

ON LOWER AND UPPER SOLUTIONS WITHOUT ORDERING ON TIME SCALES

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ABSTRACT. In order to enlarge the set of boundary value problems on time scales, for which we can use the lower and upper solutions technique to get existence of solutions, we extend this method to the case when the pair lacks ordering. We use the degree theory and *a priori* estimates to obtain the existence of solutions for the second-order Dirichlet boundary value problems. To illustrate a wider application of this result, we conclude with an example which shows that a combination of well and non-well ordered pairs can yield the existence of multiple solutions.

1. INTRODUCTION

The method of lower and upper solutions is widely used concept in the study of nonlinear boundary value problems (further abbreviated by BVP). Three quarters of the century after the pioneering work of Scorza Dragoni [8] this method still belongs among the basic tools and is frequently employed in applied analysis or mechanics. Dragoni's basic idea was to transform the BVP with an unbounded right-hand side into a problem with a bounded right-hand side (this transformation is possible thanks to the existence of lower and upper solutions) and, in the second step, to show that a solution of the modified problem is also a solution of the original problem. Together with the later introduced Nagumo conditions for the derivative dependent right-hand sides this basic scheme forms the foundations of this method.

On the other hand, the time scales calculus, with its concept to unify and extend discrete and continuous worlds, is a recent idea (the seminal work is due to Hilger and Aulbach, see e.g. [9]). In spite of this, this calculus is already broadly used. It is not surprising that, after the Schauder fixed point theorem for bounded right-hand sides, the lower and upper solutions technique was used to investigate the problems with unbounded right-hand sides. The first results for Dirichlet boundary conditions are due to Elvan Akın-Bohner [2], or [3, Section 6.6]. Later, similar statements were obtained also for periodic conditions, see e.g. Cabada [5], Stehlík [12] or Topal [13]

The main drawback of the concept of lower and upper solutions, which often hinders its practical application, is the assumption on their existence. Logical reaction to this objection was a successful attempt to include also the case when the lower and upper solutions don't satisfy the common ordering, i.e. the lower solution is above the upper solution in some points of the considered interval. The

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so-called non-well ordered case for differential equations was first studied in 1970s, see e.g. Sattinger [11].

The traditional ways to deal with the non-well ordered pairs rely on the periodicity and boundedness of trigonometric functions, properties of Fučík spectrum and the existence of intersections of lower and upper solutions (for the survey on lower and upper solutions, see e.g. De Coster, Habets [6]). Unfortunately, one cannot straightforwardly extend these concepts to the discrete or time scales context. Therefore, we avoid these approaches by relying on the degree theory.

We recall the basic definitions and notations concerning time scales calculus, the reader acquainted with the basic concepts (within the scope of the first chapters of Bohner, Peterson [3] and [4]) can jump over to the equation (1).

Time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . The natural numbers \mathbb{N} , the integers \mathbb{Z} or the union of intervals $[0, 1] \cup [2, 3]$ are the most natural examples.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ and the *backward jump operator* $\varrho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \varrho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. We say that a point $t \in \mathbb{T}$ is *right-scattered*, *left-scattered*, *right-dense*, *left-dense* if $\sigma(t) > t$, $\varrho(t) < t$, $\sigma(t) = t$, $\varrho(t) = t$, respectively. Moreover, we define the *forward graininess* function $\mu : \mathbb{T} \rightarrow [0, \infty)$ by

$$\mu(t) := \sigma(t) - t.$$

In the above references one can find the definition of the so-called *delta-derivative* x^Δ , which is equivalent to x' if $\mathbb{T} = \mathbb{R}$, or to Δx if $\mathbb{T} = \mathbb{Z}$. Similarly, several concepts of integration have been extended as well, ranging from Cauchy-Newton [3, Section 1.4] to Henstock-Kurzweil [10] integration.

For the sake of clarity we introduce the closed time scale interval by

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T},$$

with the note that other types of intervals are defined in the analogous way.

To simplify complicated formulae, we use the abridged notations:

$$x^\sigma(t) = x(\sigma(t)), \text{ and } x^\varrho(t) = x(\varrho(t)).$$

We define an *rd-continuous* function as a function that is continuous in all right-dense points and left-sided limits exist in left-dense points. The set of all rd-continuous functions will be denoted by C_{rd} . The set of twice differentiable functions whose second derivative is rd-continuous will be denoted by C_{rd}^2 . Finally, we define the following function space:

$$C_{\text{rd},0}^2([0, \sigma^2(1)]_{\mathbb{T}}) := \{x \in C_{\text{rd}}^2 : x(0) = x(\sigma^2(1)) = 0\}.$$

For the sake of brevity, we often use $C_{\text{rd},0}^2$ instead.

In this paper we consider a non-well ordered couple of lower and upper solutions for the following Dirichlet BVP:

$$(1) \quad \begin{cases} -x^{\Delta\Delta}(t) = f(t, x^\sigma(t)) & \text{on } [0, 1]_{\mathbb{T}} \\ x(0) = x(\sigma^2(1)) = 0. \end{cases}$$

The solution of (1) is a function $x \in C_{\text{rd},0}^2$ which satisfies the equation for all $t \in [0, 1]_{\mathbb{T}}$.

We base our work on the existence theorems for the well ordered case which are presented in Akin-Bohner [2]. Therefore, we start, in Section 2, with a slight modification of one of these results. Namely, we provide further information about the degree of the corresponding operator.

Next, in Section 3, we use this extension to prove the existence in non-well ordered setting. Aside from the degree theory, *a priori* estimate and the properties of first eigenvalue and eigenfunction are our main tools. If f satisfies certain growth and limit conditions, we obtain the existence of a solution. The similar approach for the p -Laplacian can be found in Drábek, Girg and Manásevich [7].

Finally, by combining these results we suggest how to acquire the existence of multiple solutions. This idea is illustrated, in Section 4, on the existence of three solutions.

2. WELL ORDERED CASE

In this section we present the basic definitions and notations for lower and upper solutions and we amend the existing results for well ordered pairs. Let us first define lower and upper solutions for BVP (1).

Definition 1. A function $\alpha \in C_{rd}^2([0, \sigma^2(1)]_{\mathbb{T}})$ is called a *lower solution* of (1) if

$$\alpha(0) \leq 0, \quad \alpha(\sigma^2(1)) \leq 0,$$

and

$$-\alpha^{\Delta\Delta}(t) \leq f(t, \alpha^\sigma) \quad \text{for all } t \in [0, \sigma^2(1)]_{\mathbb{T}}.$$

Similarly, a function $\beta \in C_{rd}^2([0, \sigma^2(1)]_{\mathbb{T}})$ is called an *upper solution* of (1) if

$$\beta(0) \geq 0, \quad \beta(\sigma^2(1)) \geq 0,$$

and

$$-\beta^{\Delta\Delta}(t) \geq f(t, \beta^\sigma) \quad \text{for all } t \in [0, \sigma^2(1)]_{\mathbb{T}}.$$

Next, let us define the ordering in $C_{rd,0}^2$.

Definition 2. We say that x is strictly smaller than y (we denote it by $x \ll y$), if

$$x(t) < y(t), \quad \text{for } t \in (0, \sigma^2(1))_{\mathbb{T}},$$

and following conditions holds on the boundary:

- (i) either $x(0) < y(0)$, or $x^\Delta(0) < y^\Delta(0)$, and
- (ii) either $x(\sigma^2(1)) < y(\sigma^2(1))$, or $x^\Delta(\sigma(1)) < y^\Delta(\sigma(1))$.

Using this ordering we can define an important subclass of lower and upper solutions.

Definition 3. We say that α is a *strict* lower solution of (1) if:

- α is a lower solution of (1),
- every possible solution x of (1), satisfying $\alpha \leq x$ satisfies $\alpha \ll x$.

Reversing the above inequality we can get the corresponding definition of a *strict* upper solution.

As usual, we introduce the solution operator $T : C_{rd,0}^2 \rightarrow C_{rd,0}^2$ defined by

$$(2) \quad Tx(t) := \int_0^{\sigma(1)} G(t, s) f(s, x^\sigma(s)) \Delta s,$$

where $G(t, s)$ is a Green's function for (1) with $f(t, s) = 0$ (see e.g. [3, Corollary 4.76]).

At this stage, we are ready to state an expanded existence theorem.

Theorem 4. *Let f be a continuous function. Let α, β be a lower and an upper solution, respectively, for which $\alpha \leq \beta$ holds. Then the problem (1) has at least one solution x satisfying:*

$$\alpha \leq x \leq \beta \quad \text{in } [0, \sigma^2(1)]_{\mathbb{T}}.$$

Furthermore, if α and β are strict and $\alpha \ll \beta$ holds then there exists $R_0 > 0$, such that for all $R > R_0$:

$$\deg(I - T; \Omega_1, o) = 1,$$

where

$$\Omega_1 := \{x \in C_{rd}^2([0, \sigma^2(1)]_{\mathbb{T}}) : \alpha \ll x \ll \beta\} \cap B(o, R).$$

Proof. With the purpose of applying the Schauder fixed point theorem, Akin-Bohner, in [3, Theorem 6.54], defines a modified right-hand side function by:

$$\widehat{f}(t, x) = \begin{cases} f(t, \beta^\sigma(t)) + \frac{x - \beta^\sigma(t)}{1 + |x|} & \text{if } x \geq \beta^\sigma(t), \\ f(t, x) & \text{if } \alpha^\sigma(t) \leq x \leq \beta^\sigma(t), \\ f(t, \alpha^\sigma(t)) + \frac{x - \beta^\sigma(t)}{1 + |x|} & \text{if } x \leq \alpha^\sigma(t). \end{cases}$$

Using the continuity and boundedness of this function she obtains the compactness of \widehat{T} (defined by (2) with f replaced by \widehat{f}) and, consequently, fixed point x of this modified operator. Finally, one can show that $\alpha(t) \leq x(t) \leq \beta(t)$ which implies that x is also the fixed point of T defined by (2) and thus a solution of (1).

We define the constant $R_0 > 0$ as a bound of operator \widehat{T} (the existence of this bound is ensured by the definition of \widehat{f}), i.e. for each $y \in C_{rd,0}^2$ we have

$$\|\widehat{T}(y)\|_{C_{rd,0}^2} < R_0.$$

This guarantees the existence of an admissible homotopy:

$$H(\tau, x) = I(x) - \tau \widehat{T}(x), \quad \tau \in [0, 1],$$

which implies the following equality of degrees:

$$\deg(I - \widehat{T}; B(o, R_0), o) = \deg(I; B(o, R_0), o) = 1.$$

Moreover, since α and β are strict, there is no solution of $x = \widehat{T}(x)$ with $x(t) \leq \alpha(t)$ or $x(t) \geq \beta(t)$ for any $t \in (0, \sigma^2(1))_{\mathbb{T}}$ and we can deduce that:

$$\deg(I - \widehat{T}; \Omega_1, o) = \deg(I - \widehat{T}; B(o, R_0), o) = 1.$$

To conclude, the definition of Ω_1 yields that

$$\deg(I - T; \Omega_1, o) = \deg(I - \widehat{T}; \Omega_1, o) = 1.$$

□

3. NON-WELL ORDERED CASE

First of all, we recall the basic results concerning the eigenvalue problem:

$$(3) \quad \begin{cases} -x^{\Delta\Delta}(t) = \lambda x(t) & \text{on } [0, \sigma^2(1)]_{\mathbb{T}}, \\ x(0) = x(\sigma^2(1)) = 0. \end{cases}$$

Using the existing oscillation theorem we can prove this simple statement.

Lemma 5. *The first eigenvalue λ_1 of (3) is positive and the corresponding eigenfunction $\varphi_1(t) > 0$ for all $t \in (0, \sigma^2(1))_{\mathbb{T}}$.*

Proof. Obviously, (3) has only a trivial solution if $\lambda = 0$. Now, let us suppose that $\lambda < 0$ is an eigenvalue. The corresponding eigenfunction $\varphi(t)$ (or $-\varphi(t)$) must attain a maximum in $(0, \sigma^2(1))_{\mathbb{T}}$. Let us suppose that $m \in (0, \sigma^2(1))_{\mathbb{T}}$ is the first point where the maximum is attained. Let us distinguish between two cases.

- m is left-dense. In that case $\varphi^{\Delta\Delta}(m) \leq 0$ and $\varphi^{\Delta}(m) = 0$, which leads to the following contradiction:

$$0 \geq \varphi^{\Delta\Delta}(m) = \lambda \varphi^{\sigma}(m) = \lambda(\varphi(m) + \mu(m)\varphi^{\Delta}(m)) = \lambda\varphi(m) > 0.$$

- m is left-scattered. This implies that $x^{\Delta}(m) \leq 0$ and $x^{\Delta}(\varrho(m)) > 0$ and we can reach a contradiction by

$$0 < \lambda\varphi(m) = \varphi^{\Delta\Delta}(\varrho(m)) = \frac{\varphi^{\Delta}(m) - \varphi^{\Delta}(\varrho(m))}{\mu^{\rho}(m)} < 0.$$

The positivity of first eigenfunctions is the immediate consequence of Oscillation Theorem, which is due to Agarwal, Bohner and Wong [1, Theorem 1] or [3, Theorem 4.106]. \square

At this stage, we are ready to prove the existence result also in the case when lower and upper solutions are without ordering.

Theorem 6. *Let f be a continuous function satisfying*

- (i) *there are $c, d > 0$ such that:*

$$(4) \quad |f(t, s)| \leq c + d|s|,$$

for all $t \in [0, 1]_{\mathbb{T}}$ and for all $s \in \mathbb{R}$,

- (ii) *and*

$$(5) \quad \lim_{|s| \rightarrow \infty} \frac{f(t, s)}{s} = \lambda_1.$$

Let us assume that α, β are lower and upper solutions and that there exists $\tau \in (0, \sigma^2(1))_{\mathbb{T}}$ such that

$$\alpha(\tau) > \beta(\tau).$$

Then (1) has at least one solution in:

$$S := \overline{\left\{ x \in C_{rd,0}^2 : \exists \zeta, \eta \in (0, \sigma^2(1))_{\mathbb{T}} : x(\zeta) < \alpha(\zeta), x(\eta) > \beta(\eta) \right\}}^{C_{rd,0}^2}.$$

If we define $\Omega_2 := S \cap B(o, R)$ and assume that there is no solution on $\partial\Omega_2$ then there exists $R_0 > 0$ such that for all $R > R_0$:

$$\deg(I - T; \Omega_2, o) = -1.$$

Proof. We assume that there is no solution on ∂S (otherwise there is no reason to proceed with the proof). For the sake of lucidity, we divide our proof into three parts.

- *A priori estimate.* First, let us prove that if we define $f_r : [0, 1]_{\mathbb{T}} \times \mathbb{R} \rightarrow \mathbb{R}$ by:

$$f_r(t, y) := \begin{cases} f(t, y) & \text{if } |y| < r, \\ (1 + r - |y|)f(t, y) & \text{if } r \leq |y| \leq r + 1, \\ 0 & \text{if } |y| \geq r + 1, \end{cases}$$

then there exists $K > 0$ such that for any $r > 0$ and any solution $x \in S$ of

$$(6) \quad \begin{cases} -x^{\Delta\Delta}(t) = f_r(t, x^\sigma(t)) & \text{on } [0, 1]_{\mathbb{T}}, \\ x(0) = x(\sigma^2(1)) = 0, \end{cases}$$

the following *a priori* estimate holds:

$$\|x\|_{C_{r,d,0}^2} \leq K.$$

As usual, we suppose that this assumption is not satisfied, i.e. there exists a sequence $\{r_k\}_{k=1}^{\infty}$ with $r_k > 0$ and a corresponding sequence of solutions $\{x_k\}_{k=1}^{\infty}$ satisfying $\|x_k\|_{C_{r,d,0}^2} \geq k$ and solving

$$(7) \quad \begin{cases} -x_k^{\Delta\Delta}(t) = f_{r_k}(t, x_k^\sigma(t)) & \text{on } [0, 1]_{\mathbb{T}}, \\ x_k(0) = x_k(\sigma^2(1)) = 0. \end{cases}$$

Defining $y_k := \frac{x_k}{\|x_k\|_{C_{r,d,0}^2}}$ and dividing (7) by $\|x_k\|_{C_{r,d,0}^2}$ we obtain

$$(8) \quad \begin{cases} -y_k^{\Delta\Delta}(t) = \frac{f_{r_k}(t, x_k^\sigma(t))}{\|x_k\|_{C_{r,d,0}^2}} & \text{on } [0, 1]_{\mathbb{T}}, \\ y_k(0) = y_k(\sigma^2(1)) = 0. \end{cases}$$

The boundedness of the sequence (clearly $\|y_k\|_{C_{r,d,0}^2} = 1$) and the compactness of T provide convergence (at least for a subsequence) to some $y \in C_{r,d,0}^2$

$$y_k \rightarrow y \quad \text{in } C_{r,d,0}^2.$$

The condition (4) implies that for some sufficiently large $\varepsilon \geq 0$ the right-hand sides of (8) are bounded by a constant (and thus integrable) function $h(s) = d + \varepsilon$ and, moreover, using the limit assumption (5) we can get:

$$\frac{f_{r_k}(t, x_k^\sigma(t))}{\|x_k\|_{C_{r,d,0}^2}} = \frac{f_{r_k}(t, x_k^\sigma(t))}{\|x_k\|_{C_{r,d,0}^2}} \frac{x_k}{x_k} \rightarrow \lambda_1 y.$$

Thus the Dominated Convergence Theorem (see Peterson, Thompson [10, Theorem 2.17] for its most general form on time scales) yields that y solves the problem:

$$\begin{cases} -y^{\Delta\Delta}(t) = \lambda_1 y(t) & \text{on } [0, 1]_{\mathbb{T}}, \\ y(0) = y(\sigma^2(1)) = 0. \end{cases}$$

Taking into account $\|y\| = 1$ again we obtain that y is a nonzero multiple of the first eigenfunction $\varphi(t) > 0$ for $t \in (0, \sigma^2(1))_{\mathbb{T}}$ (see Lemma 5). If y is positive then $x_k(t) \rightarrow \infty$ for all $t \in (0, \sigma^2(1))_{\mathbb{T}}$ which implies that there is $k_0 \in \mathbb{N}$ such that for all $k > k_0$ there does not exist $\zeta \in (0, \sigma^2(1))_{\mathbb{T}}$ such that the inequality $x_k(\zeta) < \alpha(\zeta)$ holds. Therefore $x_k \notin S$, a contradiction.

Similarly, if y is negative then there is $k \in \mathbb{N}$ such that for all $k > k_0$ and for all $\eta \in (0, \sigma^2(1))_{\mathbb{T}}$ we have $x(\eta) < \beta(\eta)$, a contradiction.

- *Construction of strict well ordered lower and upper solutions.* Let us consider an arbitrary $R > 0$ satisfying

$$(9) \quad R > R_0 := \max \{K, \|\alpha\|_C, \|\beta\|_C\} + 1,$$

and the BVP (6) with $r = R$, i.e.

$$(10) \quad \begin{cases} -x^{\Delta\Delta}(t) = f_R(t, x^\sigma(t)) & \text{on } [0, 1]_{\mathbb{T}}, \\ x(0) = x(\sigma^2(1)) = 0. \end{cases}$$

We show that $u := -R - 2$ and $v := R + 2$ are lower and upper solution of (10), respectively. Obviously, u is a lower solution since

$$u(0) = -R - 2 < 0, \quad u(\sigma^2(1)) = -R - 2 < 0,$$

and

$$u^{\Delta\Delta}(t) = 0 = f_R(t, -R - 2), \text{ for all } t \in [0, 1]_{\mathbb{T}}.$$

Now assume that u is not strict, i.e. there exists $m \in (0, \sigma^2(1))_{\mathbb{T}}$ defined by

$$m := \min_{t \in (0, \sigma^2(1))_{\mathbb{T}}} \{t : x(t) = -R - 2\}.$$

Again, we divide our reasoning into two parts.

- Let us suppose that m is left-dense. The minimality of x at m implies that $x^\Delta(m) = 0$ and the left-density of m provides the existence of $\varepsilon > 0$ such that $x(t) < -R - 1$ for all $t \in (m - \varepsilon, m)_{\mathbb{T}}$, i.e. $f_R(t, x^\sigma(t)) = 0$ for these t . These two facts suggest that $x^{\Delta\Delta}(t) = 0$ for all $t \in [m, 1]_{\mathbb{T}}$. Thus $x(\sigma^2(1)) = -R - 2$ and x is not a solution of (10), a contradiction.
- Now, we assume that m is left-scattered. Since x achieves its minimum first at m , we obtain the following two conditions:

$$x^\Delta(m) \geq 0, \text{ and } x^\Delta(\varrho(m)) < 0.$$

But this leads to the following contradiction:

$$0 = f_R(\varrho(m), x(m)) = x^{\Delta\Delta}(\varrho(m)) = \frac{x^\Delta(m) - x^\Delta(\varrho(m))}{\mu(\varrho(m))} > 0.$$

Similarly, one can derive that $v = R + 2$ is a strict supersolution of (10).

- *Computation of the degree.* We define $T_R : C_{rd,0}^2 \rightarrow C_{rd,0}^2$ by

$$T_R(x) := \int_0^{\sigma(1)} G(t, s) f_R(s, x^\sigma(s)).$$

Obviously, if we find a fixed point x_0 of T_R then x_0 is a solution of (10). If, moreover, $x_0 \in B(o, R)$, then x_0 is solution of (1) as well since, thanks to the definition of f_R , operators T and T_R coincide on this ball. We define three sets by

$$\Omega_u^v := \{x \in C_{rd,0}^2 : u \ll x \ll v\},$$

$$\Omega_\alpha^v := \{x \in C_{rd,0}^2 : \alpha \ll x \ll v\}, \quad \Omega_u^\beta := \{x \in C_{rd,0}^2 : u \ll x \ll \beta\}.$$

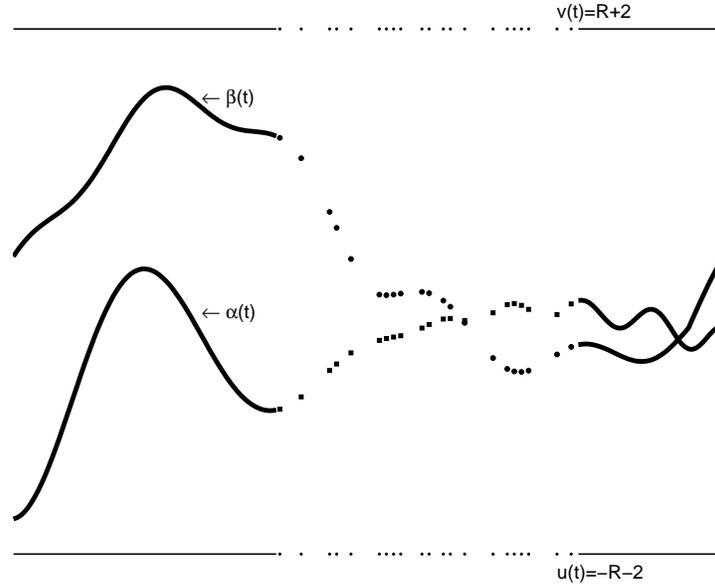


FIGURE 1. Non-well ordered case.

Clearly, Ω_α^v , Ω_u^β and Ω_2 are pairwise disjoint subsets. Thus the properties of the degree and the well-ordered result (Theorem 4) enable the following computation:

$$\begin{aligned}
 1 &= \deg(I - T_R; B(o, R) \cap \Omega_u^v, o) \\
 &= \deg(I - T_R; B(o, R) \cap \Omega_u^\beta, o) + \deg(I - T_R; B(o, R) \cap \Omega_\alpha^v, o) \\
 &\quad + \deg(I - T_R; \Omega_2, o) \\
 &= 2 + \deg(I - T_R; \Omega_2, o).
 \end{aligned}$$

Therefore (T and T_R coincide on $B(o, R)$), we obtain the required result

$$\deg(I - T; \Omega_2, o) = -1.$$

□

The statement of Theorem 6 and the strict pair u and v from its proof are illustrated in Figure 1.

Remark 7. In contrast to Theorem 4, Theorem 6 gives less transparent information about the existing solution. This is mainly due to the opaque structure of S (and consequently of Ω_2). In fact, we know only that there exists a bound R on the norm of this solution (R is closer specified in the proof in (9)) and that there exist $\zeta, \eta \in (0, \sigma^2(1))_{\mathbb{T}}$ such that $x(\zeta) < \alpha(\zeta)$ and $x(\eta) > \beta(\eta)$, respectively.

With respect to this remark, we add one simple example to illustrate the above statement.

Example 8. Let us deal with $\mathbb{T} = \frac{1}{5}\mathbb{Z}$ and a continuous function

$$\tilde{f}(t, y) = \begin{cases} \lambda_1(y + 5) & \text{if } y \leq -5, \\ -y(y - 5)(y + 5) & \text{if } -5 < y < 5, \\ \lambda_1(y - 5) & \text{if } y \geq 5. \end{cases}$$

Clearly, \tilde{f} satisfies condition (i) and (ii) of Theorem 6. Let us consider the corresponding BVP:

$$(11) \quad \begin{cases} -x^{\Delta\Delta}(t) = \tilde{f}(t, x^\sigma(t)) & \text{on } [0, 1]_{\mathbb{T}} \\ x(0) = x(\frac{7}{5}) = 0. \end{cases}$$

It is easy to verify that α and β , defined by

t	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	$\frac{6}{5}$	$\frac{7}{5}$
$\alpha(t)$	-1	$-\frac{3}{2}$	$-\frac{1}{2}$	1	2	$\frac{3}{2}$	0	$-\frac{3}{2}$
$\beta(t)$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1

are lower and upper solutions of (11). Obviously, for all $t \in [\frac{3}{5}, 1]_{\mathbb{T}}$ we have $\alpha(t) > \beta(t)$. Therefore, we can apply Theorem 6 and claim that the problem (11) has a solution $x \in C_{rd,0}^2$. The only additional information about this solution is that there exist $\zeta, \eta \in [\frac{1}{5}, \frac{6}{5}]_{\mathbb{T}}$ such that $x(\zeta) \leq \alpha(\zeta)$ and $x(\eta) \geq \beta(\eta)$.

4. MULTIPLE SOLUTIONS

The combination of the results for well-ordered case and non-well-ordered counterpart opens the way for the existence of multiple solution. As an example of such a process we state a simple result for the existence of three solutions, which can be generalized to other cases.

Theorem 9. Let f satisfy the assumption (i) and (ii) from Theorem 6. Let us assume that α_1 and α_2 are lower solutions and β_1 and β_2 are upper solutions of (1) which satisfy

$$\alpha_1 \ll \beta_1, \quad \alpha_2 \ll \beta_2,$$

and let there exist $\tau \in (0, \sigma^2(1))_{\mathbb{T}}$ such that

$$(12) \quad \alpha_2(\tau) > \beta_1(\tau).$$

Then the problem (1) has at least three distinct solutions.

Proof. First, we apply twice Theorem 4 to obtain the existence of solutions x_1 and x_2 satisfying:

$$\alpha_1 \ll x_1 \ll \beta_1, \quad \alpha_2 \ll x_2 \ll \beta_2.$$

Furthermore, (12) allows us to apply Theorem 6 to obtain the existence of a solution x_3 . Since $x_3 \in \mathcal{S}$, we know that there exists $\zeta, \eta \in (0, \sigma^2(1))_{\mathbb{T}}$ such that $x_3(\zeta) \leq \alpha_2(\zeta)$ and $x_3(\eta) \geq \beta_1(\eta)$ and this implies that x_3 is different to x_1, x_2 . \square

The statement of Theorem 9 and the solutions $x_i, i = \{1, 2, 3\}$, from its proof are illustrated in Figure 2.

Theorem 9 basically claims that if you have a non-well ordered couple and for each function from this pair you are able to find a function with which it forms an ordered couple, you have at least three solutions. We illustrate this idea on the extension of Example 8.

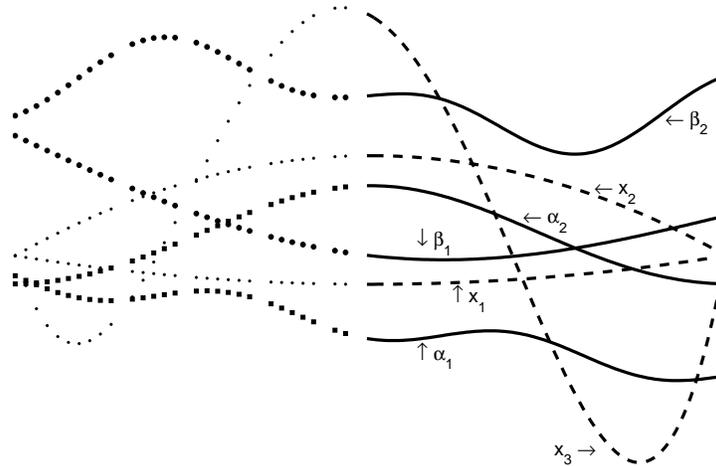


FIGURE 2. Existence of three solutions.

Example 10. Let us assume that \tilde{f} , α and β are defined as in Example 8. Obviously $\underline{\alpha}(t) \equiv -5$ and $\overline{\beta}(t) \equiv 5$ are another lower and upper solutions of (11) (note that $\tilde{f}(t, \pm 5) = 0$). Thus we can claim that the problem (11) has at least three distinct solutions where x_3 is described in Example 8, $\underline{\alpha} \ll x_1 \ll \beta$ and $\alpha \ll x_2 \ll \overline{\beta}$.

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