Maximum Principles for Elliptic Dynamic Equations

Petr Stehlík
Department of Mathematics
University of West Bohemia
Univerzitní 22, Pilsen
Czech Republic 30614
E-mail: pstehlik@kma.zcu.cz

Abstract. We consider second order partial dynamic operators of the elliptic type on time scales. We establish basic maximum principles and apply them to obtain the uniqueness of Dirichlet boundary value problems for dynamic elliptic equations, eg Poisson equation. Our special cases include the situation in which several variables are continuous and the other discrete. We conclude with open problems, we discuss the role of regressive groups in more dimensions.

AMS subject classification: 34B15 39A10 34B10

Key Words: partial dynamic equations, elliptic operators, maximum principles, time scales, Laplace operator, Poisson equation.

1. Introduction

Maximum principles are a basic tool to study partial differential equations. They enable to obtain *a priori* information about solutions of a class of differential inequalities. They can be used to approximate solutions, errors of these approximations or to obtain existence and uniqueness results. The essential survey of standard maximum principles can be found in Protter, Weinberger [13] and references therein.

Naturally, the numerical analysis takes interest in discrete maximum principles and their relation to their continuous counterparts. The validity of discrete maximum principles strongly depends on the fineness of the chosen discretization and have been consequently appropriately studied, see eg Cheng [7] or Kuo and Trudinger [11].

In 1988, Stefan Hilger [8] created an analytical tool, *time scales*, in order to unify the discrete and continuous calculus and make the transition between them smoother. His effort incited the study of the *dynamic* equations, ie unification and generalization of difference and differential equations.

Whereas ordinary dynamic equations have been studied extensively (see Bohner, Peterson [5], [6]), the first contributions to the area of partial dynamic equations (Bohner, Guseinov [4], Jackson [9] and Ahlbrandt, Morian [1]) have appeared only recently. These papers establish basic concepts (eg partial derivatives, partial integrations, mean value theorems and existence of simple heat and wave dynamic equations).

Taking into account the importance and the distinct behavior of maximum principles in partial differential and difference equations, it seems natural to study them in the time scale setting. In this paper, we state basic maximum principles for elliptic dynamic operators and consequently apply them to obtain existence of at most one solution to elliptic dynamic boundary value problems.

This paper can also be regarded as a generalization of our previous work in the one-dimensional case, see Stehlík, Thompson [14].

This paper is organized as follows. Following Bohner, Guseinov [4], we establish basic notation in Section 2. In Sections 3-5 we prove weak and strong maximum principles for elliptic dynamic operators. Consequently, we apply these results in Section 6 to obtain the existence of at most one solution of elliptic boundary value problems. Finally, in Section 7 we state several open problems and possible extensions.

2. Partial Dynamic Equations

In this section we present some basic definitions concerning partial dynamic equations. We essentially follow those of Bohner and Guseinov [4].

We fix $n \in \mathbb{N}$ and denote $I = \{1, 2, \ldots, n\}$. We consider time scales $\mathbb{T}_i$, $i \in I$ (time scale is a closed nonempty subset of the real numbers $\mathbb{R}$). Throughout this paper we assume that $\mathbb{T}_i$ is bounded for all $i \in I$. Furthermore we define the n-dimensional time scale by the Cartesian product

$$\Lambda = \mathbb{T}_1 \times \mathbb{T}_2 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : t_i \in \mathbb{T}_i\},$$

equipped with a conventional topology of $\mathbb{R}^N$. Following standard one-dimensional concepts, we endow each dimension $i \in I$ with jump operators. The i-th forward jump operator $\sigma_i : \mathbb{T}_i \to \mathbb{T}_i$ is defined by

$$\sigma_i(u) := \begin{cases} \inf\{v \in \mathbb{T}_i : v > u\} & \text{if} \quad u \neq \max \mathbb{T}_i, \\ \max \mathbb{T}_i & \text{if} \quad u = \max \mathbb{T}_i. \end{cases}$$

Similarly, we introduce the i-th backward jump operator $\varrho_i : \mathbb{T}_i \to \mathbb{T}_i$.

$$\varrho_i(u) := \begin{cases} \sup\{v \in \mathbb{T}_i : v < u\} & \text{if} \quad u \neq \min \mathbb{T}_i, \\ \min \mathbb{T}_i & \text{if} \quad u = \min \mathbb{T}_i. \end{cases}$$

Straightforwardly, this allows us to define corresponding (i-th forward and i-th backward) graininess functions $\mu_i, \nu_i : \mathbb{T}_i \to \mathbb{R}$

$$\mu_i(u) := \sigma_i(u) - u, \quad \nu_i := u - \nu_i(u).$$

Moreover, we sort the points $t \in \Lambda$ according to the values of ϱ_i and σ_i.

Definition 1. We say that a point $t \in \Lambda$ is

- i-left scattered, if $\varrho_i(t_i) < t_i$,
- i-left dense, if $\varrho_i(t_i) = t_i$,
- i-right scattered, if $\sigma_i(t_i) > t_i$,
- i-right dense, if $\sigma_i(t_i) = t_i$,
- i-scattered, if it is both i-left scattered and i-right scattered,

- i-dense, if it is both i-left dense and i-right dense.

In order to define partial derivatives properly, we introduce modified time scales:

$$(\mathbb{T}_i)^\kappa = \begin{cases} \mathbb{T}_i & \text{if } \mathbb{T}_i \text{ has a } i\text{-left dense maximum,} \\ \mathbb{T}_i \setminus \max \mathbb{T}_i & \text{if } \mathbb{T}_i \text{ has a } i\text{-left scattered maximum.} \end{cases}$$

$$(\mathbb{T}_i)_\kappa = \begin{cases} \mathbb{T}_i & \text{if } \mathbb{T}_i \text{ has a } i\text{-right dense minimum,} \\ \mathbb{T}_i \setminus \min \mathbb{T}_i & \text{if } \mathbb{T}_i \text{ has a } i\text{-right scattered minimum.} \end{cases}$$

Similarly, we restrict the n-dimensional time scale Λ by:

$$\Lambda_\kappa^\kappa := (\mathbb{T}_1)_\kappa^\kappa \times (\mathbb{T}_2)_\kappa^\kappa \times \ldots (\mathbb{T}_n)_\kappa^\kappa .$$

At this point we are ready to define partial delta and nabla derivatives

Definition 2. Let $u : \Lambda_\kappa^\kappa \to \mathbb{R}$ be a function. The partial delta derivative at $t \in \Lambda_\kappa^\kappa$ with respect to t_i is defined as

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{u(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - u(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i},$$

provided that it exists, in which case we denote it by $u^{\Delta_i}(t)$.

Similarly partial nabla derivative at $t \in \Lambda_\kappa^\kappa$ with respect to t_i is defined as

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \varrho_i(t_i)}} \frac{u(t_1, \ldots, t_{i-1}, \varrho_i(t_i), t_{i+1}, \ldots, t_n) - u(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\varrho_i(t_i) - s_i},$$

provided that it exists, in which case we denote it by $u^{\nabla_i}(t)$.

In the same way we can define higher-order partial derivatives.

3. Weak Maximum Principles

We define the dynamic Laplace operator by:

$$\Delta_{\mathbb{T}} u := \sum_{i=1}^n u^{\nabla_i \Delta_i}.$$

In contrast to our work in one dimension [14], we direct our attention to second-order operators with mixed derivatives. Whereas in one-dimensional case the choice of the second order derivative does not appear to be significant, in more dimensions we prefer to *center* the derivatives as is usual eg in difference schemes in numerical analysis.

In the following we consider all operators and maximum principles on an n-dimensional time scale which is induced by n bounded time scales, i.e. Λ can be written as

$$\Lambda = [\varrho_1(a_1), \sigma_1(b_1)]_{\mathbb{T}_1} \times \ldots \times [\varrho(a_n), \sigma(b_n)]_{\mathbb{T}_n}$$

We consider functions on Λ from the set

$$\mathcal{D}(\Lambda) := \left\{ u : \Lambda \to \mathbb{R} : u^{\nabla_i} \text{ are continuous and } u^{\nabla_i \Delta_i} \text{ exists.} \right\}$$

First we find necessary conditions in the point m where a maximum of $u \in \mathcal{D}(\Lambda)$ is attained.

Lemma 3. *If $u \in \mathcal{D}(\Lambda)$ attains a maximum at a point m, then for all $i = 1, 2, \dots, n$*
(i) $u^{\nabla_i \Delta_i}(m) \leq 0$,
(ii) $u^{\nabla_i}(m) \geq 0$,
(iii) $u^{\Delta_i}(m) \leq 0$.
The strict inequality in the last two inequalities can occur only at i-right scattered points.

Proof. Let us divide our proof according to the point type of m with respect to the i-th variable.

- m is i-dense. In this case the maximality of u at m and standard continuous necessary conditions imply that
$$u^{\nabla_i \Delta_i}(m) \leq 0, u^{\nabla_i}(m) = u^{\Delta_i}(m) = 0.$$
- m is i-left scattered and i-right dense. In this case, we have that $u^{\nabla_i}(m) \geq 0$ and concurrently $u^{\nabla_i}(m+h) \leq 0$ for all $h \in \mathbb{R}^N$ which are sufficiently small, satisfy $h_j = 0$ for all $j \neq i$ and $m + h \in \Lambda$. This yields
$$u^{\nabla_i \Delta_i}(m) = \lim_{h \to 0} \frac{u^{\nabla_i}(m+h) - u^{\nabla_i}(m)}{h} \leq 0.$$
Moreover, the continuity of the nabla derivative implies that
$$0 \leq u^{\nabla_i}(m) = \lim_{h \to 0} u^{\nabla_i}(m+h) \leq 0,$$
and consequently $u^{\nabla_i}(m) = 0$.
- m is i-right scattered (ie $\mu_i > 0$). The maximality of u at m implies that $u^{\nabla_i}(m) \geq 0$ and $u^{\Delta_i}(m) \leq 0$ and consequently
$$u^{\nabla_i \Delta_i}(m) = \frac{u^{\nabla_i}(\sigma_i(m)) - u^{\nabla_i}(m)}{\mu_i} = \frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i} \leq 0.$$

□

Necessary conditions yield directly the first simple maximum principle for the time scale Laplace operator.

Corollary 4. *If $u \in \mathcal{D}(\Lambda)$ satisfies*

$$\Delta_{\mathbb{T}} u > 0, \quad in\ \Lambda_\kappa^\kappa, \tag{1}$$

then u cannot attain its maximum at an interior point of Λ.

Proof. We assume that u attains its maximum at an interior point m ($u(m) = M$). Lemma 3 implies that $u^{\nabla_i \Delta_i}(m) \leq 0$ holds independently of the point type of m. This implies that at m
$$\Delta_{\mathbb{T}} u(m) = \sum_{i=1}^{n} u^{\nabla_i \Delta_i}(m) \leq 0,$$
a contradiction with (1). □

Natural extension of this maximum principle is to consider also operators which contain also first-derivative terms. Let us take into account the operator of the following type

$$\mathcal{L}[u] := \sum_{i=1}^{n} \left(u^{\nabla_i \Delta_i} + b_i u^{\Delta_i} + c_i u^{\nabla_i} \right) = \Delta_{\mathbb{T}} u + \sum_{i=1}^{n} \left(b_i u^{\Delta_i} + c_i u^{\nabla_i} \right)$$

Again, we can directly use Lemma 3 to obtain:

Corollary 5. *Let $u \in \mathcal{D}(\Lambda)$ satisfy*

$$\mathcal{L}[u] > 0, \quad \text{in } \Lambda_\kappa^\kappa, \tag{2}$$

and let b_i and c_i be bounded and satisfy

$$b_i \geq 0 \text{ at all } i\text{-right scattered points.} \tag{3}$$

$$c_i \leq 0 \text{ at all } i\text{-right scattered points.} \tag{4}$$

Then u cannot attain its maximum at an interior point of Λ.

Proof. Similarly as in the previous case, we assume that u attains its maximum at an interior point m ($u(m) = M$). Lemma 3 yields that at m we have $u^{\Delta_i}(m) \leq 0$, $u^{\nabla_i}(m) \geq 0$ and $u^{\nabla_i \Delta_i}(m) \leq 0$. Consequently, assumptions (3), (4) imply that

$$\mathcal{L}[u](m) = \sum_{i=1}^{n} \left(u^{\nabla_i \Delta_i}(m) + b_i(m) u^{\Delta_i}(m) + c_i(m) u^{\nabla_i}(m) \right) \leq 0,$$

which contradicts (2). □

However, this direct application of necessary conditions and corresponding sign conditions on b_i and c_i does not provide the optimal result. Our final result in this section reveals the best conditions on b_i and c_i which ensure the validity of the maximum principle. Following the statement of Lemma 3 we split the index set in each point into those indices $i \in I$, in which the point t is i-right dense and those in which it is i-right scattered. In other words, at each t we define auxiliary index sets

$$I_{RD}^t := \{i \in I, \sigma_i(t_i) = t_i\}, \tag{5}$$

$$I_{RS}^t := \{i \in I, \sigma_i(t_i) > t_i\}. \tag{6}$$

Theorem 6. *Let $u \in \mathcal{D}(\Lambda)$ satisfy the inequality* (2) *and let b_i and c_i be bounded and satisfy*

$$1 + b_i(t)\mu_i(t) \geq 0, \tag{7}$$

$$-1 + c_i(t)\mu_i(t) \leq 0, \tag{8}$$

for each $t \in \Lambda_\kappa^\kappa$. Then u cannot attain its maximum at an interior point of Λ.

Proof. If u attains its maximum at an interior point m, then Lemma 3 implies that

$$u^{\Delta_i}(m) = 0,\ u^{\nabla_i}(m) = 0 \text{ and } u^{\nabla_i \Delta_i}(m) \leq 0, \text{ if } i \in I_{RD}^m \tag{9}$$

$$u^{\Delta_i}(m) \leq 0,\ u^{\nabla_i}(m) \geq 0 \text{ and } u^{\nabla_i \Delta_i}(m) \leq 0, \text{ if } i \in I_{RS}^m \tag{10}$$

Therefore, we can rewrite $\mathcal{L}[u](m)$ in the following way (first-order terms vanish in right-dense dimensions):

$$\begin{aligned}
\mathcal{L}[u](m) &= \sum_{i=1}^{n} \left(u^{\nabla_i\Delta_i}(m) + b_i(m)u^{\Delta_i}(m) + c_i(m)u^{\nabla_i}(m)\right) \\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m} \left(u^{\nabla_i\Delta_i}(m) + b_i(m)u^{\Delta_i}(m) + c_i(m)u^{\nabla_i}(m)\right) \\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m} \left(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m)} + b_i(m)u^{\Delta_i}(m) + c_i(m)u^{\nabla_i}(m)\right) \qquad (11)
\end{aligned}$$

If $I = I_{RD}^m$ then $\mathcal{L}[u](m) \le 0$, a contradiction. Otherwise, let us define auxiliary functions $\widehat{\mu}$, $\widehat{\mu}_{-i}$ by

$$\widehat{\mu}(t) := \prod_{j\in I_{RS}^t} \mu_j(t), \quad \widehat{\mu}_{-i}(t) := \prod_{\substack{j\in I_{RS}^t \\ j\neq i}} \mu_j(t).$$

Obviously, if $i \in I_{RS}^t$ we have

$$\widehat{\mu}(t) = \mu_i(t)\widehat{\mu}_{-i}(t). \qquad (12)$$

We multiply both sides of the equality (10) by $\widehat{\mu}(m) > 0$ (ie by all denominators $\mu_i(m)$ in the latter sum in (11)) to obtain

$$\begin{aligned}
\widehat{\mu}(m)\mathcal{L}[u](m) = \widehat{\mu}(m) &\sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) \\
&+ \sum_{i\in I_{RS}^m} \left(\left(\widehat{\mu}_{-i}(m) + b_i(m)\widehat{\mu}(m)\right) u^{\Delta_i}(m) + \left(-\widehat{\mu}_{-i}(m) + c_i(m)\widehat{\mu}(m)\right) u^{\nabla_i}(m)\right),
\end{aligned}$$

and use (12) to get

$$\begin{aligned}
\widehat{\mu}(m)\mathcal{L}[u](m) = \widehat{\mu}(m) &\sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) \\
&+ \widehat{\mu}_{-i}(m) \sum_{i\in I_{RS}^m} \left(\left(1 + b_i(m)\mu_i(m)\right) u^{\Delta_i}(m) + \left(-1 + c_i(m)\mu_i(m)\right) u^{\nabla_i}(m)\right).
\end{aligned}$$

Necessary conditions (9)-(10) together with the assumptions (7)-(8) and positivity of $\widehat{\mu}(m)$ and $\widehat{\mu}_{-i}(m)$ imply then that

$$\widehat{\mu}(m)\mathcal{L}[u](m) \le 0,$$

which contradicts (2) and consequently u cannot attain a maximum at an interior point. □

Remark 7. In other words, in each $t \in \Lambda_\kappa^\kappa$ we require $2|I_{RS}^t|$ inequalities to hold ($|I_{RS}^t|$ inequalities corresponding to (7) and similarly $|I_{RS}^t|$ inequalities corresponding to (8), since both inequalities are trivially satisfied if $\mu_i(m) = 0$.

To illustrate the necessity of assumptions (7),(8), we include a simple example.

Example 8. Let us consider the two dimensional time scale $\Lambda = \{0,1,2\} \times \{0,1,2\}$. In this case, conditions (7),(8) reduce to

$$b_i \geq -1, \text{ and } c_i \leq 1.$$

Let us assume that $b_1 = -(1+\varepsilon)$. We construct a simple function which satisfy $\mathcal{L}[u] = \Delta_{\mathbb{T}} u - (1+\varepsilon) u^{\Delta_1} > 0$ and attains a maximum in interior point $a = (1,1)$. Indeed, let us consider a function $u(t_1,t_2)$ given by the following table:

$u(t_1,t_2)$	0	1	2
0	0	0	0
1	0	0	0
2	0	ξ	0

where $\xi < 0$. Obviously, $\Delta_{\mathbb{T}} u(a) = u^{\Delta_1}(a) = \xi$. Thus

$$\mathcal{L}[u](a) = \Delta_{\mathbb{T}} u(a) - (1+\varepsilon) u^{\Delta_1}(a) = \xi\varepsilon > 0,$$

but the function attains its maximum at the interior point $a = (1,1)$.

4. Strong Maximum Principles

The natural extension of the weak maximum principles is to remove the strict inequality from (2). Unfortunately, in this case necessary conditions for maximality of a considered function do not suffice to provide contradiction. Generally, one has to study the behavior of the operator not only in one direction but in the whole neighborhood of the assumed maximum.

Typically, this leads to the use of multiple integrals which are consequently subjected to the divergence theorem and transformed into curve integrals. Unfortunately, we don't have counterparts of such tools in a general time scale case

In this section we consider only n-dimensional time scales without dense points whose density is caused by an infinite sequence of scattered points.

Definition 9. We say that a time scale $\mathbb{T}$ is *regular*, if following conditions hold

- for each right dense $t \neq \max \mathbb{T}$ there exists $\delta > 0$ such that for each $\varepsilon < \delta$ the point $t + \varepsilon \in \mathbb{T}$ is right dense as well,
- for each left dense $t \neq \min \mathbb{T}$ there exists $\delta > 0$ such that for each $\varepsilon < \delta$ the point $t - \varepsilon \in \mathbb{T}$ is left dense as well.

An n-dimensional time scale $\Lambda = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n$ is called *regular* if $\mathbb{T}_i$ is regular for all $i \in I$.

All purely discrete, purely continuous time scales, unions of intervals or unions of intervals with (finite) discrete time scales are regular. On the other hand,

$$\mathbb{T} = \{0\} \cup \left\{\frac{1}{m}\right\} \cup \left\{-\frac{1}{m}\right\}, \quad m \in \mathbb{N},$$

is not regular because of its behavior in the neighborhood of 0.

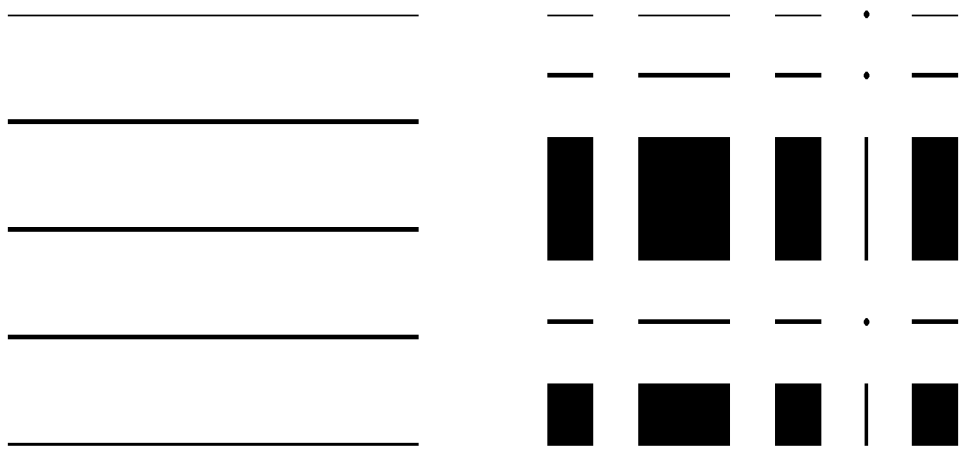

Figure 1. Examples of regular 2-dimensional time scales.

For regular n-dimensional time scales we are able to circumvent the application of divergence theorems and prove the strong maximum principles in the following form:

Theorem 10. *Let Λ be a regular n-dimensional time scale. Let $u \in \mathcal{D}(\Lambda)$ satisfy*

$$\mathcal{L}[u] \geq 0, \quad \text{in } \Lambda_\kappa^\kappa, \tag{13}$$

and let b_i and c_i be bounded and satisfy

$$1 + b_i(t)\mu_i(t) > 0, \tag{14}$$

$$-1 + c_i(t)\mu_i(t) < 0, \tag{15}$$

for each $t \in \Lambda_\kappa^\kappa$. Then u cannot attain its maximum at an interior point of Λ, unless it is constant.

Proof. We suppose that u attains its strict maximum M at some interior point m and we show that it must then be constant.

If we consider the index sets I_{RD}^m and I_{RS}^m defined in (5)-(6) and the necessary conditions (9) and (10), then the inequality (13) in m can be rewritten, similarly as in the proof of Theorem 6 into

$$\begin{aligned} &\widehat{\mu}(m) \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) + \\ &\qquad \widehat{\mu}_{-i}(m) \sum_{i \in I_{RS}^m} \big((1 + b_i(m)\mu_i(m))\, u^{\Delta_i}(m) + (-1 + c_i(m)\mu_i(m))\, u^{\nabla_i}(m) \big) > 0. \end{aligned} \tag{16}$$

If $u^{\Delta_i}(m) < 0$ or $u^{\nabla_i}(m) > 0$ for some $i \in I^m_{RS}$, then

$$\sum_{i \in I^m_{RD}} u^{\nabla_i \Delta_i}(m) > 0,$$

a contradiction with maximality of u at m. Therefore, it must be

$$u^{\Delta_i}(m) = 0, \quad u^{\nabla_i}(m) = 0 \text{ for all } i \in I^m_{RS}. \tag{17}$$

There are two possibilities now,

- m is i-scattered for all $i \in I$. Then (17) implies that for all $i \in I$
$$u(\varrho^i(m)) = u(m) = u(\sigma^i(m)).$$
- m is not i-scattered for all $i \in I$. Then m is an element of a closed connected rectangular subset Ω whose dimension d_Ω satisfies
$$1 \leq d_\Omega \leq n.$$
Since Ω is connected (i.e. all interior points of Ω are i-dense for all $i \in I^m_{RD}$ and i-scattered for all $i \in I^m_{RS}$), for all interior points of Ω the inequality (13) becomes
$$\sum_{i \in I^m_{RD}} \left(\frac{\partial^2 u}{\partial x_i^2} + (b_i + c_i) \frac{\partial u}{\partial x_i} \right) \geq 0.$$
But we can of course use a standard continuous maximum principle (Protter and Weinberger [13, Theorem 2.6]) to get that u must be constant on Ω or must attain maximum on the boundary $\partial\Omega$ and the accompanying boundary point lemma (Protter and Weinberger [13, Theorem 2.7]) implies that outward directional derivatives are positive. We argue that if the maximum of u is attained at $m \in \Omega$, then u must be constant. If that is not the case then the maximum of u is attained at the boundary $\partial\Omega$ and the outward derivative should be positive. However, the boundary points of Ω are either i-right scattered or i-left scattered points of our domain Λ. Therefore, the positivity of the outward derivative implies that u cannot attain its maximum at boundary points of Ω. Indeed, this would contradict either necessary conditions at i-left scattered points (Lemma 3), or the derived conditions (17) at i-right scattered points. This implies that u cannot attain its maximum at any point of Ω, unless it is constant on Ω. In this case the outward derivatives are zero.

$\square$

As in the previous section, we add a simple counterexample. In this case, we illustrate that it is necessary to consider strict inequalities in (14)-(15).

Example 11. Let us consider two dimensional time scale $\Lambda = [0,4] \times [0,4]_{\mathbb{Z}} = [0,4] \times \{0,1,2,3,4\}$ (see Figure 1, left), the operator

$$\mathcal{L}[u] = \sum_{i=1}^{n} \left(u^{\nabla_i \Delta_i} + u^{\nabla_i} \right)$$

and the function $u : \Lambda \to \mathbb{R}$ defined by:

$$u(t_1, t_2) = \begin{cases} t_2 & \text{if } t_2 \le 2, \\ 2 & \text{if } t_2 > 2. \end{cases}$$

Obviously, u attains its maximum at all (t_1, t_2), where $t_1 \in [0, 4]$ and $t_2 \in \{2, 3, 4\}$. Note that

$$\mathcal{L}[u](t_1, 2) = 0, \text{ for } t_1 \in (0, 4).$$

5. Corollaries and extensions

First, we consider second-order dynamic operators with non-derivative terms. The procedure of the proof of Theorem 10 can be repeated to obtain

Theorem 12. *Let Λ be a regular n-dimensional time scale. Let $u \in \mathcal{D}(\Lambda)$ satisfy*

$$\sum_{i=1}^{n} \left(u^{\nabla_i \Delta_i} + b_i u^{\Delta_i} + c_i u^{\nabla_i} \right) + hu \ge 0, \tag{18}$$

in Λ_κ^κ and let b_i, c_i and h be bounded, satisfy (14), (15) *and*

$$h \le 0. \tag{19}$$

for each $t \in \Lambda_\kappa^\kappa$. Then u cannot attain a nonnegative maximum at an interior point of Λ, unless it is constant and $h \equiv 0$

Finally, one can use this extension and consider the problem where non-derivative terms of all neighbours of t are included as well. In order to make the result more transparent we define the operator $\widehat{\sigma}_i : \Lambda \to \Lambda$, $i \in I$ by

$$\widehat{\sigma}_i(t) = \left(t_1, t_2, \dots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \dots, t_n\right).$$

The operator $\widehat{\varrho}_i : \Lambda \to \Lambda$ is defined analogically

$$\widehat{\varrho}_i(t) = \left(t_1, t_2, \dots, t_{i-1}, \varrho_i(t_i), t_{i+1}, \dots, t_n\right).$$

Corollary 13. *Let Λ be a regular n-dimensional time scale. Let $u \in \mathcal{D}(\Lambda)$ satisfy*

$$\sum_{i=1}^{n} \left(u^{\nabla_i \Delta_i} + b_i u^{\Delta_i} + c_i u^{\nabla_i} + \beta_i u(\widehat{\sigma}_i(t)) + \gamma_i u(\widehat{\varrho}_i(t)) \right) + hu \ge 0. \tag{20}$$

Further, let $b_i + \beta_i \mu_i$, $c_i + \gamma_i \nu_i$ and $h + \sum_{i=1}^{n} (\beta_i + \gamma_i)$ be bounded and satisfy

$$1 + \left(b_i(t) + \beta_i(t)\mu_i(t)\right) \mu_i(t) > 0, \tag{21}$$

$$-1 + \left(c_i(t) + \gamma_i(t)\nu_i(t)\right) \mu_i(t) < 0, \tag{22}$$

$$h + \sum_{i=1}^{n} (\beta_i + \gamma_i) \le 0, \tag{23}$$

for all $i \in I$ and $t \in \Lambda_\kappa^\kappa$. Then u cannot attain a nonnegative maximum at an interior point of Λ, unless it is constant.

Proof. Obviously, one can use the following two equalities (analogues of well-known one-dimensional equalities sometimes nicknamed as "simple useful formulas"):

$$u(\widehat{\sigma}_i(t)) = u(t) + \mu_i(t)u^{\Delta_i}(t)$$
$$u(\widehat{\sigma}_i(t)) = u(t) + \mu_i(t)u^{\nabla_i}(t)$$

Substituting these into (18), we obtain

$$\sum_{i=1}^{n}\left(u^{\nabla_i\Delta_i} + (b_i + \beta_i\mu_i)u^{\Delta_i} + (c_i + \gamma_i\nu_i)u^{\nabla_i}\right) + \left(h + \sum_{i=1}^{n}(\beta_i + \gamma_i)\right)u \geq 0.$$

Obviously, this operator has the form of (18) and assumptions (21)-(23) ensure that the inequalities (14), (15) and (19) hold. Consequently, one can use Theorem 12 to verify the statement. ☐

6. Uniqueness of Dirichlet boundary value problems

In this section we consider the Dirichlet boundary value problem of the form

$$\begin{cases} \sum\limits_{i=1}^{n}\left(u^{\nabla_i\Delta_i} + b_iu^{\Delta_i} + c_iu^{\nabla_i}\right) + hu = f(t) \quad \text{on } \Lambda_\kappa^\kappa, \\ u(t) = g(t) \quad \text{on } \partial\Lambda. \end{cases} \tag{24}$$

The above proven maximum principles yield directly the existence of at most one solution of this problem.

Theorem 14. *Let us suppose that b_i, c_i and h are bounded function on regular Λ and satisfy inequalities (14), (15) and (19) at each $t \in \Lambda_\kappa^\kappa$. If u_1 and u_2 are solutions of the boundary value problem (24) then $u_1 \equiv u_2$.*

Proof. We proceed in the standard way and define the auxiliary function $v = u_1 - u_2$. Since both u_1 and u_2 satisfy (24), the function v then satisfies

$$\begin{cases} \sum\limits_{i=1}^{n}\left(v^{\nabla_i\Delta_i} + b_iv^{\Delta_i} + c_iv^{\nabla_i}\right) + hv = 0 \quad \text{on } \Lambda_\kappa^\kappa, \\ v(t) = 0 \quad \text{on } \partial\Lambda. \end{cases}$$

Taking into account the closedness and boundedness of the domain Λ and the continuity of v we deduce that v attains a maximum and a minimum in Λ. However, the maximum principle Theorem 12 implies that

- v cannot have a positive maximum in Λ (note that $v = 0$ in the boundary),
- $-v$ cannot have a positive maximum in Λ. In other words, v cannot have a negative minimum in Λ.

Consequently $v = u_1 - u_2 \equiv 0$. ☐

As a special case of this result we obtain the existence of at most one solution to the Dirichlet problem for the dynamic Poisson equation.

Corollary 15. *Let Λ be regular. If u_1 and u_2 are solutions of the boundary value problem*

$$\begin{cases} \Delta_{\mathbb{T}} u(t) = f(t) & \text{on } \Lambda_\kappa^\kappa \\ u(t) = g(t) & \text{on } \partial\Lambda. \end{cases}$$

then $u_1 \equiv u_2$.

7. Open Problems and Extensions

Note that the necessary conditions which ensure the validity of maximum principles (14) and (15) can be rewritten as

$$b_i \in \mathcal{R}_i^+, \text{ and } -c_i \in \mathcal{R}_i^+,$$

where $\mathcal{R}_i^+$ denotes the group of positively regressive functions on i-th time scale $\mathbb{T}_i$. (see eg [5, Chapter 2] for more details). It implies, for example, that the corresponding exponential function on $\mathbb{T}_i$ is always positive. The natural question arises whether the regressivity groups are going to play similar role in more dimensions as our results suggest.

Next, we have proved the strong maximum principles for all regular n-dimensional time scales (see Theorem 10). Can we find a proof that would include all n-dimensional time scales?

Finally, it seems natural now to consider other types of partial dynamic operators (ie parabolic and hyperbolic) and study the corresponding maximum principles.

8. Acknowledgements

The author gratefully acknowledges the support by the Ministry of Education, Youth and Sports of the Czech Republic, Research Plan No. MSM 4977751301.

The author is also very grateful to the referee of this paper for his valuable remarks and revelation of inaccuracies.

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