ON BOUNDARY VALUE PROBLEMS FOR SECOND ORDER DISCRETE INCLUSIONS

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Abstract. In this paper we prove some existence theorems regarding solutions to boundary value problems for systems of second order discrete inclusions. For a certain class of right-hand sides, we present some lemmas showing that all solutions to discrete second order inclusions satisfy an *a priori* bound. Then we apply these *a priori* bounds, in conjunction with an appropriate fixed point theorem for inclusions, to obtain the existence of solutions. The theory is highlighted with several examples.

1. Introduction

The theory of differential inclusions has recieved much attention due to its versatility and generality. For example, differential inclusions can accurately model discontinuous processes, such as: systems with dry friction; the work of an electric oscillator; and autopilot (and other) control systems [9]. When considering these (or other) situations in discrete time, the modelling process gives rise to a discrete (or difference) inclusion, rather than a differential inclusion. In many cases, considering the model in discrete time gives a more precise or realistic description [1].

Let X and Y be two normed spaces. A *set-valued map* $G : X \to Y$ is a map that associates with any $x \in X$ a set $G(x) \subset Y$. By $CK(E)$ we denote the set of nonempty, convex and closed subsets of a Banach space E. We say that $G : \mathbb{R}^n \to CK(\mathbb{R}^n)$ is *upper semicontinuous* if for all sequences $\{u_i\} \subseteq \mathbb{R}^n$, $\{v_i\} \subseteq \mathbb{R}^n$, where $i \in \mathbb{N}$, the conditions $u_i \to u_0$, $v_i \to v_0$ and $v_i \in G(u_i)$ imply that $v_0 \in G(u_0)$. Since the upper semicontinuity plays an essential role in this paper, we illustrate this notion by the simple example, [6, Example 4.1.1].

Example 1. The set-valued map $f_1 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_1(t) = \begin{cases} \{0\}, & \text{for } t = 0, \\ [0,1]\,, & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases}$$

is not upper semicontinuous. On the other hand, the set-valued map $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(t) = \begin{cases} [0,1]\,, & \text{for } t = 0, \\ \{0\}, & \text{for } t \in \mathbb{R} \setminus \{0\}, \end{cases}$$

is upper semicontinuous.

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For more information about set-valued maps and differential inclusions see Aubin and Cellina [4] or Smirnov [9].

We are interested in the following boundary value problem (abbreviated further as BVP) for second order discrete inclusions

$$\begin{cases} \Delta^2 y(k-1) \in F(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \\ y(0) = A, \quad y(T+1) = B, \end{cases} \tag{1}$$

where $A, B \in \mathbb{R}^d$ are constants and $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ is a set-valued map. A solution $\overline{y} = \{y(k)\}_{k=0}^{T+1} \in \mathbb{R}^{(T+2)d}$ to (1) is a vector $\overline{y} = \{y(0), \dots, y(T+1)\}$ such that each element $y(k) \in \mathbb{R}^d$ satisfies the discrete inclusion for $k = 1, \dots, T$ and the boundary conditions for $k = 0$ and $k = T+1$.

In Section 2, we show that under certain conditions on the right-hand side F, all solutions of (1) are bounded. The inequalities employed rely on growth conditions on F and an appropriate discrete maximum principles.

Section 3 contains the appropriate operator formulations for (1) to be considered as a fixed point problem.

In Section 4 we apply the results of Sections 2 and 3, to prove the existence of solutions to (1), in conjunction with the following fixed-point theorem, [3, Theorem 1.2].

Theorem 2. *Let E be a Banach space, U an open subset of E and $0 \in U$. Suppose that $P : \overline{U} \to CK(E)$ is an upper semicontinuous and compact map. Then either*

(A1) *P has a fixed point in $\overline{U}$ or*

(A2) *$\exists u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda P(u)$.*

To prove the compactness of the image of an upper semicontinuous map we shall use a criterion which can be found in Berge [5, Théorème VI.3].

Theorem 3. *Let $P : X \to Y$ be an upper semicontinuous map. If K is a compact set in X then $P(K)$ is a compact set in Y.*

In [3], Agarwal et al. gave conditions under which the following BVP has at least one solution

$$\begin{cases} \Delta^2 y(k-1) \in F(k, y(k)), \quad k = 1, \dots, T, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases}$$

where $F : \{1, \dots, T\} \times \mathbb{R} \to CK(\mathbb{R})$. In comparison with the results and conditions in [3], we introduce new inequalities unrelated to those (Theorem 11, herein) and we also extend some of the results (Theorem 14, herein).

2. A priori bound

In this section we prove two different *a priori* bound results for the following system of BVPs for second order discrete inclusions:

$$\begin{cases} \Delta^2 y(k-1) \in \lambda F(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \\ y(0) = \lambda A, \quad y(T+1) = \lambda B, \end{cases} \tag{2}$$

where $\lambda \in [0, 1]$.

The study of the above family of BVPs is motivated by the family of inclusions in Theorem 2, $u \in \lambda P(u)$.

We denote $\langle \cdot, \cdot \rangle$ as the Euclidean inner product and by $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^n$.

Lemma 4. *Let $F : \{1,\dots,T\}\times\mathbb{R}^d\times\mathbb{R}^d \to CK(\mathbb{R}^d)$ be a set-valued map. If there exist constants $\alpha \geq 0$ and $K \geq 0$ such that*

(3)
$$\|\phi\| \leq \alpha(2\langle p,\phi\rangle + \|q\|^2) + K, \text{ for } k = 1,\dots,T \text{ and all } (p,q)\in\mathbb{R}^{2d}, \phi\in F(k,p,q),$$

then all solutions $\bar{y}$ of BVP for the system of discrete inclusions (2) *satisfy*

$$\|y(k)\| < R, \quad k = 0,\dots,T+1,$$

for $\lambda\in[0,1]$, and R is defined by

$$R := \alpha\beta^2 + \beta + K\frac{(T+1)^2}{8} + 1, \quad \text{where } \beta := \max\{\|A\|,\|B\|\}. \tag{4}$$

Proof. Let us suppose that $\bar{y}$ is a solution of (2). Since we work on a discrete topology, every solution of (2) is a solution of a system of discrete BVP

$$\begin{cases} \Delta^2 y(k-1) = \lambda\hat{f}(k,y(k),\Delta y(k)), & k = 1,\dots,T, \\ y(0) = \lambda A, \quad y(T+1) = \lambda B, \end{cases} \tag{5}$$

where $\hat{f} : \{1,\dots,T\}\times\mathbb{R}^d\times\mathbb{R}^d \to CK(\mathbb{R}^d)$ is a single-valued function such that $\hat{f}(k,p,q)\in F(k,p,q)$ for every $k = 1,\dots,T$ and $(p,q)\in\mathbb{R}^{2d}$. In the theory of the set-valued map $\hat{f}$ is called a *selector* of F.

Then $\bar{y}$ solves the integral equation

$$y(k) = \lambda\Phi(k) + \lambda\sum_{l=1}^{T} G(k,l)\hat{f}(l,y(l),\Delta y(l)), \quad k = 0,\dots,T+1,$$

where

$$\lambda\Phi(k) = \lambda\frac{A(T+1)+(B-A)k}{T+1}, \tag{6}$$

and $G : \{0,\dots,T+1\}\times\{1,\dots,T\}\to\mathbb{R}^d$ defined by

$$G(k,l) = \begin{cases} -\frac{1}{T+1}l(T+1-k), & l = 1,\dots,k-1, \\ -\frac{1}{T+1}k(T+1-l), & l = k,\dots,T, \end{cases} \tag{7}$$

is the Green's function for the BVP

$$\Delta^2 y(k-1) = 0, \quad k = 1,\dots,T, \quad y(0) = 0,\ y(T+1) = 0.$$

Since $\|\lambda\Phi(k)\| \leq \beta$ for each $k = 0,\dots,T+1$, we obtain that

$$\|y(k)\| \leq \beta + \sum_{l=1}^{T} |G(k,l)|\lambda\|\hat{f}(l,y(l),\Delta y(l))\|.$$

Using (3), we have that

$$\begin{aligned} \|y(k)\| &\leq \beta + \sum_{l=1}^{T} |G(k,l)|\lambda\left\{\alpha\left[2\langle y(l),\hat{f}(l,y(l),\Delta y(l))\rangle + \|\Delta y(l)\|^2\right] + K\right\} \\ &\leq \beta + \sum_{l=1}^{T} |G(k,l)|\left\{\alpha\left[2\langle y(l),\lambda\hat{f}(l,y(l),\Delta y(l))\rangle + \|\Delta y(l)\|^2\right] + K\right\} \qquad (8) \end{aligned}$$

Define

$$r(k) := \|y(k)\|^2, \quad k = 0,\dots,T+1,$$

and use the discrete product rule to calculate the second difference of r at the point $k-1$ to obtain

$$\Delta^2 r(k-1) = \|\Delta y(k)\|^2 + 2\langle y(k), \Delta^2 y(k-1)\rangle + \|\Delta y(k-1)\|^2.$$

By using the first equation in (5) we deduce that

$$\Delta^2 r(k-1) \geq \|\Delta y(k)\|^2 + 2\langle y(k), \lambda \hat{f}(k, y(k), \Delta y(k))\rangle.$$

We install this into (8) to obtain that

$$\|y(k)\| \leq \beta + \alpha \sum_{l=1}^{T} |G(k,l)|\Delta^2 r(l-1) + \sum_{l=1}^{T} |G(k,l)|K. \tag{9}$$

Using (7) we make the following computation

$$\begin{aligned}
&\sum_{l=1}^{T} |G(k,l)|\Delta^2 r(l-1) \\
&= \frac{T+1-k}{T+1} \sum_{l=1}^{k-1} l\Delta^2 r(l-1) + \frac{k}{T+1} \sum_{l=k}^{T} (T+1-l)\Delta^2 r(l-1) \\
&= \frac{T+1-k}{T+1} \left([l\Delta r(l-1)]_1^k - \sum_{l=1}^{k-1} \Delta r(l) \right) \\
&\qquad + \frac{k}{T+1} \left([(T+1-l)\Delta r(l-1)]_k^{T+1} + \sum_{l=k}^{T} \Delta r(l) \right) \\
&= \frac{T+1-k}{T+1} \left(k\Delta r(k-1) - \Delta r(0) - r(k) + r(1) \right) \\
&\qquad + \frac{k}{T+1} \left(-(T+1-k)\Delta r(k-1) + r(T+1) - r(k) \right) \\
&\qquad\qquad = \frac{T+1-k}{T+1} r(0) + \frac{k}{T+1} r(T+1) - r(k) \leq \beta^2.
\end{aligned}$$

Finally, if we consider this estimation and (see e.g. [8, Exercise 6.20])

$$\max_{k \in \{0,\ldots,T+1\}} \sum_{l=1}^{T} |G(k,l)| \leq \frac{(T+1)^2}{8},$$

we rewrite (9) as

$$\|y(k)\| \leq \beta + \alpha\beta^2 + K\frac{(T+1)^2}{8} < R, \quad k = 0, \ldots, T+1,$$

and this concludes the proof. ■

Definition 5. Let $R > 0$ be a constant. Define the set $D_R \subset \{1, \ldots, T\} \times \mathbb{R}^d \times \mathbb{R}^d$ by the set containing all triplets (k, p, q) such that

$$k = 1, \ldots, T, \ (p, q) \in \mathbb{R}^{2d} : \|p\| \geq R \text{ and } 2\langle p, q\rangle + \|q\|^2 \leq 0. \tag{10}$$

By using the same selector technique as above, but employing an unrelated inequality, we now prove the second *a priori* bound result.

Lemma 6. *Let $R > 0$ be a constant such that*

$$\max\{\|A\|, \|B\|\} < R, \tag{11}$$

and let $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ be a set-valued map. If

$$2\langle p, \phi\rangle + \|q\|^2 > 0, \quad \textit{for every } (k, p, q) \in D_R \textit{ and all } \phi \in F(k, p, q),$$

then all solutions $\bar{y}$ for the system of discrete inclusions (2) *satisfy*

$$\|y(k)\| < R, \quad k = 0, \dots, T + 1, \tag{12}$$

for $\lambda \in [0, 1]$.

Proof. Suppose that $\bar{y}$ is a solution of (2). As in the previous proof we can deduce that $\bar{y}$ is a solution of (5) for some selector $\hat{f}(k, p, q) \in F(k, p, q)$.

Assume that the conclusion is not true. Then the function $r(k) := \|y(k)\|^2 - R^2$ must have a nonnegative maximum in $\{0, \dots, T + 1\}$. From the assumption (11), this maximum must be achieved in $\{1, \dots, T\}$. Choose $c \in \{1, \dots, T\}$ such that $r(c) = \max\{r(k); k \in \{1, \dots, T\}\}$ and suppose that there is no $k < c$ for which $r(k) = r(c)$. This choice of c implies that the conditions

$$\Delta r(c) \le 0, \tag{13}$$

$$\Delta^2 r(c - 1) \le 0, \tag{14}$$

must be satisfied simultaneously. Since

$$\begin{aligned}\Delta r(c) &= \langle y(c) + y(c + 1), \Delta y(c)\rangle = \langle 2y(c) + \Delta y(c), \Delta y(c)\rangle,\\ &= 2\langle y(c), \Delta y(c)\rangle + \|\Delta y(c)\|^2,\end{aligned}$$

holds, we rewrite (13) as

$$2\langle y(c), \Delta y(c)\rangle + \|\Delta y(c)\|^2 \le 0.$$

Similarly as in the proof of Lemma 4 we obtain with the help of the product rule

$$\Delta^2 r(c - 1) \ge \|\Delta y(c)\|^2 + 2\langle y(c), \hat{f}(c, y(c), \Delta y(c))\rangle > 0,$$

which contradicts (14). Hence $\|y(k)\| < R$ for $k = 0, \dots, T + 1$.

■

Lemma 6 is a natural extension to the lower and upper solution methods used in [3] for the case $d = 1$. If the right-hand side in (1) is a single-valued map, then we have the following corollary to Lemma 6 for systems of discrete BVPs.

Corollary 7. *Let $R > 0$ be a constant. If*

$$2\langle p, F(k, p, q)\rangle + \|q\|^2 > 0, \quad \textit{for all } (k, p, q) \in D_R, \tag{15}$$

and

$$\max\{\|A\|, \|B\|\} < R,$$

then all solutions $\bar{y}$ of (5) *satisfy* (12) *for $\lambda \in [0, 1]$.*

Remark 8. Note that if $F(k, y(k), \Delta y(k)) = F(k, y(k))$ then in place of (15) we would require only

$$\langle p, F(k, p)\rangle > 0, \quad \text{for all } k = 1, \dots, T \text{ and all } p \in \mathbb{R}^d : \|p\| \ge R,$$

to be satisfied.

The advantage of (10) rather than assuming (15) for, say, all $q \in \mathbb{R}^d$, is highlighted in the following example.

Example 9. Consider the single, scalar-valued function $f(k,p,q) = p^3 - q$. For all $q \in \mathbb{R}$ we have that

$$2pf(k,p,q) + q^2 = 2p^4 - 2pq + q^2, \tag{16}$$

and for all $|p| \geq R$ we can find $q \in \mathbb{R}$ such that (16) is negative. But on the reduced set D_R, the assumption (10) implies that $2pq + q^2 \leq 0$ ($-2pq \geq q^2$ equivalently) and thus

$$2pf(k,p,q) + q^2 = 2p^4 - 2pq + q^2 \geq 2p^4 + q^2 + q^2 > 0,$$

for all $(p,q) \in \mathbb{R}^2$, $|p| \geq R$ such that (10) holds for any $R > 0$. Hence we can use Corollary 7 to prove the *a priori* bound for discrete BVP,

$$\begin{cases} \Delta^2 y(k-1) = \lambda y^3(k) - \lambda \Delta y(k), \quad k = 1, \dots, T, \\ y(0) = \lambda A, \quad y(T+1) = \lambda B, \end{cases}$$

where $\lambda \in [0,1]$.

3. Operator formulation

In this section we formulate the necessary operators to apply Theorem 2.

Solving (2) is equivalent to finding a vector $\overline{y} = \{y(k)\}_{k=0}^{T+1} \in \mathbb{R}^{(T+2)d}$ which satisfies

$$y(k) \in \lambda\Phi(k) + \lambda \sum_{l=1}^{T} G(k,l) F(l, y(l), \Delta y(l)), \quad k = 0, \dots, T+1, \tag{17}$$

where $\lambda\Phi$ is the function defined by (6) and $G : \{0, \dots, T+1\} \times \{1, \dots, T\} \to \mathbb{R}^d$, defined by (7), is the Green's function for the BVP

$$\Delta^2 y(k-1) = 0, \ k = 0 \dots, T, \quad y(0) = 0, \ y(T+1) = 0.$$

Let us suppose that $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ then we can define the operator $\mathcal{F} : \mathbb{R}^{(T+2)d} \to CK(\mathbb{R}^{Td})$ by

$$\mathcal{F}(\overline{u}) := \{\overline{v} \in \mathbb{R}^{Td} : v(k) \in F(k, u(k), \Delta u(k)), \quad k = 1, \dots, T\},$$

and the operator $\mathcal{T} : \mathbb{R}^{Td} \to \mathbb{R}^{(T+2)d}$ by

$$\mathcal{T} y(k) := \sum_{l=1}^{T} G(k,l) y(l), \quad k = 1, \dots, T.$$

The discreteness of topology implies that $\mathcal{T}$ is continuous and linear and thus we have $\mathcal{T} \circ \mathcal{F} : \mathbb{R}^{(T+2)d} \to CK(\mathbb{R}^{(T+2)d})$. We can rewrite (17) as

$$\bar{y} \in \hat{\mathcal{S}}(\bar{y}),$$

where $\hat{\mathcal{S}} : \mathbb{R}^{(T+2)d} \to CK(\mathbb{R}^{(T+2)d})$ is defined by

$$\hat{\mathcal{S}}(\overline{y}) := \lambda \mathcal{T} \circ \mathcal{F}(\overline{y}) + \lambda\Phi.$$

In order to use Theorem 2 we need to define a suitable open subset U of the Banach space $\mathbb{R}^{(T+2)d}$ and the operator $\mathcal{S} : \overline{U} \to CK(\mathbb{R}^{(T+2)d})$. Introduce $U \subset \mathbb{R}^{(T+2)d}$ by

$$U := \{\overline{u} \in \mathbb{R}^{(T+2)d} : \|u(k)\| < R, \quad k = 0, \dots, T+1\}, \tag{18}$$

where R is the constant defined either in (4) or in Lemma 6. Next, we define the operator $\mathcal{S} : \overline{U} \to CK(\mathbb{R}^{(T+2)d})$ by

$$\mathcal{S} = \hat{\mathcal{S}}|_{\overline{U}}. \tag{19}$$

To satisfy the remaining assumptions of Theorem 2 on the operator $\mathcal{S}$ we need to prove that it is upper semicontinuous and compact.

Lemma 10. *If F satisfies*

(US) $\quad F(k,p,q)$ *is upper semicontinuous for all* $(p,q) \in \mathbb{R}^{2d}$,

for $k = 1,\dots,T$ and the assumptions either of Lemma 4 *or* Lemma 6 *hold, then the operator $\mathcal{S}$ is upper semicontinuous and compact.*

Proof. Consider the sequences $\{u_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ such that $w_i \in \mathcal{S}(u_i)$ and $u_i \to u_0$ and $w_i \to w_0$ as $i \to \infty$ in $\mathbb{R}^{(T+2)d}$. To prove our assertion we must show that $w_0 \in \mathcal{S}(u_0)$. For each $i \in \mathbb{N}$ there exists $v_i \in \mathbb{R}^{Td}$ such that $w_i = \lambda \mathcal{T} v_i + \lambda \Phi$ and $v_i \in \mathcal{F}(u_i)$. Since condition (US) holds and $\overline{U}$ is a compact set we can deduce from Theorem 3 that $\mathcal{F}(\overline{U})$ is a compact set. This implies that there exists at least a subsequence $\{v_{i_n}\}_{n=1}^{\infty}$ of $\{v_i\}_{i=1}^{\infty}$ such that $v_{i_n} \to v_0 \in \mathcal{F}(u_0)$. Since $\mathcal{T}$ is linear and continuous we have that

$$w_i = \lambda \mathcal{T} v_i + \lambda \Phi \to \lambda \mathcal{T} v_0 + \lambda \Phi,$$

and noting that that $w_i \to w_0$ in $\mathbb{R}^{(T+2)d}$ we can conclude that

$$w_0 \in \mathcal{S}(u_0).$$

This proves that $\mathcal{S}$ is upper semicontinuous and we can use Theorem 3 to obtain the compactness of $\mathcal{S}$. ■

4. Existence results and examples

In this section we combine the theory of Sections 2 and 3 to formulate existence results.

Theorem 11. *If the set-valued map $F : \{1,\dots,T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ satisfies* (US) *and the assumptions of* Lemma 4 *hold then the system of discrete inclusion boundary value problems* (1) *has a solution.*

Proof. We showed in the previous section that the problem of existence of a solution of (1) (where F satisfies the assumptions of Lemma 4) is equivalent to the problem of existence of a fixed point of $\mathcal{S}(\bar{y})$, where $\mathcal{S} : \overline{U} \to CK(\mathbb{R}^{(T+2)d})$ is defined in (19) and U is defined in (18). Since $\mathcal{S}$ is compact and upper semicontinuous (cf. Lemma 10) we are ready to use Theorem 2 and thanks to the conclusion of Lemma 4 we can exclude the possibility (A2) there. Therefore the operator S has a fixed point and the problem (1) has a solution. ■

We illustrate the above result on the following example with $n = 1$.

Example 12. Let $\widetilde{F} : \{1,\dots,T\} \times \mathbb{R} \to CK(\mathbb{R})$ be a set-valued map defined by

$$\widetilde{F}(k,p) := \bigcup_{\epsilon \in [-1;1]} (k \pm \epsilon) p^5.$$

Consider the BVP:

$$\begin{cases} \Delta^2 y(k-1) \in \widetilde{F}(k, y(k)), \quad k = 1,\dots,T, \\ y(0) = A, \quad y(T+1) = B, \end{cases} \tag{20}$$

where $A, B \in \mathbb{R}$. Since $k \in \{1, \dots, T\}$ and $|\epsilon| \leq 1$ the inequality $(k \pm \epsilon)p^5 \leq (k \pm \epsilon)(p^6 + 1)$ holds and thus for every selector $\widetilde{f}(k, p) \in \widetilde{F}(k, p)$ we have that

$$\begin{aligned} |\widetilde{f}(k,p)| &\leq (k \pm \epsilon)(p^6 + 1) \\ &= p(k \pm \epsilon)p^5 + t \pm \epsilon \\ &\leq p\widetilde{f}(k,p) + T + 1. \end{aligned}$$

and the inequality (3) is satisfied with $\alpha = \frac{1}{2}$ and $K = T + 1$. The set-valued map $\widetilde{F}$ satisfies (US) and thus we can use Theorem 11 to prove that the problem (20) has a solution.

As a natural corollary to Theorem 11 we have the following result.

Corollary 13. *Let $K \geq 0$ be a constant. If the set-valued map $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ satisfies* (US) *and*

$$\|\phi\| \leq K, \text{ for } k = 1, \dots, T \text{ and all } (p, q) \in \mathbb{R}^{2d}, \phi \in F(k, p, q),$$

then the system of discrete inclusion boundary value problems (1) *has a solution.*

Proof. See that this is a special case of Theorem 11 with $\alpha = 0$. ∎

We now prove an existence result for the condtitions from Lemma 6.

Theorem 14. *If the set-valued map $F : \{1, \dots, T\} \times \mathbb{R}^d \times \mathbb{R}^d \to CK(\mathbb{R}^d)$ satisfies* (US) *and the assumptions of* Lemma 6 *hold then the system of discrete inclusion boundary value problems* (1) *has a solution.*

Proof. The proof is identical to that of Theorem 11. The only difference consists of the different definition of the constant R when defining the set $\bar{U}$, cf. Lemma 6. ∎

We illustrate the above result with an example in two dimensions.

Example 15. Let $\hat{F} : \{1, \dots, T\} \times \mathbb{R}^2 \times \mathbb{R}^2 \to CK(\mathbb{R}^2)$ be a set-valued map defined by

$$\hat{F}(k, p, q) := \bigcup_{\epsilon \in (0;1]} \begin{pmatrix} \epsilon k p_1 - q_1 \\ p_2^3(1 + q_1^2) - p_2 - q_2 \end{pmatrix},$$

where $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ and $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Consider the BVP

$$\left\{ \begin{array}{l} \Delta^2 y(k-1) \in \hat{F}(k, y(k), \Delta y(k)), \quad k = 1, \dots, T, \\ y(0) = A, \quad y(T+1) = B, \end{array} \right. \tag{21}$$

where $A, B \in \mathbb{R}^2$. Let $\hat{f}(k, p, q) \in \hat{F}(k, p, q)$ be an arbitrary selector for all $k = 1, \dots, T$, $q, p \in \mathbb{R}^d$, $\|p\| \geq R$, where $R > 1$ and p, q are such that $2\langle p, q \rangle + \|q\|^2 \leq 0$, e.g.

$$2p_1 q_1 + 2p_2 q_2 + q_1^2 + q_2^2 \leq 0. \tag{22}$$

We make the following calculation

$$\begin{aligned} 2\langle p, \hat{f}(k, p, q)\rangle + \|q\|^2 &= 2 \left\langle \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} \epsilon k p_1 - q_1 \\ p_2^3(1 + q_1^2) - p_2 - q_2 \end{pmatrix} \right\rangle + q_1^2 + q_2^2 \\ &= 2\epsilon k p_1^2 + 2p_2^4(1 + q_1^2) - 2p_2^2 - 2p_1 q_1 - 2p_2 q_2 + q_1^2 + q_2^2. \end{aligned}$$

Using (22) we can provide the following estimation

$$2\langle p, \hat{f}(k,p,q)\rangle + \|q\|^2 \geq 2\epsilon k p_1^2 + 2p_2^4(1+q_1^2) - p_2^2 + 2q_1^2 + 2q_2^2 > 0,$$

since either $\epsilon p_1^2 > 0$ or $p_2^4(1+q_1^2) - p_2^2 > 0$. The set-valued map $\hat{F}$ satisfies also the condition (US) and thus we can use Theorem 14 to prove that the problem (21) has a solution.

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