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Applications of Maximum Principles to Dynamic equations on Time Scales

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Abstract

We apply the strong maximum principle to obtain a priori bounds and uniqueness of solutions for some initial value and boundary value problems as well as to establish oscillation results for second order dynamic equations on time scales. Our results extend corresponding results for ordinary differential equations to the times scales setting. Moreover they involve different assumptions from and complement corresponding results in the time scales literature.

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1 INTRODUCTION

Recently Mawhin, Tonkes and Thompson [13] established discrete analogues of the strong maximum principle and its variants for ordinary differential equations (see Protter and Weinberger [14]). They applied their results to answer some uniqueness questions posed in Thompson [17].

In [15] the authors established the strong maximum principle and variants for second order dynamic equations on time scales that include the corresponding results for ordinary differential equations (see [14]) and difference equations (see [13]) as special cases.

In the literature recently there has been a number of results on the existence of multiple solutions for second order differential and difference equations as well as for dynamic equations on time scales which have been based on lower and upper solutions; see, for example, DeCoster and Habets [6], Drábek, Henderson, Tisdell [7] and Henderson and Thompson [10, 9, 8, 11]. These existence results depend on establishing a priori bounds on solutions. Thus it is natural to ask what are sufficient conditions for a priori bounds on solutions and what are sufficient conditions for uniqueness of solutions.

In this paper we apply our strong maximum principle and its variants to obtain a priori bounds for solutions of initial value and boundary value problems as well as to establish oscillation results for second order dynamic equations on time scales.

Our results extend the continuous results given in Protter and Weinberger, [14] and their discrete counterparts, see Agarwal [1] for general results, Agarwal et al. [2] for oscillation results, Cheng [5] for alternative applications of maximum principles and Mawhin et al. [13] for nonlinear comparisons, uniqueness results and approximations. The main contribution of this work consists in the fact that the presented results show the precise transition from the continuous to the discrete setting. Consequently, we believe that some of them can be applied in numerical analysis. In order to allow the interested reader to compare the main differences easier we try to follow the structure of Protter and Weinberger [14] and Mawhin et al. [13].

Moreover, the results appear to complement corresponding ones in the time scales literature in that we consider quite general boundary conditions, and we do not assume that equations are in self-adjoint form nor do we require the coefficients to be right dense continuous. This latter is important when establishing comparison results for nonlinear equations.

Dynamic equations on time scales is a relatively new, rapidly expanding area of research which unifies and extends the discrete and continuous calculus. See, for example, the books by Bohner and Peterson [3] and [4], together with the references therein.

2 PRELIMINARIES

To understand the notation used and the notion of a time scale some preliminary definitions are needed.

Definition A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale need not be connected, the concept of jump operators is needed to overcome this difficulty.

Definition The forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ ($\rho(t)$ at $t \in \mathbb{T}$ for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\} \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\}) \ .$$

For simplicity and clarity set $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$.

If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t = \rho(t)$. If $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$ and $\rho(t) = t - 1$.

Throughout this work it is assumed that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also it is assumed throughout that $a < b$ are points in $\mathbb{T}$ with $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$.

The jump operators σ and ρ allow the following classification of points in a time scale: If $\sigma(t) > t$ then call the point t right-scattered; while if $\rho(t) < t$ then call t left-scattered. If $\sigma(t) = t$ then call the point t right-dense; while if $\rho(t) = t$ then call t left-dense. We shall also use the notation $\mu(t) := \sigma(t) - t$, where μ is called the graininess function.

If $\mathbb{T}$ has a left-scattered maximum M, we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition Fix $t \in \mathbb{T}^\kappa$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t with

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

Call $y^\Delta(t)$ the (delta) derivative of $y(t)$. Define the second (delta) derivative by $y^{\Delta\Delta} = (y^\Delta)^\Delta$. We say that a function f is (delta) differentiable at $t \in \mathbb{T}^\kappa$ if $f^\Delta(t)$ exists.

Remark As usual, for the sake of brevity we omit the adjective *delta* when dealing with derivatives and differentiability. Consequently, the term *derivative* (differentiability) refers to *delta derivative* (delta differentiability) in the rest of the paper.

The following theorem is due to Hilger [12].

Theorem 1. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$.*
(i) If f is differentiable at t then f is continuous at t.
(ii) If f is continuous at t and t is right-scattered then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If f is differentiable and t is right-dense then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t then $f(\sigma(t)) = f(t) + (\sigma(t) - t)f^{\Delta}(t)$.

Definition Let $f : \mathbb{T} \to \mathbb{R}$. Define and denote $f \in C_{rd}(\mathbb{T};\mathbb{R})$ as right-dense continuous if for each $t \in \mathbb{T}$:
$\lim_{s\to t^+} f(s) = f(t)$, if $t \in \mathbb{T}$ is right-dense,
$\lim_{s\to t^-} f(s)$ exists and is finite if $t \in \mathbb{T}$ is left-dense.

Note that, in general, the jump operator σ is neither differentiable nor bounded, as shown in [3, Example 1.56] however $\sigma \in C_{rd}(\mathbb{T})$.

In [15] the authors established the strong maximum principle, the generalized strong maximum principle and the boundary point lemma listed below.

Theorem 2. (**The strong maximum principle**) *Assume that the functions $g_1,\ g_2,\ h_2\ :\ [a,b]_{\mathbb{T}} \to \mathbb{R}$ satisfy*

$$\mu g_1 < \quad 1, \tag{1}$$

$$\mu g_2 > -1, \tag{2}$$

$$h_2 \leq \quad 0, \tag{3}$$

$$\frac{g_1 + g_2}{1 + g_2\mu} \text{ is bounded,} \tag{4}$$

and

$$\frac{h_2}{1 + g_2\mu} \text{ is bounded.} \tag{5}$$

Moreover, let $x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}$ satisfy the dynamic inequality

$$\mathcal{L}_2[x] := x^{\Delta\Delta} + g_1 x^{\Delta} + g_2 x^{\Delta\sigma} + h_2 x^{\sigma} \geq 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}\,. \tag{6}$$

If x attains a nonnegative maximum M at $t \in \left(a, \sigma^2(b)\right)_{\mathbb{T}}$, then $x \equiv M$.

Note that all quantities in the above and subsequent results are assumed to exist. For example, since x satisfies (6), $x^{\Delta\Delta}$ is defined on $[a,b]_{\mathbb{T}}$, x is defined and continuous on $[a, \sigma^2(b)]_{\mathbb{T}}$ and x^{Δ} is defined and continuous on $[a, \sigma(b)]_{\mathbb{T}}$. In particular if $b < \sigma(b) = \sigma^2(b)$, then we assume that $x^{\Delta}(\sigma^2(b))$ exists. This may appear strange at first as one usually works with one sided derivatives at end points in the continuous case.

Theorem 3. (**The generalized strong maximum principle**) *Let $x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}$ satisfy the dynamic inequality*

$$\widetilde{\mathcal{L}}_2[x] := x^{\Delta\Delta} + g x^{\Delta} + h x^{\sigma} \geq 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}\,, \tag{7}$$

where $g,\ h : [a,b]_{\mathbb{T}} \to \mathbb{R}$, g satisfies

$$\mu g < 1, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}} \tag{8}$$

and there exists a function w satisfying

$$w > 0, \quad \text{for} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}} \tag{9}$$

while on $[a,b]_{\mathbb{T}}$

$$\widetilde{\mathcal{L}}_2[w] \leq 0, \tag{10}$$

$$\frac{gw + w^{\Delta} + w^{\sigma\Delta}}{w^{\sigma} + \mu w^{\sigma\Delta}} \text{ is bounded} \tag{11}$$

and

$$\frac{\widetilde{\mathcal{L}}_2[w]}{w^{\sigma} + \mu w^{\sigma\Delta}} \text{ is bounded.} \tag{12}$$

If $v = \dfrac{x}{w}$ attains a local maximum M at $t \in \left(a, \sigma^2(b)\right)_{\mathbb{T}}$, then $v \equiv M$.

Remark 4. Let $\overline{g}_1$, $\overline{g}_2$ and $\overline{h}_2$ be given by

$$\overline{g}_1 := \frac{gw + w^{\Delta}}{w^{\sigma}}, \quad \overline{g}_2 := \frac{w^{\sigma\Delta}}{w^{\sigma}}, \quad \overline{h}_2 := \frac{\widetilde{\mathcal{L}}_2[w]}{w^{\sigma}}.$$

It follows from the statement of Theorem 2.3 and [15, Lemma 11] that $\overline{g}_1$, $\overline{g}_2$ and $\overline{h}_2$ satisfy all the conditions required to apply Theorem 2 to

$$\overline{\mathcal{L}}_2[v] = v^{\Delta\Delta} + \overline{g}_1 v^{\Delta} + \overline{g}_2 v^{\Delta\sigma} + \overline{h}_2 v^{\sigma} \geq 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}.$$

Theorem 3 follows.

We show in [15, Remark 13] that the condition (11) is always satisfied if σ^{Δ} exists and is bounded.

We also established the following boundary point lemma.

Lemma 5. (**The boundary point lemma**) *Assume that the functions g_1, g_2 and h_2 satisfy (1), (2), (4), (3) and (5). Moreover, assume that x is a nonconstant function satisfying (6). If x attains a nonnegative maximum at a, then $x^{\Delta}(a) < 0$. If x attains a nonnegative maximum at $\sigma^2(b)$, then $x^{\Delta}(\rho(\sigma^2(b))) > 0$.*

Remark 6. We make the following observations.

(i) Whether or not one needs the existence of $w^{\sigma\Delta}$ in the generalized maximum principle is an open question.

(ii) There is an analogue of the boundary point lemma for the case $h_2(t) > 0$ for some $t \in [a,b]_{\mathbb{T}}$.

Remark 7. All quantities in the above and subsequent results are assumed to exist. For example, if x satisfies (6), then $x^{\Delta\Delta}$ is defined on $[a,b]_{\mathbb{T}}$, x is defined and continuous on $[a,\sigma^2(b)]_{\mathbb{T}}$ and x^{Δ} is defined and continuous on $[a,\sigma(b)]_{\mathbb{T}}$. In particular if $b < \sigma(b) = \sigma^2(b)$, then we assume that $x^{\Delta}(\sigma^2(b))$ exists. This may appear strange at first as one usually works with one sided derivatives at end points in the continuous case.

Definition (cf. [3, Definition 4.35]) We say that a solution x of (6) has a generalized zero at t if

$$x(t) = 0$$

or if t is left scattered and

$$x(\rho(t))x(t) < 0.$$

In the latter case, by abuse of notation, we say will say that the generalized zero is in the real interval $(\rho(t), t)$.

Note between consecutive zeroes a function w may change sign however between two consecutive generalized zeroes the function w has one sign.

3 Nonlinear Comparison Results

Let $f : [a,b]_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$f(t,y,z), \quad f_u(t,u,v), \quad f_v(t,u,v)$$

are continuous. Assume that $x : \left[a, \sigma^2(b)\right]_{\mathbb{T}} \to \mathbb{R}$ is a solution of the nonlinear equation:

$$x^{\Delta\Delta} + f(t, x^{\sigma}, x^{\Delta}) = 0 \tag{13}$$

on $[a,b]_{\mathbb{T}}$. Let us consider the function $w : \left[a, \sigma^2(b)\right]_{\mathbb{T}} \to \mathbb{R}$ which satisfies the dynamic inequality

$$w^{\Delta\Delta} + f(t, w^{\sigma}, w^{\Delta}) \geq 0, \tag{14}$$

on $[a,b]_{\mathbb{T}}$. Defining $y := w - x$ and subtracting (13) from (14) we get

$$y^{\Delta\Delta} + f(t, w^{\sigma}, w^{\Delta}) - f(t, x^{\sigma}, x^{\Delta}) \geq 0.$$

The mean value theorem for functions of two variables yields the following equivalent formulation

$$y^{\Delta\Delta} + f_v(t,\psi,\zeta)y^{\Delta} + f_u(t,\psi,\zeta)y^{\sigma} \geq 0. \tag{15}$$

for some $(\psi,\zeta) = \theta(w^{\sigma}, w^{\Delta}) + (1-\theta)(x^{\sigma}, x^{\Delta})$, where $\theta \in (0,1)$. In order to apply Theorem 2 we restrict our choice of f to those which satisfy (see (1))

$$\mu(t) f_v(t,u,v) < 1, \tag{16}$$

for all $t \in [a,b]_{\mathbb{T}}$ and $u, v \in \mathbb{R}$ and (see (3))

$$f_u(t,u,v) \leq 0. \tag{17}$$

for all $t \in [a,b]_{\mathbb{T}}$ and $u, v \in \mathbb{R}$. Then Theorem 2 provides the following result.

Theorem 8. *Suppose that f, f_u, f_v are continuous, that f satisfies* (16) *and* (17) *and that w and x satisfy*

$$w^{\Delta\Delta} + f(t, w^{\sigma}, w^{\Delta}) \geq x^{\Delta\Delta} + f(t, x^{\sigma}, x^{\Delta}). \tag{18}$$

If $w - x$ attains a nonnegative maximum M in $\left(a, \sigma^2(b)\right)_{\mathbb{T}}$, then $w - x \equiv M$.

To illustrate this result and especially the assumptions on derivatives of f, we include a simple example.

Example 9. Let us consider the function

$$f(t,u,v) = -u^3 + 3v.$$

Then, if we consider a time scale $\mathbb{T}$ such that $\mu(t) < \frac{1}{3}$ for all $t \in \mathbb{T}$, then $\mu f_v = \mu 3 < 1$ and $f_u = -3u^2 \leq 0$. Consequently, if there exist functions w and x which satisfy (18) then the difference $w - x$ cannot attain a nonnegative maximum in $\left(a, \sigma^2(b)\right)_{\mathbb{T}}$, unless $w - x \equiv M$.

4 Uniqueness of Sturm-Liouville Problem

One can extend this reasoning to show that there is at most one solution for the following boundary value problem with Sturm-Liouville boundary conditions.

Theorem 10. *Suppose that f, f_u, f_v are continuous and satisfy (16) and (17). Let $g, h \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfy*

$$\begin{aligned} g_u(u,v) \geq 0, h_u(u,v) &\geq 0 \quad, & (19)\\ g_u(u,v) + h_u(u,v) &> 0 \quad, & (20)\\ g_v(u,v) \leq 0, h_v(u,v) &\geq 0 \quad, & (21)\\ g_u(u,v) - g_v(u,v) &> 0 \quad, \ \textit{and} & (22)\\ h_u(u,v) + h_v(u,v) &> 0 & (23) \end{aligned}$$

for all $u, v \in \mathbb{R}$. Then there is at most one solution of

$$\begin{cases} x^{\Delta\Delta} + f(t, x^{\sigma}, x^{\Delta}) = 0, & \textit{for } t \in [a,b]_{\mathbb{T}}, \\ g(x(a), x^{\Delta}(a)) = 0, & h(x(\sigma^2(b)), x^{\Delta}(\sigma(b))) = 0. \end{cases} \tag{24}$$

Proof. In order to make the proof more transparent, we introduce the following notation for a function $x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}$

$$\underline{x} := x(a), \quad \underline{x}^{\Delta} := x^{\Delta}(a),$$
$$\overline{x} := x(\sigma^2(b)), \quad \overline{x}^{\Delta} := x^{\Delta}(\sigma(b)).$$

Let us assume that there exist two distinct solutions $\alpha, \beta : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}$ of (24). Let us define $\gamma := \alpha - \beta$. Applying the boundary conditions in (24) and the mean value theorem we get

$$0 \quad = g(\underline{\alpha}, \underline{\alpha}^{\Delta}) - g(\underline{\beta}, \underline{\beta}^{\Delta}) = g_u(\underline{\eta}, \underline{\xi})\underline{\gamma} + g_v(\underline{\eta}, \underline{\xi})\underline{\gamma}^{\Delta}, \tag{25}$$

$$0 \quad = h(\overline{\alpha}, \overline{\alpha}^{\Delta}) - h(\overline{\beta}, \overline{\beta}^{\Delta}) = h_u(\overline{\eta}, \overline{\xi})\overline{\gamma} + h_v(\overline{\eta}, \overline{\xi})\overline{\gamma}^{\Delta}, \tag{26}$$

where for some $\underline{\theta}, \overline{\theta} \in (0, 1)$ the new parameters satisfy equalities

$$(\underline{\eta}, \underline{\xi}) \quad = \underline{\theta}(\underline{\alpha}, \underline{\alpha}^{\Delta}) + (1 - \underline{\theta})(\underline{\beta}, \underline{\beta}^{\Delta})$$
$$(\overline{\eta}, \overline{\xi}) \quad = \overline{\theta}(\overline{\alpha}, \overline{\alpha}^{\Delta}) + (1 - \overline{\theta})(\overline{\beta}, \overline{\beta}^{\Delta}).$$

Subtracting (24) for β from the same equation for α we get

$$0 = \gamma^{\Delta\Delta} + f(t, \alpha^{\sigma}, \alpha^{\Delta}) - f(t, \beta^{\sigma}, \beta^{\Delta}).$$

For the fixed t one can apply the mean value theorem for the function $f(\cdot, u, v)$ to get

$$0 = \gamma^{\Delta\Delta} + f_u(t, \kappa, \varphi)\gamma^{\sigma} + f_v(t, \kappa, \varphi)\gamma^{\Delta}. \tag{27}$$

for some $(\kappa, \varphi) = \theta(\alpha^{\sigma}, \alpha^{\Delta}) + (1 - \theta)(\beta^{\sigma}, \beta^{\Delta})$, for some $\theta \in (0, 1)$.

Now (t, κ, φ) lies in a compact set and f_u and f_v are continuous and thus bounded.

Without loss of generality we can suppose γ attains a positive maximum in $m \in [a, \sigma^2(b)]_{\mathbb{T}}$. There are three cases.

(i) $m \in (a, \sigma^2(b))_{\mathbb{T}}$. Thanks to (16), (17), we can apply Theorem 8 to obtain that if γ satisfies (27), then $\gamma \equiv M$. This implies that $\underline{\gamma} = \overline{\gamma} > 0$ and $\underline{\gamma}^{\Delta} = \overline{\gamma}^{\Delta} = 0$. Using the inequalities (19) and (20) we get the contradiction with one of the equalities (25), (26).

(ii) $m = a$. In this case, Lemma 5 yields that $\underline{\gamma} > 0$ and that $\underline{\gamma}^{\Delta} < 0$. These inequalities together with (19), (21) and (22) provide the contradiction with (25).

(iii) $m = \sigma^2(b)$. As in the previous case, the boundary point lemma Lemma 5 yields that $\overline{\gamma} > 0$ and that $\overline{\gamma}^{\Delta} > 0$. Using (19), (21) and (23) we arrive at a contradiction with (26).

This argument implies that, if the solution of (24) exists, then it is unique. □

The general assumptions (19)-(23) are designed to encompass as many cases as possible and hence deserve a simple example.

Example 11. Let us consider the boundary value problem:

$$\begin{cases} x^{\Delta\Delta} + e^{-x^{\sigma} - x^{\Delta}} = 0, & \text{for } t \in [a, b]_{\mathbb{T}}, \\ 2x(a) - x^{\Delta}(a) = 0, & x(\sigma^2(b)) + 4x^{\Delta}(\sigma(b)) = 0. \end{cases} \tag{28}$$

Obviously, in this case we have $f(t, u, v) = e^{-u-v}$, $g(u, v) = 2u - v$ and $h(u, v) = u + 4v$. This implies that $f_u = f_v = -e^{-u-v}$, $g_u = 2$, $g_v = -1$, $h_u = 1$ and $h_v = 4$. Consequently, (16)-(17) and (19)-(23) hold. This implies that Theorem 10 can be applied to prove that there exists at most one solution of (28).

The following remark suggests that the continuity assumption can be easily replaced.

Remark 12. We can weaken the assumptions in Theorem 10 by replacing the continuity of f_u and f_v by their boundedness and in the proof, in place of applying the Mean Value Theorem we use

$$\begin{aligned}
& f(t,\alpha^\sigma,\alpha^\Delta)-f(t,\beta^\sigma,\beta^\Delta) \\
= \; & f(t,\alpha^\sigma,\alpha^\Delta)-f(t,\beta^\sigma,\alpha^\Delta)+f(t,\beta^\sigma,\alpha^\Delta)-f(t,\beta^\sigma,\beta^\Delta) \\
= \; & f_u(t,\psi,\alpha^\Delta)(\alpha^\sigma-\beta^\sigma)+f_v(t,\beta^\sigma,\chi)(\alpha^\Delta-\beta^\Delta)
\end{aligned}$$

for some ψ and χ between α^σ and β^σ and between α^Δ and β^Δ, respectively.

We illustrate the application of Theorem 10 by the following extension of [3, Theorem 6.54]. Namely, we consider the Dirichlet BVP

$$\begin{cases} x^{\Delta\Delta}(t)+f(t,x^\sigma)=0, \\ x(a)=A, \quad x(\sigma^2(b))=B. \end{cases} \tag{29}$$

We define lower solution z_1 and upper solution z_2 of (29) to be functions which satisfy $z_1^{\Delta\Delta}+f(t,z_1^\sigma)\geq 0$ and $z_2^{\Delta\Delta}+f(t,z_2^\sigma)\geq 0$ for all $t\in[a,b]_{\mathbb{T}}$.

Corollary 13. *Let $f:[a,b]_{\mathbb{T}}\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f_u(t,u)$ is continuous (or bounded) and satisfies (17). Assume that there exist lower and upper solutions, z_1 and z_2, respectively, of (29) such that*

$$z_1(a)\leq A\leq z_2(a), \text{ and } z_1(\sigma^2(b))\leq B\leq z_2(\sigma^2(b)).$$

and

$$z_1(t)\leq z_2(t) \text{ for all } t\in[a,\sigma^2(b)]_{\mathbb{T}}.$$

Then problem (29) has a unique solution x with

$$z_1(t)\leq x(t)\leq z_2(t) \text{ for all } t\in[a,\sigma^2(b)]_{\mathbb{T}}.$$

Proof. [3, Theorem 6.54] yields the existence of solution. Moreover, this solution is unique by Theorem 10. □

5 Approximation of Solutions of Boundary Value Problems

Let us suppose that x is the unique solution of the problem

$$\begin{cases} x^{\Delta\Delta}(t)+g(t)x^\Delta(t)+h(t)x^\sigma(t)=f(t), & \text{on } [a,b]_{\mathbb{T}}, \\ -x^\Delta(a)\cos\theta+x(a)\sin\theta=\gamma_1, \\ x^\Delta(\sigma(b))\cos\phi+x(\sigma^2(b))\sin\phi=\gamma_2, \end{cases} \tag{30}$$

where $g,\ h,\ f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ and $\theta,\phi\in\left[0,\frac{\pi}{2}\right]$ are constants. Furthermore, let $z_1:[a,\sigma^2(b)]_{\mathbb{T}}\to\mathbb{R}$ satisfy

$$\begin{cases} z_1^{\Delta\Delta}(t)+g(t)z_1^\Delta(t)+h(t)z_1^\sigma(t)\leq f(t), & \text{on } [a,b]_{\mathbb{T}}, \\ -z_1^\Delta(a)\cos\theta+z_1(a)\sin\theta\geq\gamma_1, \\ z_1^\Delta(\sigma(b))\cos\phi+z_1(\sigma^2(b))\sin\phi\geq\gamma_2, \end{cases} \tag{31}$$

and let its twin function $z_2:[a,\sigma^2(b)]_{\mathbb{T}}\to\mathbb{R}$ satisfy the reversed inequalities

$$\begin{cases} z_2^{\Delta\Delta}(t)+g(t)z_2^\Delta(t)+h(t)z_2^\sigma(t)\geq f(t), & \text{on } [a,b]_{\mathbb{T}}, \\ -z_2^\Delta(a)\cos\theta+z_2(a)\sin\theta\leq\gamma_1, \\ z_2^\Delta(\sigma(b))\cos\phi+z_2(\sigma^2(b))\sin\phi\leq\gamma_2. \end{cases} \tag{32}$$

Then we can prove the following relationship among x, z_1 and z_2.

Theorem 14. *Let $h \le 0$ be bounded and g be bounded with $\mu g < 1$, $\theta, \phi \in \left[0, \frac{\pi}{2}\right]$, and x be the unique solution of (30). Moreover, let z_1 and z_2 satisfy (31) and (32), respectively. If $|\theta| + |\phi| + |h(t)| \not\equiv 0$, then*

$$z_2(t) \le x(t) \le z_1(t).$$

Proof. Set $v_1 := x - z_1$ and assume that $v_1(t) > 0$ for some $t \in [a, \sigma^2(b)]_{\mathbb{T}}$. Clearly, v_1 satisfies

$$\begin{cases} v_1^{\Delta\Delta}(t) + g(t)v_1^{\Delta}(t) + h(t)v_1^{\sigma}(t) \ge 0, & \text{on } [a,b]_{\mathbb{T}}, \\ -v_1^{\Delta}(a)\cos\theta + v_1(a)\sin\theta \le 0, \\ v_1^{\Delta}(\sigma(b))\cos\phi + v_1(\sigma^2(b))\sin\phi \le 0. \end{cases} \tag{33}$$

From the maximum principle, Theorem 2, it follows that the positive maximum must be attained at a or at $\sigma^2(b)$. If it occurs at a, then the boundary point lemma Lemma 5 implies that $v_1^{\Delta}(a) < 0$ and thus

$$-v_1^{\Delta}(a)\cos\theta + v_1(a)\sin\theta > 0,$$

a contradiction with the second inequality in (33). Similarly v_1 cannot attain a positive maximum at $\sigma^2(b)$. It follows that $x \le z_1$.

Similarly, setting $v_2 := z_2 - x$ one can show that v_2 is always non-positive. The result follows. □

Remark 15. This result complements the Comparison Theorem for Conjugate BVPs [3, Theorem 4.81]. Bohner and Peterson's results apply to equations in the self-adjoint form

$$(px^{\Delta})^{\Delta}(t) + q(t)x^{\sigma}(t) = 0.$$

If in $\mathcal{L}_2$ we assume that $g_2 \equiv 0$ and g_1, and h_2 are right dense continuous then $\mathcal{L}_2$ can be put in self-adjoint form if either $1 - g_1(t)\mu(t) + h_2(t)\mu^2(t) \neq 0$ (see [3, Theorem 4.12]) or if $1 + \mu(t)g_1(t) \neq 0$ (see [3, Theorem 4.17]).

Moreover [3, Theorem 4.81] require that the coefficients are right dense continuous.

We establish the theorem for linear differential equations which need not be self-adjoint, and for more general boundary conditions. Moreover we do not assume that the coefficients are right dense continuous. We assume that $h \le 0$ and $\mu g < 1$ whereas [3, Theorem 4.81] assumes the equation is disconjugate on $\left[a, \sigma^2(b)\right]_{\mathbb{T}}$.

As far as the authors know the precise relationship between existence of solutions and uniqueness of solutions of equations of the form

$$(px^{\Delta})^{\Delta}(t) + f(t, x^{\sigma}, x^{\Delta}) = 0$$

and smoothness of p and f is still not fully resolved even for the case $f(t, x^{\sigma}, x^{\Delta}) = q(t)x^{\sigma}(t)$. For an excellent discussion of existence in this latter case see [3, Chapter 4].

6 Uniqueness of the Solution of Initial Value Problems

In this section we apply the generalized maximum principle from the previous section to obtain the uniqueness of soltuions of the initial value problem:

$$\begin{cases} \widetilde{\mathcal{L}}_2[x] = x^{\Delta\Delta} + gx^{\Delta} + hx^{\sigma} = f(x), \\ x(a) = \gamma, \quad x^{\Delta}(a) = \delta, \end{cases} \tag{34}$$

where $\gamma, \delta \in \mathbb{R}$ and g, h satisfy the assumptions of Theorem 3.

Theorem 16. *Suppose that x_1 and x_2 are solutions of (34) in an interval $\left[a, \sigma^2(b)\right]_{\mathbb{T}}$. Then $x_1 \equiv x_2$ in $\left[a, \sigma^2(b)\right]_{\mathbb{T}}$.*

Proof. Set $x = x_1 - x_2$. Obviously x is the solution of the initial value problem

$$\begin{cases} x^{\Delta\Delta} + gx^{\Delta} + hx^{\sigma} = 0, \\ x(a) = 0, \quad x^{\Delta}(a) = 0. \end{cases} \tag{35}$$

Assume that x is not identically zero.

Let $C = \{t \in [a,b]_{\mathbb{T}} : x(s) = x^{\Delta}(s) = 0, \text{ for all } s \in [a,t]_{\mathbb{T}}\}$. Now $\sup C = c \in [a,b]_{\mathbb{T}}$ is well defined since $a \in C$. It suffices to show that $c = b$ so assume that $c < b$.

Since $x(c) = x^{\Delta}(c) = 0$, we have that $x(\sigma(c)) = x(c) = 0$. If c is right scattered then we can rewrite the first equation in (35) into

$$x^{\Delta}(\sigma(c)) = (1 - g(c)\mu(c))x^{\Delta}(c) - \mu(c)h(c)x(\sigma(c)) = 0.$$

Thus $b \geq \sigma(c) > c$ and $\sigma(c) \in C$, a contradiction, and c is right dense.

Since c is right dense there is $d \in (c,b]_{\mathbb{T}}$ such that $\sigma^2(d) \in [d,b]_{\mathbb{T}}$ and the distance $(\sigma^2(d) - c)$ is sufficiently small so that there is a function w defined on $\left[c, \sigma^2(d)\right]_{\mathbb{T}}$ satisfying (9)–(12) (see [15, Lemma 14] for more details). By Theorem 3 the function $\frac{x}{w}$ attains a maximum either at c or at d. Clearly, $-x$ is the solution of (35) as well. Therefore, the maximum of $-\frac{x}{w}$ is also attained at one of the endpoints. Since $x(c) = 0$, the function $\frac{x}{w}$ attains either a maximum or a minimum at d. Using the initial conditions and the positivity of w we get

$$\left(\frac{x}{w}\right)^{\Delta}(c) = \frac{x^{\Delta}w - xw^{\Delta}}{ww^{\sigma}}(c) = 0.$$

It follows from the boundary point lemma Lemma 5 and the initial condition that $\frac{x}{w} \equiv 0$ on $[c,d]_{\mathbb{T}}$, a contradiction with either a maximality or minimality of $\frac{x}{w}$ at d.

Thus $c = b$, $x \equiv 0$ and $x_1 \equiv x_2$, as required. □

Remark 17. In contrast to the traditional uniqueness theorem for IVPs [3, Theorem 4.5], we do not establish the existence of solutions. On the other hand, we consider a wider class of problems in the sense of Remark 15.

7 Approximation of Solutions of Initial Value Problems

In this section, we provide the analogues for initial value problems of the results in Section 5. These results can be viewed as extensions of [3, Theorem 4.26]. We consider the problem where x is the solution of (34), z_1 satisfies

$$\begin{cases} z_1^{\Delta\Delta}(t) + g(t)z_1^{\Delta}(t) + h(t)z_1^{\sigma}(t) \geq f(t), & \text{on } [a,b]_{\mathbb{T}}, \\ z_1(a) \geq \gamma, z_1^{\Delta}(a) \geq \delta, \end{cases} \tag{36}$$

and its twin, z_2, satisfies

$$\begin{cases} z_2^{\Delta\Delta}(t) + g(t)z_2^{\Delta}(t) + h(t)z_2^{\sigma}(t) \leq f(t), & \text{on } [a,b]_{\mathbb{T}}, \\ z_2(a) \leq \gamma, z_2^{\Delta}(a) \leq \delta. \end{cases} \tag{37}$$

We apply Theorem 2 to obtain the following approximation result.

Theorem 18. *Let g be a bounded function satisfying $\mu g < 1$ and let h be a non-positive bounded function. If x is the solution of (34) and z_1 and z_2 satisfy (36) and (37), respectively, then*

$$z_2 \leq x \leq z_1, \text{ for all } t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}},$$

and

$$z_2^{\Delta} \leq x^{\Delta} \leq z_1^{\Delta}, \text{ for all } t \in [a, \sigma(b)]_{\mathbb{T}}.$$

Proof. Set $v_1 = z_1 - x$. Clearly, v_1 satisfies

$$\begin{cases} v_1^{\Delta\Delta}(t) + g(t)v_1^{\Delta}(t) + h(t)v_1^{\sigma}(t) \geq 0, & \text{on } [a,b]_{\mathbb{T}}, \\ v_1(a) \geq 0, \quad v_1^{\Delta}(a) \geq 0. \end{cases}$$

Considering the problem on $[a,t]_{\mathbb{T}}$ for $t > a$, we can use the maximum principle, Theorem 2, and the boundary point lemma, Lemma 5, to conclude that for all $t \in (a,b)_{\mathbb{T}}$

$$v_1(t) \geq v_1(a), \tag{38}$$

and

$$v_1^{\Delta}(t) \geq 0. \tag{39}$$

Using the inequality $z_1(a) \geq \gamma$, we can rewrite (38) as

$$x(t) \leq z_1(t),$$

and the inequality (39) trivially yields

$$x^{\Delta}(t) \leq z_1^{\Delta}(t).$$

Reversing the inequalities in the above a similar argument shows that z_2 is the lower bound, that is,

$$x(t) \geq z_2(t), \quad x^{\Delta}(t) \geq z_2^{\Delta}(t).$$

□

Finally, we consider problem (34) with $h(t) > 0$ for some $t \in [a,b]_{\mathbb{T}}$. We employ the generalized maximum principle, Theorem 3, and assume that there exists a function w such that the conditions (9)-(11) are satisfied (its existence is guaranteed for small intervals by [15, Lemma 14]). If u is a solution of

$$\widetilde{\mathcal{L}}_2[u] \geq f, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}$$

setting

$$v := \frac{u}{w}, \quad g_1 := \frac{gw + w^{\Delta}}{w^{\sigma}}, \quad g_2 := \frac{w^{\sigma\Delta}}{w^{\sigma}}, \text{ and } h_2 := \frac{\widetilde{\mathcal{L}}_2[w]}{w^{\sigma}},$$

it follows that

$$\mathcal{L}_2[v] = v^{\Delta\Delta} + g_1 v^{\Delta} + g_2 v^{\Delta\sigma} + h_2 v^{\sigma} \geq \frac{f}{w^{\sigma}}, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}.$$

Theorem 19. *Let g and h be bounded functions on $[a,b]_{\mathbb{T}}$ such that $\mu g < 1$. Let x be the solution of (34) and assume that there exists a function z_1 satisfying*

$$\widetilde{\mathcal{L}}_2[z_1] \geq f, \ z_1(a) \geq \gamma, \ z_1^{\Delta}(a)w(a) - z_1(a)w^{\Delta}(a) \geq \delta w(a) - \gamma w^{\Delta}(a),$$

a function z_2 satisfying

$$\widetilde{\mathcal{L}}_2[z_2] \leq f, \ z_2(a) \leq \gamma, \ z_2^{\Delta}(a)w(a) - z_2(a)w^{\Delta}(a) \leq \delta w(a) - \gamma w^{\Delta}(a),$$

and a function w satisfying (9)-(11). Then

$$z_2 \leq x \leq z_1, \ \textit{for all } t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \tag{40}$$

and

$$\left(\frac{z_2}{w}\right)^{\Delta} \leq \left(\frac{x}{w}\right)^{\Delta} \leq \left(\frac{z_1}{w}\right)^{\Delta} \ \textit{for all } t \in [a, \sigma(b)]_{\mathbb{T}}. \tag{41}$$

Proof. Clearly

$$\frac{z_2(a)}{w(a)} \leq \frac{x(a)}{w(a)} \leq \frac{z_1(a)}{w(a)},$$

and it follows from the positivity of w and initial conditions for z_1 and z_2 that

$$\left(\frac{z_2(a)}{w(a)}\right)^{\Delta} \leq \left(\frac{x(a)}{w(a)}\right)^{\Delta} \leq \left(\frac{z_1(a)}{w(a)}\right)^{\Delta}.$$

It follows from the assumptions on z_1, x and z_2 that

$$\mathcal{L}_2\left[\frac{z_2}{w}\right] \leq \mathcal{L}_2\left[\frac{x}{w}\right] \leq \mathcal{L}_2\left[\frac{z_1}{w}\right], \text{ for all } t \in [a,b]_{\mathbb{T}}$$

and by [15, Lemma 11], $\mathcal{L}_2$ satisfies the conditions of the strong maximum principle and the boundary point lemma. Arguing as in the proof of Theorem 18 it follows that for all $t \in \left(a, \sigma^2(b)\right)_{\mathbb{T}}$

$$\frac{z_2}{w} \leq \frac{x}{w} \leq \frac{z_1}{w}, \tag{42}$$

and that (41) holds. Since $w > 0$ it follows from (42) that

$$z_2 \leq x \leq z_1,$$

as required. □

Naturally, one can extract the bound on x^{Δ} from (41) as well.

Remark 20. Since $w > 0$ and $w^{\sigma} > 0$ the inequality (41) yields

$$wz_2^{\Delta} - z_2 w^{\Delta} \leq wx^{\Delta} - xw^{\Delta} \leq wz_1^{\Delta} - z_1 w^{\Delta}.$$

Furthermore, dividing through by w we get

$$z_2^{\Delta} - \frac{w^{\Delta}}{w}\left(z_2 - x\right) \leq x^{\Delta} \leq z_1^{\Delta} - \frac{w^{\Delta}}{w}\left(z_1 - x\right).$$

If $w^{\Delta} \leq 0$ we can use (40) to estimate x in this expression. Otherwise, we estimate the derivative only with the help of z_2^{Δ} and z_1^{Δ}. That is, it follows from (41) that

$$\begin{cases} z_2^{\Delta} + \frac{w^{\Delta}}{w}\left(z_1 - z_2\right) \leq x^{\Delta} \leq z_1^{\Delta} - \frac{w^{\Delta}}{w}\left(z_1 - z_2\right) & \text{if } w^{\Delta} \leq 0 \\ z_2^{\Delta} \leq x^{\Delta} \leq z_1^{\Delta} & \text{if } w^{\Delta} > 0. \end{cases}$$

8 Oscillation results

There are several ways to approach the Sturmian oscillation theorem. For the self-adjoint case see, for example, the discussion in [3, Chapter 7] and [4, Chapter 10] of the Riccati and variational methods as well as the use of Pruefer transformation and IVPs. The standard version for self-adjoint and disconjugate problems can be found in [3, Theorem 4.53].

In the case of ordinary differential equations [14] give a proof of the Sturmian oscillation theorem using the maximum principle. As an application of our results we have the following result for dynamic equations on time scales. As mentioned earlier we do not assume that the coefficients are right dense continuous nor do we assume that the operator is in self-adjoint form.

Note we will say a function u is non-trivial on $[a,b]_{\mathbb{T}}$ if it is not identically zero on any nonempty open subinterval of $[a,b]_{\mathbb{T}}$.

Theorem 21. *Let the functions u and w be non-trivial solutions of*

$$\widetilde{\mathcal{L}}_2[x] = 0, \text{ for } t \in \mathbb{T}$$

where $\widetilde{\mathcal{L}}_2$ is given by (7), g and $w^{\sigma\Delta}$ are bounded and g satisfies (8). Then between any two consecutive generalized zeroes of w the function u can have at most one generalized zero.

Proof. Let t_1 and t_2 be two consecutive generalized zeroes of w. Thus at t_1 either $w(t_1) = 0$ or $w(\rho(t_1))w(t_1) < 0$ while at t_2 either $w(t_2) = 0$ or $w(\rho(t_2))w(t_2) < 0$. Assume that u has two consecutive generalized zeroes in $(t_1, t_2)_{\mathbb{T}}$ at $c < d$, say. Without loss of generality we may assume that $w > 0$ on $(t_1, t_2)_{\mathbb{T}}$. If $\rho(c) < c$ and u has a generalized zero in $(\rho(c), c)$ we set $j = \rho(c)$, otherwise we set $j = c$. Since w is a solution of (7), $w^{\Delta\Delta}$ exists so w and w^{Δ} are continuous. Moreover the function $v := \dfrac{u}{w}$ satisfies

$$\begin{aligned}
\mathcal{L}_2[v] &= v^{\Delta\Delta} + \frac{gw + w^{\Delta}}{w^{\sigma}} v^{\Delta} + \frac{w^{\sigma\Delta}}{w^{\sigma}} v^{\Delta\sigma} && (43)\\
&= v^{\Delta\Delta} + g_1 v^{\Delta} + g_2 v^{\Delta\sigma} \geq 0, \quad \text{for} \quad t \in [j, d)_{\mathbb{T}} && (44)
\end{aligned}$$

where $g_1 = \frac{gw+w^{\Delta}}{w^{\sigma}}$, $g_2 = \frac{w^{\sigma\Delta}}{w^{\sigma}}$, and either $j = c$ and $v(j) = 0$ or $j = \rho(c)$ and $v(j)v(c) < 0$ while at d either $v(d) = 0$ or $v(\rho(d))v(d) < 0$. Similarly, v is continuous on $(t_1, t_2)_{\mathbb{T}}$.

Since u is non-trivial and v is continuous, considering $-v$ if necessary, it follows that v attains a positive maximum, L, on $[j, d]_{\mathbb{T}}$ at $l \in (j, d)_{\mathbb{T}}$, say. Since $L > 0$, $v(d) \leq 0$, and v is continuous on $[j, d]_{\mathbb{T}}$ we may choose $l \in (j, d)_{\mathbb{T}}$ so that $v(t) < L = v(l)$ for $t \in (l, d]_{\mathbb{T}}$.

Now either $\sigma(l) = l$ or $\sigma(l) > l$.

First we consider the case $\sigma(l) = l$. Choose $m \in (l, d)_{\mathbb{T}}$ such that $\sigma^2(m) < t_2$. Since $t_1 < j$ it follows that $w > 0$ on $\left[j, \sigma^2(m)\right]_{\mathbb{T}}$. Moreover, since g satisfies (8), it follows from the proof of [15, Lemma 11] applied to $\mathcal{L}_2[v]$ on $[j, m]_{\mathbb{T}}$ that $\mu g_1 < 1$. Moreover if $\mu(t) = 0$, then $\mu(t)g_2(t) = 0$ while if $\mu(t) > 0$, then $\mu g_2 = \mu \frac{w^{\sigma\Delta}}{w^{\sigma}} = \frac{w^{\sigma\sigma} - w^{\sigma}}{w^{\sigma}}$.

We show that $\mu g_2 = \frac{w^{\sigma\sigma} - w^{\sigma}}{w^{\sigma}} \geq \delta - 1$ for some $\delta > 0$; here again we use $t_1 < j$ and $\sigma^2(m) < t_2$. Let $C = \left[j, \sigma^2(m)\right]_{\mathbb{T}}$. Now $\sigma([j, m]_{\mathbb{T}}) \subset C \subset [j, t_2)_{\mathbb{T}}$ so that

$$\begin{aligned}
w(t_{max}) &= \max\{w(t) : t \in C\} \geq \sup\{w^{\sigma}(t) : t \in [j, m]_{\mathbb{T}}\}\\
&\geq \inf\{w^{\sigma}(t) : t \in [j, m]_{\mathbb{T}}\} \geq \min\{w(t) : t \in C\} = w(t_{min}) > 0
\end{aligned}$$

for some t_{min}, $t_{max} \in C$, since w is continuous and $w > 0$ on C which is compact.

Similarly $\sigma^2([j, m]_{\mathbb{T}}) \subset C$ so that

$$\inf\{w^{\sigma\sigma}(t) : t \in [j, m]_{\mathbb{T}}\} \geq \min\{w(t) : t \in C\} = w(t_{min}) > 0.$$

Thus $\mu g_2 = \frac{w^{\sigma\sigma} - w^{\sigma}}{w^{\sigma}} \geq \frac{w^{\sigma\sigma}}{w^{\sigma}} - 1 \geq \delta - 1$ where $\delta = \frac{w(t_{min})}{w(t_{max})} > 0$. Since g and $w^{\sigma\Delta}$ are bounded it follows that (11) holds on $[j, m]_{\mathbb{T}}$. Therefore Theorem 3 is applicable on $[j, m]_{\mathbb{T}}$ so that v is identically constant on $[j, m]_{\mathbb{T}}$. Now $j \leq l < m < d$ so $v(m) = v(l) = L$, a contradiction, since $v(t) < L$ for $t \in (l, d]_{\mathbb{T}}$.

Next consider the case $l < \sigma(l) \leq d$. Since v attains a positive maximum at l and $v(\sigma(l)) < v(l)$, it follows by a standard argument that l is left scattered. A simple calculation yields $\mathcal{L}_2[v](\rho(l)) < 0$, contradicting (44). The result follows. □

Remark 22. In the proof of Theorem 21 we use the boundedness of g and $w^{\sigma\Delta}$ only to show that (11) holds on $[c, \rho(d))_{\mathbb{T}}$. Thus we could rephrase Theorem 21 without assuming the boundedness of g and $w^{\sigma\Delta}$ assuming that $w > 0$ on $(t_1, t_2)_{\mathbb{T}}$ and (11) holds on $[c, \rho(d))_{\mathbb{T}}$ where c, $d \in (t_1, t_2)_{\mathbb{T}}$.

9 Conclusion and Future Directions

We conclude with some open questions.

Is the following analogue of [14, Theorem 19], true?

$$L[x] = x^{\Delta\Delta} + g_1 x^{\Delta} + g_2 x^{\Delta\sigma}$$

and let u and w satisfy

$$L[u] + h_1 u^{\sigma} = 0 = L[w] + h_2 w^{\sigma}$$

with $h_2 \geq h_1$ on $[a,b]_{\mathbb{T}}$. If $w > 0$ on the open interval $(a,b)_{\mathbb{T}}$ then u can have at most one generalized zero in the closed interval $[a,b]_{\mathbb{T}}$ unless w has generalized zeroes at a and b and $h_1 \equiv h_2$ and u is a constant multiple of w.

Are conditions (11) and (12) needed in the generalized strong maximum principle, Theorem 3?

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