MAXIMUM PRINCIPLES FOR SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES

PETR STEHLIK* AND BEVAN THOMPSON[2]

*Department of Mathematics
The University of Rostock,
Universitätsplatz 1, Rostock
GERMANY 18051
E-mail: petr.stehlik@uni-rostock.de
[2] Centre for Mathematical Modelling and Analysis,
Department of Mathematics
The University of Queensland
Brisbane, Queensland
AUSTRALIA 4072
E-mail: hbt@maths.uq.edu.au
Fax: +61 7 33651477

Abstract. We establish the strong and the generalized maximum principle as well as the boundary point lemma for second order dynamic equations on time scales. These include the corresponding results for ordinary differential and difference equations as special cases.

AMS(MOS) subject classification: 34B15 39A10 34B10
Key Words and Phrases: maximum principles on time scales, generalized maximum principle, boundary point lemma.

The first author gratefully acknowledges the support by the Ministry of Education, Youth and Sports of the Czech Republic, Research Plan No. MSM 4977751301. The second author extends his thanks to Prof Pavel Drábek and the Department of Mathematics of the University of West Bohemia for their support and hospitality during this work.

1. INTRODUCTION

The strong maximum principle for second order ordinary differential equations and its variants are well known and have many applications. They yield *a priori* bounds which can be used to establish existence and uniqueness of solutions for initial and boundary value problems as well as to establish oscillation results. The base strong maximum principle is as follows.

Theorem 1. (*Protter and Weinberger* [6, Theorem 1]) *Suppose* $u \in C^2(a,b)$ *satisfies the differential inequality*

$$u'' + g(x)u' \geq 0 \tag{1}$$

for $a < x < b$, *where* g *is a bounded real valued function. If* u *attains a local maximum value,* M, *at an interior point of* (a,b), *then* $u(x) \equiv M$.

Recently Mawhin, Tonkes and Thompson [5] established discrete analogues of the results of Protter and Weinberger and applied their results to answer some uniqueness questions posed in Thompson [8].

Dynamic equations on time scales is a relatively new, rapidly expanding area of research which unify and extend the discrete and continuous calculus. See, for example, the books by Bohner and Peterson [1] and [2], together with the references therein.

In the literature recently there has been a number of results on the existence of multiple solutions for second order differential and difference equations as well as for dynamic equations on time scales which have been based on lower and upper solutions.

This raises the question as to what conditions guarantee uniqueness for second order dynamic equations on time scales. Thus it is natural to study maximum principles in this context. The importance of maximum principles is further highlighted by the fact that the method of lower and upper solution is based on a weak version of the maximum principle.

Simple maximum principles for time scales have been already used by Drábek, Henderson, Tisdell [3] to prove the existence of multiple solutions and by Stehlík [7] to construct monotone sequences.

In this paper we establish time scales analogues of the results of [5] and [6] as well as analogues of other maximum principles. In a forthcoming paper we will apply these to obtain a priori bounds, uniqueness and oscillation results for boundary value problems for second order dynamic equations on time scales. In particular we show under fairly weak assumptions on the coefficients that if x is a solution of

$$x^{\Delta\Delta} + g_1 x^{\Delta} + g_2 x^{\Delta\sigma} + h_2 x^{\sigma} \geq 0$$

on a time scale $[a, b]_{\mathbb{T}}$, then x cannot have a non-negative interior maximum unless x is identically constant.

Our results unify and generalize the results of Mawhin, Tonkes and Thompson [5] and those given in Protter and Weinberger [6].

2. PRELIMINARIES

To understand the notation used and the notion of a time scale some preliminary definitions are needed.

Definition A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale need not be connected, the concept of jump operators is needed to overcome this difficulty.

Definition The forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ ($\rho(t)$ at $t \in \mathbb{T}$ for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\},) .$$

For simplicity and clarity set $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$.

If $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t = \rho(t)$. If $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$ and $\rho(t) = t - 1$.

Throughout this work it is assumed that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also it is assumed throughout that $a < b$ are points in $\mathbb{T}$ with $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$.

The jump operators σ and ρ allow the following classification of points in a time scale. If $\sigma(t) > t$ then call the point t right-scattered; while if $\rho(t) < t$ then call t left-scattered. If $\sigma(t) = t$ then call the point t right-dense; while if $\rho(t) = t$ then call t left-dense. We shall also use the notation $\mu(t) := \sigma(t) - t$, where μ is called the graininess function.

If $\mathbb{T}$ has a left-scattered maximum m then define $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t with

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

Call $y^\Delta(t)$ the derivative of $y(t)$. Define the second derivative by $y^{\Delta\Delta} = (y^\Delta)^\Delta$.

The following theorem is due to Hilger [4].

Theorem 2. *Assume that $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$.*
(i) If f is differentiable at t then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If f is differentiable and t is right-dense then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t then $f(\sigma(t)) = f(t) + (\sigma(t) - t) f^{\Delta}(t)$.

Definition If $F^{\Delta}(s) = f(s)$ for all $s \in [a,t]_{\mathbb{T}}$ then define the integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a).$$

Definition Let $f : \mathbb{T} \to \mathbb{R}$. Define and denote $f \in C_{rd}(\mathbb{T}; \mathbb{R})$ as right-dense continuous if for each $t \in \mathbb{T}$:
$\lim_{s \to t^+} f(s) = f(t)$, if $t \in \mathbb{T}$ is right-dense,
$\lim_{s \to t^-} f(s)$ exists and is finite if $t \in \mathbb{T}$ is left-dense.

Note that, in general, the jump operator σ is neither differentiable nor bounded, as shown in [1, Example 1.56]. However, it is right-dense continuous.

3. Derivatives-only Maximum Principles

We present first the weak version of the maximum principle for operators which do not contain non-derivative terms.

Lemma 3. *Assume that the functions g_1, $g_2 : [a,b]_{\mathbb{T}} \to \mathbb{R}$ satisfy*

$$\mu g_1 \leq 1, \tag{2}$$

$$\mu g_2 \geq -1. \tag{3}$$

If x satisfies the dynamic inequality

$$\mathcal{L}_1[x] := x^{\Delta\Delta} + g_1 x^{\Delta} + g_2 x^{\Delta\sigma} > 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}, \tag{4}$$

and attains a maximum M at an interior point of $(a, \sigma^2(b))_{\mathbb{T}}$, then $x \equiv M$.

Remark 4. All quantities in the above and subsequent results are assumed to exist. For example, since x satisfies (4), $x^{\Delta\Delta}$ is defined on $[a,b]_{\mathbb{T}}$, x is defined and continuous on $[a, \sigma^2(b)]_{\mathbb{T}}$ and x^{Δ} is defined and continuous on $[a, \sigma(b)]_{\mathbb{T}}$. In particular if $b < \sigma(b) = \sigma^2(b)$, then we assume that $x^{\Delta}(\sigma^2(b))$ exists. This may appear strange at first as one usually works with one sided derivatives at end points in the continuous case.

Proof. Assume that x is not a constant function. Let us choose $c \in (a, \sigma^2(b))_{\mathbb{T}}$ such that $x(c) = M$. Let us divide our proof into three parts:

(i) c is left- and right-dense. In this case the maximality of x at c provides $x^{\Delta}(c) = 0 = x^{\Delta\sigma}(c)$ and $x^{\Delta\Delta}(c) \le 0$ so that (4) reduces to
$$x^{\Delta\Delta}(c) > 0,$$
a contradiction.

(ii) c is left-dense and right-scattered. If $x^{\Delta}(c) = 0$ then the above conditions imply that
$$0 \ge x^{\Delta\Delta}(c) = \frac{x^{\Delta}(\sigma(c))}{\mu(c)},$$
and consequently $x^{\Delta}(\sigma(c)) \le 0$. Multiplying (4) at c by $\mu(c)$ yields a following contradiction
$$\mu(c)\left(x^{\Delta\Delta}(c) + g_2(c)x^{\Delta}(\sigma(c))\right) = x^{\Delta}(\sigma(c))\,(1 + \mu(c)g_2(c)) \le 0.$$
If $x^{\Delta}(c) < 0$ then there is $\delta > 0$ such that $x^{\Delta}(t) < 0$ for $(c-\delta, c)_{\mathbb{T}}$ so that $x(t) > M$ on $(c-\delta, c)_{\mathbb{T}}$, a contradiction.

(iii) c is left-scattered. The necessary condition in this case is
$$x^{\Delta}(c) \le 0, \quad x^{\Delta}(\varrho(c)) \ge 0.$$
These immediately give $x^{\Delta\Delta}(\varrho(c)) \le 0$. Multiplying (4) by $\mu^{\varrho}(c)$ and using (2) and (3) we have
$$\begin{aligned}&\mu^{\varrho}(c)\left(x^{\Delta\Delta}(\varrho(c)) + g_1^{\varrho}(c)x^{\Delta}(\varrho(c)) + g_2^{\varrho}(c)x^{\Delta}(c)\right)\\ &= x^{\Delta}(c)\,(1 + g_2^{\varrho}(c)\mu^{\varrho}(c)) + x^{\Delta}(\varrho(c))\,(g_1^{\varrho}(c)\mu^{\varrho}(c) - 1) \le 0,\end{aligned}$$
a contradiction.

Thus $x \equiv M$. □

We give a variant of this result where we weaken the inequality (4) and strengthen the inequalities (2) and (3).

Theorem 5. *Assume that the functions $g_1,\ g_2 : [a,b]_{\mathbb{T}} \to \mathbb{R}$ satisfy*

$$\mu g_1 < 1, \tag{5}$$

$$\mu g_2 > -1, \tag{6}$$

and

$$k = \frac{g_1 + g_2}{1 + g_2\mu} \text{ is bounded.} \tag{7}$$

If x satisfies the dynamic inequality

$$\mathcal{L}_1[x] \ge 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}, \tag{8}$$

and attains a maximum M *at* $t \in (a, \sigma^2(b))_{\mathbb{T}}$*, then* $x \equiv M$*.*

Proof. Assume the result is false. Thus we may choose $c \in (a, \sigma^2(b))_{\mathbb{T}}$ such that $x(c) = M$. Since $x(t) \not\equiv M$ there is $d \in (a, b)_{\mathbb{T}}$ such that $x(d) < M$. Let us assume first that $c < d$. Let us take arbitrary $\alpha > 0$ such that

$$\alpha > -\frac{g_1 + g_2}{1 + g_2 \mu},$$

and let us define a function z by

$$z(t) := e_\alpha(t, c) - 1,$$

where $e_\alpha(\cdot, c)$ is an exponential function on $\mathbb{T}$. Exploiting the way α was chosen and the positivity of e_α we obtain

$$\begin{aligned} \mathcal{L}_1[z] &= z^{\Delta\Delta} + g_1 z^{\Delta} + g_2 z^{\Delta\sigma} \\ &= \left(\alpha^2 + g_1\alpha + g_2\alpha\,(1 + \mu\alpha)\right) e_\alpha(\cdot, c) \\ &= \alpha\,(g_1 + g_2 + \alpha\,(1 + \mu g_2))\, e_\alpha(\cdot, c) \\ &> 0. \end{aligned}$$

Let us define a function w by

$$w(t) := x(t) + \beta z(t),$$

where $\beta > 0$ is chosen so that

$$\beta < \frac{M - x(d)}{z(d)}.$$

Since $e_\alpha(\xi, c) < 1$ for any $\xi \in (a, c)_{\mathbb{T}}$ we have

$$w(\xi) < M.$$

Moreover, the definition of β yields that

$$w(d) = x(d) + \beta z(d) < x(d) + M - x(d) = M.$$

Finally, $e_\alpha(c, c) = 1$ provides

$$w(c) = M.$$

It follows that w has a maximum inside the interval $(\xi, d)_{\mathbb{T}}$. However

$$\mathcal{L}_1[w] = \mathcal{L}_1[x] + \beta\mathcal{L}_1[z] > 0,$$

which contradicts the statement of Lemma 3.

A more complicated situation occurs if $x(t) = M$ for all $t \geq c$ so that $d < c$ (note that $e_{-\alpha}$ is not generally positive). In this case, we define

$$m := \inf\left\{s \geq a : x(\xi) = M \text{ and } s \leq \xi \leq c\right\}.$$

If m is left-scattered then $x^\Delta(m) \le 0$, and $x^\Delta(\varrho(m)) > 0$ and multiplying $\mathcal{L}_1(\varrho(m))$ by $\mu(\varrho(m))$ we get

$$\mathcal{L}_1(\varrho(m))\mu(\varrho(m)) = x^\Delta(\varrho(m))\,(g_1^\varrho(m)\mu^\varrho(m) - 1) < 0,$$

a contradiction and the result follows. As in the proof of Lemma 1, if m is left-dense, then $x^\Delta(m) = 0$. Similarly, as in the previous case we define functions z and w by

$$z(t) = e_{-\alpha}(t, m) - 1, \text{ and } w(t) = x(t) + \beta z(t),$$

where α, β are positive constants satisfying

$$\alpha > \frac{g_1 + g_2}{1 + g_2\mu}, \text{ and } \beta < \frac{M - x(d)}{z(d)}.$$

Thus, we can choose d_1 so that $1-\mu\alpha > 0$ and consequently $e_{-\alpha}(t, m) > 0$ for all $t \in [d_1, m)_\mathbb{T}$. Therefore, on this interval we have

$$\begin{aligned}\mathcal{L}_1[z] &= \left(\alpha^2 - g_1\alpha - g_2\alpha\,(1 - \mu\alpha)\right) e_{-\alpha}(\cdot, c)\\ &= \alpha\,(-g_1 - g_2 + \alpha\,(1 + \mu g_2))\, e_{-\alpha}(\cdot, c)\\ &> 0.\end{aligned}$$

Using Lemma 3 we see that the maximum of w must occur at boundary points of $[d_1, m]_\mathbb{T}$. Since $w(d_1) < M = w(m)$, we see that w attains its maximum at m. Thus

$$0 \le w^\Delta(m) = x^\Delta(m) + \beta z^\Delta(m).$$

Taking into account $z^\Delta(m) = -\alpha < 0$ we see that $x^\Delta(m) > 0$ which contradicts the maximality of x at m. The result follows. □

We include a simple example on $\mathbb{T} = \mathbb{Z}$ to illustrate that the assumptions (5), (6) cannot be improved.

Example 6. Let us assume that $\mathbb{T} = \mathbb{Z}$ and that x is defined by

$$x(t) = \begin{cases} t, & t \le 4,\\ 4, & t > 4.\end{cases}$$

Obviously, x satisfies

$$x^{\Delta\Delta} + x^\Delta \ge 0, \qquad t \in [0, 10]_\mathbb{Z},$$

but x attains its local maximum at $t = 4$. This does not contradict Lemma 2 since $\mu g_1 = 1$ in this case.

Similarly, y defined by

$$y(t) = \begin{cases} 4 & t \le 4,\\ 8 - t & t > 4,\end{cases}$$

attains a local maximum at $t = 4$, even if it satisfies

$$x^{\Delta\Delta} - x^{\Delta\sigma} \geq 0, \qquad t \in [0,10]_{\mathbb{Z}}.$$

Note that $\mu g_2 = 1$ now.

The presence of both x^{Δ} and $x^{\Delta\sigma}$ in (4) is motivated by the desire to obtain the most general result. However, (v) in the following remark indicates that this case can be transformed to that of an operator where only one of these derivatives is present.

Remark 7. Following Protter and Weinberger, [6], we make the following observations.

(i) Theorem 5 shows that a function x satisfying the dynamic inequality (8) cannot have a strict local maximum in $(a,b)_{\mathbb{T}}$. Indeed if x has such a local maximum at $c \in (a,b)_{\mathbb{T}}$, then there is $d < c < e \in (a,b)_{\mathbb{T}}$ such that x has an absolute maximum in $[d,e]_{\mathbb{T}}$ at c so that $x \equiv x(c)$, a contradiction.

(ii) A function x satisfying the dynamic inequality (8) cannot have two strict local minima in $(a,b)_{\mathbb{T}}$. If it did then there would be a strict local maximum between the local minima, a contradiction.

(iii) If in place of the dynamic inequality (8) in Theorem 5 we assume that

$$\mathcal{L}_1[x] \leq 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}, \tag{9}$$

then we obtain the corresponding minimum principle that if a function x attains a non positive minimum m at $t \in (a, \sigma^2(b))_{\mathbb{T}}$, then $x \equiv m$.

(iv) An examination of the proof of Theorem 5 shows that it is sufficient that k is bounded on every closed subinterval of $(a,b)_{\mathbb{T}}$. This local boundedness of $k = \frac{g_1+g_2}{1+g_2\mu}$ is necessary for the conclusion of Theorem 5. In the case $\mathbb{T} = \mathbb{R}$, $g_2 \equiv 0$, and

$$g_1(t) = \begin{cases} -\frac{5}{t}, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0 \end{cases}$$

$x(t) = 1 - t^6$ is a solution of (8) on $[-1,1]$ which has a maximum at $t = 0$.

(v) Assuming that (6) holds and using the fact that $x^{\Delta\sigma} = x^{\Delta} + \mu x^{\Delta\Delta}$, one can rewrite (8) (or similarly (4)) as

$$x^{\Delta\Delta} + \frac{g_1+g_2}{1+\mu g_2} x^{\Delta} \geq 0.$$

Note, that in this case (5) is equivalent to

$$\mu\frac{g_1+g_2}{1+\mu g_2}<1.$$

(vi) If $\mu g_2 > -1$ then the boundedness of g_1 and g_2 imply that $\dfrac{g_1+g_2}{1+g_2\mu}$ is bounded however the converse is not true. To see this let $[a,b]_{\mathbb{T}} = [-1,1]_{\mathbb{T}}$ and $\mathbb{T} = \{\pm 2^{-n} : n \in \mathbb{N}\} \cup \{0\}$ so that $\mu(2^{-n}) = 2^{-n}$. If $g_1(2^{-n}) = n-1$ and $g_2(2^{-n}) = -n+1$, then $\dfrac{g_1+g_2}{1+g_2\mu}$ is bounded however neither g_1 nor g_2 is bounded.

4. Full Maximum Principles

We extend the above results to operators which also contain non-derivative terms.

Lemma 8. *Assume that the functions* $g_1,\ g_2,\ h_2 : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *satisfy* (2), (3) *and*

$$h_2 \le 0. \tag{10}$$

Moreover, let x *satisfy the dynamic inequality*

$$\mathcal{L}_2[x] := x^{\Delta\Delta} + g_1 x^{\Delta} + g_2 x^{\Delta\sigma} + h_2 x^{\sigma} > 0, \quad \textit{for} \quad t \in [a,b]_{\mathbb{T}}. \tag{11}$$

If x *attains a nonnegative maximum* M *at* $t \in (a, \sigma^2(b))_{\mathbb{T}}$, *then* $x \equiv M$.

Proof. The proof is similar to Lemma 3 and thus omitted. □

Theorem 9. *Assume that the functions* $g_1,\ g_2,\ h_2 : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *satisfy* (5), (6), (7), (10) *and*

$$\frac{h_2}{1+g_2\mu} \text{ is bounded.} \tag{12}$$

Moreover, let x *satisfy the dynamic inequality*

$$\mathcal{L}_2[x] \ge 0, \quad \textit{for} \quad t \in [a,b]_{\mathbb{T}}. \tag{13}$$

If x *attains a nonnegative maximum* M *at* $t \in (a, \sigma^2(b))_{\mathbb{T}}$, *then* $x \equiv M$.

Proof. The proof follows that of Theorem 5. If $d > c$ it suffices to choose $\alpha > 0$ large enough, such that

$$\alpha^2(1+g_2\mu) + \alpha(g_1+g_2) + h_2(1+\mu\alpha - e_{\ominus\alpha}(t,c)) > 0.$$

Note that this is possible by (3) and (7) since $h_2 \le 0$.

In the case when $d < c$, we again define m by

$$m := \inf\{s \ge a : x(\xi) = M \text{ and } s \le \xi \le c\}.$$

If m is left-scattered then

$$\mathcal{L}_2(\varrho(m))\mu(\varrho(m)) = x^{\Delta}(\varrho(m))\left(g_1^{\varrho}(m)\mu^{\varrho}(m) - 1\right) - h_2(m)g_1^{\varrho}(m)M < 0.$$

If m is left-dense, we can follow the argument from Theorem 5, except for choosing large enough $\alpha > 0$ so that

$$\begin{aligned}\mathcal{L}_2[z] &= \left(\alpha^2\left(1 + g_2\mu\right) - \alpha\left(g_1 + g_2\right) + h_2\left(1 - \mu\alpha - e_{\ominus-\alpha}(\cdot, c)\right)\right) e_{-\alpha}(\cdot, c) \\ &> 0,\end{aligned}$$

for all $t \in [d_1, m)_{\mathbb{T}}$, where d_1 is close enough to m. The same argument as above yields that $x^{\Delta}(m) > 0$, a contradiction. □

5. Shifted maximum principles

Since $x^{\Delta\Delta}(t)$ depends on $x(t)$, $x^{\sigma}(t)$ and $x^{\sigma\sigma}(t)$ a more natural form of operator for the full maximum principle is

$$\mathcal{L}_3[x] := x^{\Delta\Delta} + g_1 x^{\Delta} + g_2 x^{\Delta\sigma} + h_1 x + h_2 x^{\sigma} + h_3 x^{\sigma\sigma}.$$

In the spirit of Remark 7 we transform $\mathcal{L}_3$ into $\mathcal{L}_2$ and then apply Theorem 9.

Corollary 10. *Assume that the functions g_i, $i = 1, 2$ and h_j, $j = 1, 2, 3$ satisfy*

$$\mu\left(g_1 - h_1\mu\right) < 1, \tag{14}$$

$$\mu\left(g_2 + h_3\mu^{\sigma}\right) > -1, \tag{15}$$

$$h_1 + h_2 + h_3 \leq 0, \tag{16}$$

and that

$$\frac{\left(g_1 - h_1\mu\right) + \left(g_2 + h_3\mu^{\sigma}\right)}{1 + \left(g_2 + h_3\mu^{\sigma}\right)\mu} \text{ is bounded,} \tag{17}$$

and that

$$\frac{h_1 + h_2 + h_3}{1 + \left(g_2 + h_3\mu^{\sigma}\right)\mu} \text{ is bounded.} \tag{18}$$

If x satisfies the inequality

$$\mathcal{L}_3[x] \geq 0, \tag{19}$$

and attains its nonnegative maximum M at $t \in (a, \sigma^2(b))_{\mathbb{T}}$, then $x \equiv M$.

Proof. It is easy to see that

$$x = x^{\sigma} - \mu x^{\Delta}, \tag{20}$$

and

$$x^{\sigma\sigma} = \mu^{\sigma} x^{\Delta\sigma} + x^{\sigma}. \tag{21}$$

Thus one can rewrite (19) in the form

$$x^{\Delta\Delta} + (g_1 - h_1\mu)\, x^{\Delta} + (g_2 + h_3\mu^{\sigma})\, x^{\Delta\sigma} + (h_1 + h_2 + h_3)\, x^{\sigma} \geq 0, \tag{22}$$

that is, the form of (13). The assumptions (14)-(18) ensure that the operator (22) satisfies the assumptions of Theorem 9 and the result follows. □

6. Generalized Maximum Principles

In this section we extend Theorem 9 also to the case where condition (12) is not satisfied, that is, $h(t) > 0$ for some $t \in [a,b]_{\mathbb{T}}$. We consider here the operator

$$\widetilde{\mathcal{L}}_2[x] := x^{\Delta\Delta} + g x^{\Delta} + h x^{\sigma}, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}. \tag{23}$$

Thanks to Remark 7, $\widetilde{\mathcal{L}}_2$ and $\mathcal{L}_2$ can be transformed into each other. We assume that there exists a function w such that on $[a,b]_{\mathbb{T}}$,

$$w > 0, \quad \text{for} \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}} \tag{24}$$

while for $t \in [a,b]_{\mathbb{T}}$

$$\widetilde{\mathcal{L}}_2[w] \leq 0, \qquad \text{while} \tag{25}$$

$$\frac{g w + w^{\Delta} + w^{\sigma\Delta}}{w^{\sigma} + \mu w^{\sigma\Delta}} \qquad \text{and} \tag{26}$$

$$\frac{\widetilde{\mathcal{L}}_2[w]}{w^{\sigma} + \mu w^{\sigma\Delta}} \quad \text{are bounded.} \tag{27}$$

Let us define v by

$$v = \frac{x}{w}, \quad \text{for} \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}.$$

Differentiating x, one obtains

$$x^{\Delta} = v w^{\Delta} + w^{\sigma} v^{\Delta},$$
$$x^{\Delta\Delta} = v^{\sigma} w^{\Delta\Delta} + w^{\Delta} v^{\Delta} + w^{\sigma} v^{\Delta\Delta} + v^{\Delta\sigma} w^{\sigma\Delta},$$

on $[a,b]_{\mathbb{T}}$. Substituting these into (23) provides

$$\widetilde{\mathcal{L}}_2[x] = w^{\sigma} v^{\Delta\Delta} + v^{\Delta}\left(g w + w^{\Delta}\right) + v^{\Delta\sigma} w^{\sigma\Delta} + v^{\sigma} \widetilde{\mathcal{L}}_2[w] \geq 0.$$

Dividing by w^σ yields

$$\overline{\mathcal{L}}_2[v] := v^{\Delta\Delta} + v^{\Delta}\frac{gw + w^{\Delta}}{w^{\sigma}} + v^{\Delta\sigma}\frac{w^{\sigma\Delta}}{w^{\sigma}} + v^{\sigma}\frac{\widetilde{\mathcal{L}}_2[w]}{w^{\sigma}} \geq 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}.$$

Defining auxiliary functions $\overline{g}_1$, $\overline{g}_2$ and $\overline{h}_2$ by

$$\overline{g}_1 := \frac{gw + w^{\Delta}}{w^{\sigma}}, \quad \overline{g}_2 := \frac{w^{\sigma\Delta}}{w^{\sigma}}, \quad \overline{h}_2 := \frac{\widetilde{\mathcal{L}}_2[w]}{w^{\sigma}},$$

we see that

$$\overline{\mathcal{L}}_2[v] = v^{\Delta\Delta} + \overline{g}_1 v^{\Delta} + \overline{g}_2 v^{\Delta\sigma} + \overline{h}_2 v^{\sigma} \geq 0, \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}.$$

Lemma 11. *Let us suppose that*

$$\mu g < 1, \quad \textit{for} \quad t \in [a,b]_{\mathbb{T}}, \tag{28}$$

and that w satisfies (24)-(26)*. Then on $[a,b]_{\mathbb{T}}$*

(a) $\overline{h}_2 \leq 0$,
(b) $\mu\overline{g}_1 < 1$,
(c) $\mu\overline{g}_2 > -1$, *and*
(d) $\frac{\overline{g}_1 + \overline{g}_2}{1 + \overline{g}_2\mu}$ *is bounded.*

Proof. By continuity of w, $w \geq c > 0$ on $[a, \sigma^2(b)]_{\mathbb{T}}$ so that $w^\sigma \geq c > 0$ on $(a, \sigma^2(b))_{\mathbb{T}}$. Thus (d) is an immediate consequence of (26). If $\mu(t) = 0$, then (b) and (c) are trivially satisfied at t, so we assume that $\mu(t) > 0$.

(a) Non-positivity of $\overline{h}_2$ is an immediate consequence of (24) and (25).

(b) (28) and (24) yield that

$$0 > (\mu g - 1)\, w = \mu g w - w.$$

Adding w^σ to this inequality, we get

$$\mu\,(gw + w^{\sigma}) < w^{\sigma},$$

which is equivalent to

$$\mu\overline{g}_1 = \mu\frac{gw + w^{\Delta}}{w^{\sigma}} < 1.$$

(c) Finally,

$$\overline{g}_2(t) = \frac{w^{\sigma\Delta}}{w^{\sigma}} = \frac{w^{\sigma\sigma} - w^{\sigma}}{w^{\sigma}\mu}.$$

Thus

$$\mu(t)\overline{g}_2(t) = \frac{w^{\sigma\sigma}(t) - w^{\sigma}(t)}{w^{\sigma}(t)} > -1,$$

since $w^\sigma(t) > 0$ and $w^{\sigma\sigma}(t) > 0$ on $[a,b]_{\mathbb{T}}$.

□

Combining Lemma 11 and Theorem 9 we obtain the generalized maximum principle.

Theorem 12. *Assume that there exists w satisfying* (24)-(27). *Moreover, let g satisfy* (28). *If x is such that $\widetilde{\mathcal{L}}_2[x] \geq 0$ and $\frac{x}{w}$ attains a local maximum at $t \in (a, \sigma^2(b))_{\mathbb{T}}$, then $\frac{x}{w}$ is a constant.*

Remark 13. The assumption (26) is always satisfied if σ^Δ exists and is bounded. Indeed, if $\mu = 0$ then the statement follows from (24) since $w^{\sigma\Delta} = w^{\Delta\sigma}\sigma^\Delta$ exists and so $\overline{g}_2$ is bounded and $\overline{g}_1$ is bounded. If $\mu > 0$, then assumption (26) is equivalent to $\frac{gw+w^\Delta+w^\sigma(\sigma(t))-w^\sigma(t)}{w^\sigma(\sigma(t))}$ bounded which follows from the assumptions on g and w.

This raises the question as to whether or not the generalized maximum principle requires the existence of $w^{\sigma\Delta}$.

Finally, we give some conditions, under which there exist a suitable function w.

Lemma 14. *If g, h satisfy the assumption of Theorem 12, σ^Δ exists and is bounded, $b > a$ and $\sigma^2(b) - a$ is sufficiently small, then there exists a function w satisfying* (24)-(26).

Proof. We show that under the stated assumptions the function

$$w(t) = (1 - \beta \int_a^t (\int_a^s \Delta q)\Delta s) \tag{29}$$

is the solution of (24)-(26).

To see that (24) and (25) are satisfied note that

$$w^{\Delta\Delta} + gw^\Delta + hw^\sigma = -\beta\left(1 + g\int_a^t \Delta s + h\int_a^t (\int_a^s \Delta q)\Delta s)\right) + h \leq 0$$

for $\beta > 0$ sufficiently large and $b - a$ sufficiently small that

$$1 + g\int_a^t \Delta s + h\int_a^t (\int_a^s \Delta q)\Delta s > 0$$

and $w > 0$ for any $\beta > 0$ provided $b - a$ is correspondingly small. To see that (26) is satisfied note that w, w^Δ, $w^{\sigma\Delta}(t) = w^{\Delta\sigma}(t)\sigma^\Delta(t) = -\beta\int_a^{\sigma(t)} \Delta s \sigma^\Delta(t)$, and g are bounded and $w^\sigma + \mu w^{\sigma\Delta} = w^{\sigma\sigma} \geq k > 0$ for a constant k.

□

Note that in the case $\mathbb{T} = \mathbb{R}$ then (29) gives $w(t) = 1 - \beta(t-a)^2/2$ as in Protter and Weinberger, [6], while if $\mathbb{T} = h\mathbb{Z}$ and $[a, b]_{\mathbb{T}} = \{0, h, 2h, \ldots, hn\}$ then (29) gives $w_k = 1 - \beta k^{[2]}/2$ as in Mawhin, Thompson, Tonkes [5].

7. Boundary Point Lemma

All of the above results investigate the behaviour of the function inside the considered interval. In this final section we complete the description by providing the information about the boundary points.

Lemma 15. *Assume that the functions g_1, g_2 and h_2 satisfy* (5), (6), (7), (10) *and* (12). *Moreover, assume that x is a nonconstant function satisfying*

$$\mathcal{L}_2[x] \geq 0,$$

on $[a,b]_{\mathbb{T}}$. If x attains a nonnegative maximum at a, then $x^{\Delta}(a) < 0$. If x attains a nonnegative maximum at $\sigma^2(b)$, then $x^{\Delta}(\rho(\sigma^2(b))) > 0$.

Proof. Assume that the result is false.

First assume that a is right-scattered. Then $x^{\sigma}(a) \geq x(a) = M$, which contradicts the statement of Theorem 9. Similarly if $\sigma(b)$ is right-scattered (or b is right-scattered and $\sigma(b)$ is right-dense) then $x^{\sigma}(b) \geq x(\sigma^2(b)) = M$ (or $x(b) \geq x(\sigma(b)) = M$) which again contradicts the statement of Theorem 9.

If a is right-dense, then we choose d close enough to a and define w and z by

$$z(t) = e_{\alpha}(t,a) - 1, \text{ and } w(t) = x(t) + \beta z(t),$$

where β satisfies

$$0 < \beta < \frac{x(a) - x(d)}{z(d)},$$

and $\alpha > 0$ is big enough to ensure that

$$\alpha^2 (1 + g_2\mu) + \alpha (g_1 + g_2) + h (1 + \mu\alpha - e_{\ominus\alpha}(t,a)) > 0. \tag{30}$$

Then $\mathcal{L}_2[z] > 0$ for $t \in (a,d]_{\mathbb{T}}$ and Lemma 8 and $w(d) < x(a) = w(a)$ imply that

$$0 \geq w^{\Delta}(a) = x^{\Delta}(a) + \beta z^{\Delta}(a).$$

Now since $z^{\Delta}(a) = \alpha > 0$ we obtain the desired result $x^{\Delta}(a) < 0$. The proof for the other end is similar. □

In the spirit of Remark 7, one can find in the special case weaker assumptions on the boundedness of considered functions. We sum up this reasoning in the following refinement of the above boundary point lemma.

Lemma 16. *Let g_1, g_2 and h_2 satisfy* (5), (6),(10) *and that*

$$\frac{g_1 + g_2}{1 + \mu g_2} \text{ and } \frac{h_2}{1 + \mu g_2}$$

are bounded on every closed subinterval of $(a,b)_{\mathbb{T}}$. *Let* x *be a nonconstant function satisfying*

$$\mathcal{L}_2[x] \geq 0,$$

on $(a,b)_{\mathbb{T}}$.

(i) *If* a *is right-dense and* x *has a nonnegative maximum at* a *and the function*

$$k_a = \frac{g_1 + g_2 + h_2(t+\mu-a)}{1+\mu g_2}$$

is bounded from below in a neighborhood of a, *then* $x^{\Delta}(a) < 0$.

(ii) *Similarly, if* b *is right- and left-dense and* x *has a nonnegative maximum at* b *and the function*

$$k_b = \frac{g_1 + g_2 + h_2(b-t-\mu)}{1+\mu g_2}$$

is bounded from above in a neighborhood of b, *then* $x^{\Delta}(b) > 0$.

Proof. We prove only the part (i). As in the previous case, assume the result is false. Since x is non constant we may choose d close enough to a so that $x < x(a)$ and k_a is bounded from below in $(a,d]_{\mathbb{T}}$. Then for $\alpha > 0$ and $t \in (a,d)_{\mathbb{T}}$ we have the following inequality

$$1 - e_{\ominus\alpha}(t,a) \leq -\ominus\alpha(t-a) \leq \alpha(t-a).$$

Thus, for a large $\alpha > 0$ the inequality (30) becomes (note that thanks (10) we have $h \leq 0$)

$$\begin{aligned}\alpha^2(1+g_2\mu) + \alpha(g_1+g_2) + h_2(1+\mu\alpha - e_{\ominus\alpha}(t,a)) \geq \\ \alpha^2(1+g_2\mu) + \alpha(g_1+g_2) + \alpha h_2(t+\mu-a) = \\ \alpha^2(1+g_2\mu) + \alpha(g_1+g_2+h_2(t+\mu-a)) > 0.\end{aligned}$$

And the argument follows that of Lemma 15. Similarly, we get the statement for the other case. □

Remark 17. We have the following variant of Lemma 15 where we weaken the assumptions on g_1, g_2 and h_2 while strengthening the assumptions on $\mathcal{L}_2$.

Assume that the functions g_1, g_2 satisfy (2), (3) while the function h_2 satisfies (10). Moreover, assume that x is a nonconstant function satisfying

$$\mathcal{L}_2[x] > 0,$$

on $[a,b]_{\mathbb{T}}$. If x attains a nonnegative maximum at a, then $x^{\Delta}(a) < 0$. If x has a nonnegative maximum at $\sigma^2(b)$, then $x^{\Delta}(\rho(\sigma^2(b))) > 0$.

As in the real line case (see Protter and Weinberger, [6]) a function x cannot have a point of horizontal inflection at $c \in (a,b)_{\mathbb{T}}$. In particular, there cannot exist $d < c < e \in (a,b)_{\mathbb{T}}$ such that x is strictly increasing on $[d,e]_{\mathbb{T}}$ while $x^{\Delta}(c) = 0$. Indeed if x has a point of inflection then x has maximum in $[c,e]_{\mathbb{T}}$ at c while $x^{\Delta}(c) = 0$, a contradiction.

References

[1] Martin Bohner and Allan Peterson. *Dynamic equations on time scales. An introduction with applications.* Birkhäuser Boston Inc., Boston, MA, 2001.

[2] Martin Bohner and Allan Peterson, editors. *Advances in dynamic equations on time scales.* Birkhäuser Boston Inc., Boston, MA, 2003.

[3] Pavel Drábek, Johnny Henderson, and Christopher C. Tisdell. Multiple solutions to dynamic equations on time scales. *Commun. Appl. Nonlinear Anal.*, 11(4):25–42, 2004.

[4] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[5] Jean Mawhin, Elliot Tonkes, and H.B. Thompson. Uniqueness for boundary value problems for second order finite difference equations. *J. Differ. Equations Appl.*, 10:749–757, 2004.

[6] Murray H. Protter and Hans F. Weinberger. *Maximum Principles in Differential Equations.* Prentice-Hall, New Jersey, 1967.

[7] Petr Stehlík. Periodic boundary value problems on time scales. *Adv. Difference Equ.*, 1:81–92, 2005.

[8] H.B. Thompson. Topological methods for some boundary value problems. *Comput. Math. Appl.*, 42:487–495, 2001.