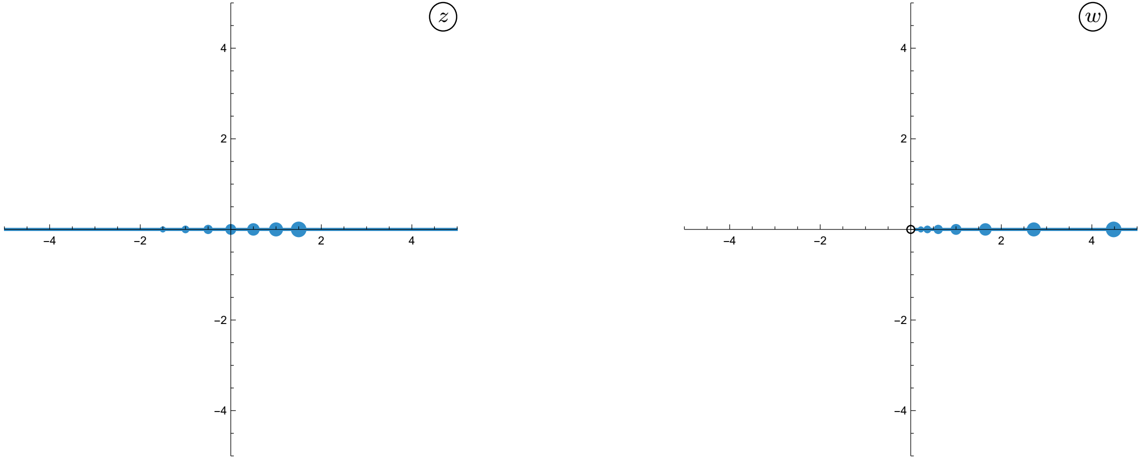
$$x = \pm\sqrt{\tfrac{1}{2}u + \tfrac{1}{2}\sqrt{u^2 + v^2}}$$



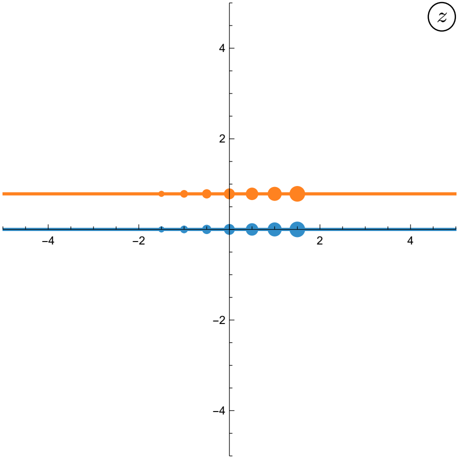
$$y = \frac{\pm v}{\sqrt{2u + 2\sqrt{u^2 + v^2}}}$$

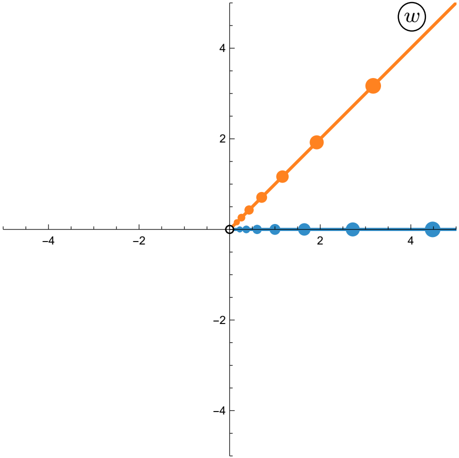


Exponenciální funkce $f : w = \exp(z)$, $D(f) = \mathbb{C}$, $H(f) = \mathbb{C} \setminus \{0\}$

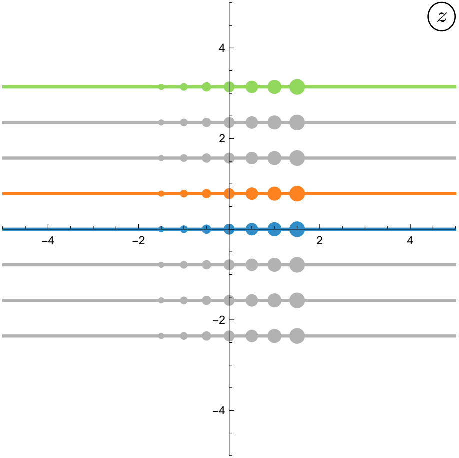


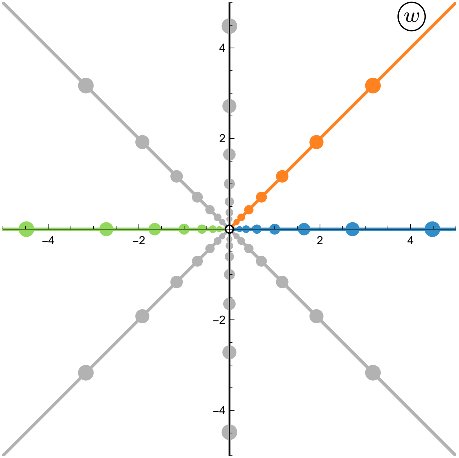
Exponenciální funkce $f : w = \exp(z)$, $D(f) = \mathbb{C}$, $H(f) = \mathbb{C} \setminus \{0\}$



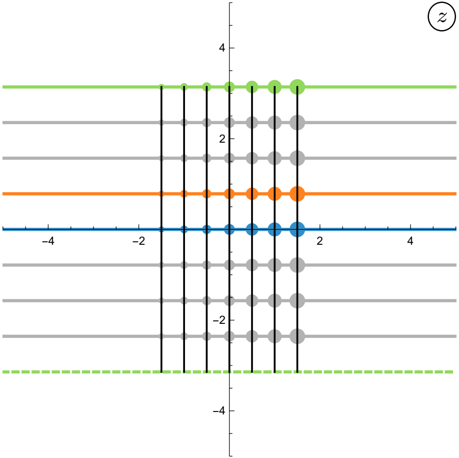


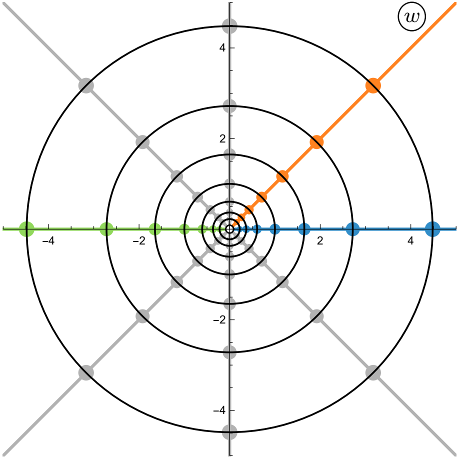
Exponenciální funkce $f : w = \exp(z)$, $D(f) = \mathbb{C}$, $H(f) = \mathbb{C} \setminus \{0\}$



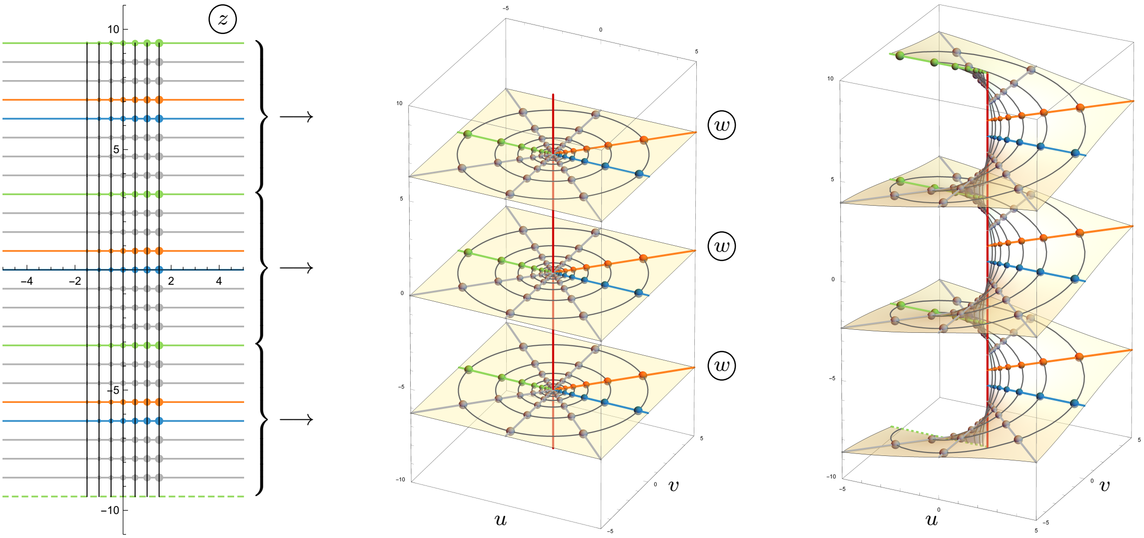


Exponenciální funkce $f : w = \exp(z)$, $D(f) = \mathbb{C}$, $H(f) = \mathbb{C} \setminus \{0\}$





Exponenciální funkce $f : w = \exp(z)$, $D(f) = \mathbb{C}$, $H(f) = \mathbb{C} \setminus \{0\}$



Georg Friedrich Bernhard Riemann (✳ 17. září 1826, † 20. července 1866)

- Riemannův integrál
- Riemannova koule
- Riemannovy plochy
- Riemannova geometrie
- Riemannův tenzor
- Riemannova zeta funkce
- Riemannova hypotéza

Všechny netriviální nulové body Riemannovy funkce zeta mají reálnou část rovnu $\frac{1}{2}$.

„. . . je velmi pravděpodobné, že všechny kořeny jsou reálné. Samozřejmě bych si zde přál uvést řádný důkaz. Po několika marných pokusech jsem však svoje snažení prozatím odložil. Zdá se, že k dosažení bezprostředního cíle mého výzkumu je hledání důkazu postradatelné."

Riemannova funkce zeta

Pro $\operatorname{Re} z > 1$ definujeme

$$\zeta(z) = \sum_{n=1}^{+\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \dots$$

- $\zeta(\overline{z}) = \overline{\zeta(z)}$
- $0 = \zeta(-2) = \zeta(-4) = \zeta(-6) = \dots$
- $\lim\limits_{x \to +\infty} \zeta(x + \mathrm{i}\, y) = 1, \quad y \in \mathbb{R}$
- $\lim\limits_{z \to 1} \zeta(z) = \infty$
- pro $z \in \mathbb{C} \setminus \{0, 1\}$

$$\zeta(z) \cdot \pi^{-\frac{z}{2}} \cdot \Gamma\left(\frac{z}{2}\right) = \zeta(1-z) \cdot \pi^{-\frac{1-z}{2}} \cdot \Gamma\left(\frac{1-z}{2}\right)$$