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ON WEAK NON-LINEARITY OF MODELS OF PHYSICAL SYSTEMS

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1. INTRODUCTION TO THE PROBLEM

The call for penetrating the theory of non-linear technical and physical systems as deeply as possible has been more and more urgent lately. We distinguish non-linear systems whose very action is based on their non-linear properties and those which, though being non-linear, do not necessarily require the non-linearity for their function. The former include *systems with functional non-linearities*, the latter *systems with non-functional (parasitic) non-linearities*. For example, the telecommunications use various types of electric networks with functional non-linearities (e.g. rectifiers, oscillators, stabilizers, clipping networks, flip-flop circuits, clamping networks, frequency multipliers and divisors, AM demodulators, voltage and current limiters, parametric amplifiers, magnetic memory devices e.t.c.). The other group includes e.g. various devices of heavy current engineering with strongly saturated magnetic circuits (e.g. electric machines and apparatus). A theory of systems with non-functional non-linearities is usually established by formulating first a non-linear mathematical model. Using our experience or merely the intuitive imagination, we presume that the behaviour of the non-linear model is "reasonable" enough to allow linearization. This argument, quite common for instance in the theory of electric machines, is often fully justified and we do not commit significant errors by applying it. Nevertheless, it can be completely inadmissible and its application can lead to serious blunders in other cases, since generally the behaviour of the non-linear model is qualitatively different from that of the linear one.

In this paper we try to describe a class of non-linear models whose certain important properties coincide with the corresponding properties of the linearized models. We call such models *weakly non-linear* ones. When dealing with weakly non-linear models we make use of the above mentioned feature: we linearize them and, if the errors originating in this process are not negligible, suppress them by a suitable numerical method. It is a fundamental problem whether the physical system considered, which has a non-functional non-linearity, can be described by a weakly non-linear model. Obviously the character of the non-linearities of the physical system is decisive here, but frequently also the way in which we approximate the characteristics of its non-linear elements. This approximation should be done so as to express with a sufficient adequacy the properties of the physically real non-linear element, at the same time possessing such "suitable" properties which guarantee the model to be weakly non-linear. We shall give a mathematical definition of weakly non-linear models and deduce a criterion which enables us in many particular cases to decide whether the model may be considered weakly non-linear. One of effective methods of the analysis of weakly non-linear electric networks is suggested in [1].

2. DEFINITION OF A WEAKLY NON-LINEAR MODEL OF A PHYSICAL SYSTEM AND ITS PROPERTIES

Let us consider a model of a non-linear physical system with lumped parameters, for instance an electric network. Such a system is mathematically described by a system of n differential equations of the first order solved with respect to the derivative (e.g. by the method of state variables [2], [3], [4]). This system of equations together with the initial conditions is written as usual in the vector form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

where $\mathbf{x}(t) : E^1 \to E^n$ is the vector of responses while $\mathbf{f}(\mathbf{x}, t) : E^{n+1} \to E^n$ is the vector whose components contain the parameters of both the inputs and the passive elements.

Definition. *A model of the physical system described by the equation* (1) *is called weakly non-linear* (*quasilinear*) *if any two solutions* $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ *of the equation* (1) *satisfy*

$$\lim_{t\to\infty} \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\| = \mathbf{0}\,.^1) \tag{2}$$

Remark. Let us assume that the system considered has been already associated with a fixed model. Then the properties of the model can be transferred to the system itself and the notions "solution of the system" etc. can be used as is usual in applications and technology. The following properties, which are of great importance from the technical view-point, are immediately seen from the definition:

a) If a weakly non-linear system has at least one bounded solution [on the interval $\langle t_0, +\infty)$], then all its solutions are bounded. This property is in accordance with the well-known properties of asymptotically stable systems.

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[1]) The term "weakly non-linear" is introduced because of the terminology used in the technical practice. From the mathematical view-point we have here the global quasiasymptotical stability.

b) If, moreover, there exists a periodic solution of the equation (1), then, provided the system is weakly non-linear, this solution represents the "steady state" of the physical system, this steady state being independent on the choice of the initial conditions. For example, if a weakly non-linear system is excited by periodical inputs, then its responses in the steady state are periodic as well, any they are uniquely determined: the system is monostable. This means that weakly non-linear systems behave also in this respect in the same way as asymptotically stable linear systems.

It is evident that the introduction of the notion of weak non-linearity is motivated by the effort to determine and master a certain class of non-linear systems whose properties important in technical application coincide with those of asymptotically stable linear systems[2]).

The above described properties of a weakly non-linear model of a physical system may be evidently of advantage when investigating the behaviour of the physical system, particularly in its analysis or synthesis. Therefore it is clear that in particular cases we shall be interested in the problem whether the given model may be considered weakly non-linear.

Now we shall proceed to the formulation and then to the proof of a theorem which gives a sufficient condition for weak non-linearity. This theorem can be used as a criterion whether a certain model is weakly non-linear.

Theorem 1. *Let the right hand side of the equation* (1) *be expressed in the form*

$$f(x, t) = Ax + B(t)\,x + f_0(x, t) + g(t)\,, \tag{3}$$

where $A \in E^{n \times n}$ *is a constant matrix and there exist such positive constants* α, ν *that*

$$\|e^{At}\| \leqq \alpha e^{-\nu t} \tag{4}$$

(*we assume that the matrix norm* $\| \;\|$ *is induced by the vector norm chosen in* E^n),

$B(t) : E^1 \to E^{n \times n}$ *is a bounded continuous function on the interval* $\langle t_0, +\infty)$, *i.e. there exists a constant* $b > 0$ *such that*

$$\|B(t)\| \leqq b \quad for \quad t \geqq t_0\,, \tag{5}$$

$f_0(x, t) : E^{n+1} \to E^n$ *is continuous on* $E^n \times \langle t_0, +\infty)$.

Let there exist a function $a(t)$ *which is non-negative and continuous on* $\langle t_0, +\infty)$ *and such that*

$$\|f_0(x_1, t) - f_0(x_2, t)\| \leqq a(t)\,\|x_1 - x_2\| \tag{6}$$

[2]) Consequently, weakly non-linear systems are a certain generalization of asymptotically stable linear systems. Evidently, a linear system which is not asymptotically stable need not be weakly non-linear.

holds for all $t \geqq t_0$ *and* $\mathbf{x}_1, \mathbf{x}_2 \in E^n$,

$\mathbf{g}(t) : E^1 \to E^n$ *is continuous on the interval* $\langle t_0, +\infty)$.

Then to every initial condition $\mathbf{x}_0 \in E^n$ *there exists one and only one solution of the equation* (1) *on the interval* $\langle t_0, +\infty)$.

If, moreover, there exists $\varepsilon > 0$ *such that*

$$\int_{t_0}^{t} a(\tau)\, \mathrm{d}\tau < \left(\frac{v}{\alpha} - b - \varepsilon\right)(t - t_0) \tag{7}$$

for all $t > t_0$, *then the system described by the equation* (1) *is weakly non-linear.*

Remark. In particular, if $a(t) = a$ (= constant), then the condition (7) assumes the form

$$v < \alpha(a + b)\,. \tag{8}$$

Proof. The uniqueness and the local existence of a solution of the equation (1) is guaranteed by the Lipschitz condition (6). Let J be the right maximal interval of existence of solution. The function $\mathbf{f}_0(\mathbf{x}, t)$ is continuous on $E^n \times \langle t_0, +\infty)$ by the assumption and hence the case b) of the theorem on the prolongation of a solution cannot occur (see Appendix, Lemma 3).

Let us choose an arbitrary but fixed $\mathbf{x}^1 \in E^n$. Then (6) implies

$$\|\mathbf{f}_0(\mathbf{x}, t)\| - \|\mathbf{f}_0(\mathbf{x}^1, t)\| \leqq \|\mathbf{f}_0(\mathbf{x}, t) - \mathbf{f}_0(\mathbf{x}^1, t)\| \leqq$$
$$\leqq a(t)\,\|\mathbf{x} - \mathbf{x}^1\| \leqq a(t)\,\|\mathbf{x}\| + a(t)\,\|\mathbf{x}^1\|$$

for every $\mathbf{x} \in E^n$ and $t \geqq t_0$. Hence there exists a non-negative function $\vartheta(t)$ continuous on $\langle t_0, +\infty)$ and such that

$$\|\mathbf{f}_0(\mathbf{x}, t)\| \leqq a(t)\,\|\mathbf{x}\| + \vartheta(t)\,. \tag{9}$$

Let $\mathbf{x}(t)$ be a solution of the equation (1). Then the relation (13) (see Appendix, Lemma 1) together with (4), (5), (6) and (9) yields the estimate

$$\|\mathbf{x}(t)\|\, \mathrm{e}^{vt} \leqq \alpha\|\mathbf{x}_0\|\, \mathrm{e}^{vt_0} + \int_{t_0}^{t} \{\alpha[b + a(\tau)]\, \mathrm{e}^{v\tau}\|\mathbf{x}(\tau)\| +$$
$$+ \alpha\, \mathrm{e}^{v\tau}[\vartheta(\tau) + \|\mathbf{g}(\tau)\|]\}\, \mathrm{d}\tau$$

which again implies by virtue of the Gronwall lemma an inequality

$$\|\mathbf{x}(t)\| = K\mathrm{e}^{\xi(t)}$$

with K a positive constant and $\xi(t)$ a continuous function defined on the interval $\langle t_0, +\infty)$. This excludes also the case c) of Lemma 3 in Appendix and hence necessarily $J = \langle t_0, +\infty)$.

Now let us assume that the inequality (7) is satisfied in addition. Let us choose $\mathbf{x}_0^1, \mathbf{x}_0^2 \in E^n$, let $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ be solutions of the equation (1) on $\langle t_0, +\infty)$ such that $\mathbf{x}_i(t_0) = \mathbf{x}_0^i$, $i = 1, 2$. Put $\mathbf{x}_0 = \mathbf{x}_0^1 - \mathbf{x}_0^2$. Then in virtue of the equation (13), the function $\mathbf{x}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ satisfies

$$\|\mathbf{x}(t)\| \leqq \|e^{\mathbf{A}(t-t_0)}\| \cdot \|\mathbf{x}_0\| + \int_{t_0}^{t} \|e^{\mathbf{A}(t-\tau)}\| \{\|\mathbf{B}(\tau)\| \cdot \|\mathbf{x}(\tau)\| +$$

$$+ \|\mathbf{f}_0[\mathbf{x}_1(\tau), \tau] - \mathbf{f}_0[\mathbf{x}_2(\tau), \tau]\|\} \, d\tau .$$

This implies by means of (4), (5) and (6) the relation

$$\|\mathbf{x}(t)\| \, e^{\nu t} \leqq \alpha e^{\nu t_0} \|\mathbf{x}_0\| + \int_{t_0}^{t} \alpha[b + a(\tau)] \, \|\mathbf{x}(\tau)\| e^{\nu\tau} \, d\tau$$

and hence we obtain by the Gronwall lemma applied to

$$\varphi(t) = \|\mathbf{x}(t)\| \, e^{\nu t} , \quad \gamma = \alpha e^{\nu t_0} \|\mathbf{x}_0\| ,$$

$$\psi(t) = \alpha[b + a(t)] , \quad \Theta(t) = 0$$

the estimate

$$\|\mathbf{x}(t)\| \leqq \alpha \|\mathbf{x}_0\| e^{-\nu(t-t_0)} \exp\left\{\int_{t_0}^{t} \alpha[b + a(\tau)] \, d\tau\right\} ,$$

that is,

$$\|\mathbf{x}(t)\| \leqq \alpha \|\mathbf{x}_0\| \exp\left[(\alpha b - \nu)(t - t_0) + \alpha \int_{t_0}^{t} a(\tau) \, d\tau\right] .$$

By the inequality (7) this yields immediately

$$\|\mathbf{x}(t)\| \leqq \alpha \, \|\mathbf{x}_0\| \, e^{-\alpha\varepsilon(t-t_0)}$$

and, consequently,

$$\lim_{t\to\infty} \|\mathbf{x}(t)\| = 0 .$$

3. EXAMPLE OF INVESTIGATING THE WEAK NON-LINEARITY

Let us find whether the model of a physical system described by the system of differential equations

$$(10) \qquad \frac{dx_1}{dt} = -3x_1 - 2x_2 + \arctan x_2$$

$$\frac{dx_2}{dt} = \qquad - 2x_2$$

is weakly non-linear. The application of Theorem 1 depends essentially on the way in which we write the right-hand sides of the system (10) in the form (3). We shall show that this may affect in particular cases the validity of the condition (8).

I. Let us write the right-hand sides of the system (10) in the form (3) with

$$\mathbf{A} = \begin{bmatrix} -3; & -2 \\ 0; & -2 \end{bmatrix}, \quad \mathbf{f}_0(\mathbf{x}, t) = \begin{bmatrix} \arctan x_2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}(t) = \mathbf{0}, \quad \mathbf{g}(t) = \mathbf{0}.$$

If E^n is equipped with the norm

$$\|x_1, \ldots, x_n\| = \max |x_i| \quad (i = 1, \ldots, n) \tag{11}$$

and $E^{n \times n}$ with the norm induced by (11), then

$$\|e^{\mathbf{A}t}\| = \left\| \begin{bmatrix} e^{-3t}; & -2e^{-2t} + e^{-3t} \\ 0; & e^{-2t} \end{bmatrix} \right\| < 2e^{-2t}$$

and consequently, $\alpha = 2$, $\nu = 2$.

Further, for every $\mathbf{x}_1, \mathbf{x}_2 \in E^n$

$$\|\mathbf{f}_0(\mathbf{x}_1, t) - \mathbf{f}_0(\mathbf{x}_2, t)\| = |\arctan (\mathbf{x}_1)_2 - \arctan (\mathbf{x}_2)_2| \leqq$$

$$\leqq |(\mathbf{x}_1)_2 - (\mathbf{x}_2)_2| \leqq \|\mathbf{x}_1 - \mathbf{x}_2\|$$

and thus $a = 1$. Since $\mathbf{B}(t) = \mathbf{0}$, we have $b = 0$. Substituting these values into (8) we find the inequality does not hold (instead we have an identity).

If we replace the norm in E^n by

$$\|(x_1, \ldots, x_n)\| = \sum_{i=1} |x_i| \tag{12}$$

we obtain $\alpha = 3$, $\nu = 2$, $a = 1$ and these values again do not satisfy the inequality (8).

II. Let

$$\mathbf{A} = \begin{bmatrix} -3; & 0 \\ 0; & -2 \end{bmatrix}, \quad \mathbf{f}_0(\mathbf{x}, t) = \begin{bmatrix} -2x_2 + \arctan x_2 \\ 0 \end{bmatrix},$$

$$\mathbf{B}(t) = \mathbf{0}, \quad \mathbf{g}(t) = \mathbf{0}.$$

Then

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-3t}; & 0 \\ 0; & e^{-2t} \end{bmatrix}.$$

Both the norms (11) and (12) yield the same values: $\alpha = 1$, $\nu = 2$, $a = 2$; again the condition (8) is not satisfied.

III. Let

$$\boldsymbol{A} = \begin{bmatrix} -3; & -\frac{3}{2} \\ 0; & -2 \end{bmatrix}, \quad \boldsymbol{f}_0(\boldsymbol{x}, t) = \begin{bmatrix} -\frac{1}{2}x_2 + \arctan x_2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{B}(t) = \boldsymbol{0}, \quad \boldsymbol{g}(t) = \boldsymbol{0}.$$

(In this case the Lipschitz constant of the function $\boldsymbol{f}_0$ assumes its least value possible.) Then

$$e^{\boldsymbol{A}t} = \begin{bmatrix} e^{-3e}; & -\frac{3}{2}e^{-2t} + \frac{3}{2}e^{-3t} \\ 0\,; & e^{-2t} \end{bmatrix}.$$

Using the norm (11) we obtain

$$\|e^{\boldsymbol{A}t}\| \leqq \tfrac{3}{2}e^{-2t},$$

that is,

$$\alpha = \tfrac{3}{2}, \quad \nu = 2$$

and

$$\|\boldsymbol{f}_0(\boldsymbol{x}_1, t) - \boldsymbol{f}_0(\boldsymbol{x}_2, t)\| \leqq \tfrac{1}{2}\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$$

which yields $a = \frac{1}{2}$. The condition (8) is now satisfied.

Similarly, the norm (12) gives the values

$$\alpha = \tfrac{5}{2}, \quad \nu = 2, \quad a = \tfrac{1}{2}$$

which again satisfy the condition (8).

In the conclusion, let us point out once more that a model of the physical system described by the equation (1) is weakly non-linear if there exists a way of expressing the right-hand side $\boldsymbol{f}(\boldsymbol{x}, t)$ of this equation in the form (3) so that the inequality (8) is satisfied. The above example shows that even for weakly non-linear models not every way of expressing the right-hand side is suitable. It would be of interest to study this problem in more detail in order to obtain some results suggesting how to deal with particular cases.

4. NOTE TO THE APPROXIMATION OF NON-LINEAR CHARACTERISTICS OF THE ELEMENTS OF THE SYSTEM CONSIDERED

When formulating the equations of non-linear physical systems it is necessary to approximate the characteristics of their non-linear elements, which had been as a rule obtained by measurement, by a suitable analytic formula. For instance, when formulating the equations of an electric network which includes a coil with ferro-magnetic cores, it is necessary to approximate the dependence of the flux linkages Φ on the current i passing through the coil, that is, to find a suitable function $\Phi = \Phi(i)$ or its inverse function $i = i(\Phi)$. Here the graph of the function $\Phi = \Phi(i)$ reminds by its shape a magnetization curve. When investigating the weak non-linearity, the choice of the approximation formulae for non-linear characteristics must be done by applying both the general view-point (see e.g. [4]) and the requirement that the assumptions of Theorem 1 be fulfilled. For example, in the above-mentioned case of non-linear coils we approximate by a one-to one continuous function whose graph has the shape of a "magnetization curve", and which satisfies the Lipschitz condition with respect to Φ. For instance, we can use the function

$$i = a_1\Phi - a_2 \arctan(a_3\Phi)$$

($a_1 = \tan \alpha_1$, a_2, a_3 constants) which has the Lipschitz constant

$$a = \max(a_1, |a_1 - a_2a_3|) .$$

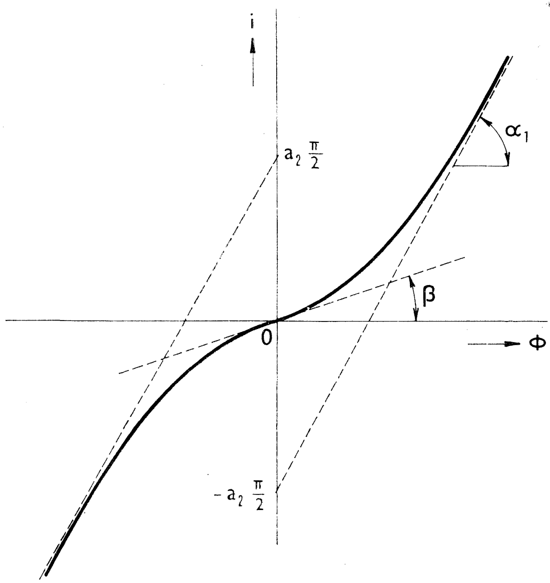

Fig. 1

Its asymptotes obey the equation

$$i = a_1\Phi \pm a_2 \frac{\pi}{2}$$

and its derivative at the origin is

$$\left.\frac{di}{d\Phi}\right|_{=0} = a_1 - a_2a_3 = \tan \beta \quad \text{(see Fig. 1).}$$

To determine the constants a_1, a_2, a_3 in a particular case, we use the equations of the asymptotes to obtain a_1, a_2 and the value of the derivative at the origin to find a_3.

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APPENDIX

Lemma 1. *Let the function* $\mathbf{x}(t)$ *in the equation* (1) *have a continuous first derivative; let the function* $\mathbf{f}(\mathbf{x}, t)$ *have the form of* (3). *Then the differential equation* (1) *is equivalent to the integral equation*

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\,\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\{\mathbf{B}(\tau)\,\mathbf{x}(\tau) + \mathbf{f}_0[\mathbf{x}(\tau), \tau] + \mathbf{g}(\tau)\}\,d\tau\,. \tag{13}$$

Proof is well-known.

Lemma 2. (*Gronwall*). *Let*

$$\varphi(t) \leqq \gamma + \int_{t_0}^{t} [\psi(\tau)\,\varphi(\tau) + \Theta(\tau)]\,d\tau \tag{14}$$

where $\varphi(\tau)$, $\psi(\tau)$ *and* $\Theta(\tau)$ *are non-negative continuous functions and* γ *is a positive constant. Then*

$$\varphi(t) \leqq \gamma \exp\left\{\int_{t_0}^{t}\left[\psi(\tau) + \frac{\Theta(\tau)}{\gamma}\right] d\tau\right\}. \tag{15}$$

Proof may be found e.g. in [3].

Lemma 3. (*Theorem on the prolongation of solutions*). *Let* $\mathbf{f}(\mathbf{x}, t)$ *be continuous on the closure* $\bar{E}$ *of an open set* $E \in E^{n+1}$ *and let the equation* (1) *have a solution* *on a right maximal interval J. Then there occurs precisely one of the following three possibilities:*

a) $J = \langle t_0, +\infty)$;

b) $J = \langle t_0, \delta)$, $\delta < +\infty$ and $[\delta, \mathbf{x}(\delta)] \in \partial E$;[1])

c) $J = \langle t_0, \delta)$, $\delta < +\infty$ and $\lim_{t \to \infty} \|\mathbf{x}(t)\| = +\infty$.

Proof may be found e.g. in [5].

[1]) The symbol ∂E denotes the boundary of a set E.

Souhrn

O SLABÉ NELINEARITĚ MODELŮ FYZIKÁLNÍCH SYSTÉMŮ

DANIEL MAYER, ZDENĚK RYJÁČEK

Zavedení pojmu slabé nelinearity je motivováno snahou vyčlenit jistou třídu nelineárních systémů, jejichž jisté, pro technické aplikace významné vlastnosti jsou shodné s vlastnostmi asymptoticky stabilních lineárních systémů. Model fyzikálního systému, popsaný rovnicí (1), nazýváme slabě nelineárním (kvazilineárním), jestliže pro každá dvě řešení $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$ rovnice (1) platí rovnice (2). Model fyzikálního systému popsaný rovnicí (1) je slabě nelineární, existuje-li pro pravou stranu $\mathbf{f}(\mathbf{x}, t)$ této rovnice takové vyjádření ve tvaru (3), aby byla splněna nerovnost (8). Při posuzování slabé nelinearity je třeba uplatňovat při volbě aproximačních formulí pro charakteristiky nelineárních prvků jednak všeobecná hlediska, jednak požadavek, aby tyto vztahy splňovaly předpoklady věty 1.

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