INVESTIGATION OF STEADY STATE OF PHYSICAL SYSTEMS WITH PERIODIC INPUTS

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1. SETTING THE PROBLEM

Let us consider a linear physical system (e.g. an electric network) with lumped parameters whose mathematical model assumes the form

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \tag{1}$$

where $\mathbf{A} = [a_{ij}]$ is a square matrix of a type (s, s) with constant elements which are determined by the parameters of the passive elements of the system; $\mathbf{x}(t) = {}^t[x_1(t), \dots, x_s(t)]$ is a matrix of the type $(s, 1)$, whose elements are the responses of the system, $\mathbf{f}(t) = {}^t[f_1(t), \dots, f_s(t)]$ is a matrix of type $(s, 1)$, whose elements are continuous periodic functions all with the same period T which are determined by the parameters of both passive and active elements of the system; $i, j = 1, \dots, s$. (A method of constructing this mathematical model for an electric network was suggested e.g. in [2] and [3]). Our aim is to find a periodic solution $\mathbf{x}_p(t)$ of the equation (1) which means – from the physical point of view – to find the responses of the system in its steady state.

A solution of the equation (1) which starts from general initial conditions $\mathbf{x}(0)$ includes some transient phenomena that damp sooner or later so that the solution eventually represents the response in the steady state. If the system is asymptotically stable (for details see [2] and [4]) and if it has sinusoidal inputs with the same frequency, it is easy to analyze its steady state by the method of phasor representation of sinusoidal functions. If the input is periodic but not sinusoidal we can retain the advantages of this method by developing the inputs into Fourier series, analyzing the system separately for each sinusoidal component of the input taking into account a "sufficient" number of the terms and then using the results to find the responses of the system in its steady state (see e.g. [2]). If we decide the number of sinusoidal components of the input taken into account only by the rate of convergence of its Fourier series, we may commit an uncontrollable error since the rate of convergence of the series representating the responses need not coincide with that of the input series. In particular, if the frequencies of some sinusoidal components of the input are close to the eigenfrequencies of the system then some resonance phenomena may appear so that the corresponding components of the responses are strongly developed even if the amplitudes of the input components are small. The method sugested above is suitable for application to such systems whose input differs only slightly from sinusoidal function. It is unsuitable for systems with a strongly non-sinusoidal input (for example for electric networks with diodes and thyristors) not only because of the uncontrollable error but also because of the necessity to consider a great number of sinusoidal terms which makes the calculation cumbersome and lengthy.

A simple idea which offers itself is to integrate the differential equation (1) for a "sufficiently" long period until the transient phenomena vanish and the system passes to the steady state. However, if the transient state lasts for a long time, for example some tens or even hundreds of periods, this method turns out to be not only uneconomical (regarding the necessary computer time) but also unreliable (due to the possibility of cumulation of errors).

2. METHOD OF ANALYSIS OF THE STEADY STATE OF A SYSTEM: THEORETICAL BACKGROUND

We seek a T-periodic solution of the equation (1) in the interval $\langle 0, \infty)$, T being the least positive number such that $\mathbf{x}(t) = \mathbf{x}(t + T)$ for all $t \geqq 0$. It was shown on [5] that this solution may be found on the basis of the following argument: It is well known that for any vector of initial conditions $\mathbf{x}_0$ there exists exactly one solution $\mathbf{x}(t)$ of the equation (1) such that $\mathbf{x}(0) = \mathbf{x}_0$. If $\mathbf{x}_p(t)$ is the required periodic solution then this assertion is obviously valid for it, too. Consequently, if we knew the vector of initial conditions $\mathbf{x}_{p0}$ corresponding to the solution $\mathbf{x}_p(t)$, i.e. satisfying $\mathbf{x}_p(0) = \mathbf{x}_{p0}$, then solving the equation (1) numerically by any known method of numerical integration of differential equations we determine one period of the function $\mathbf{x}_p(t)$; the full response of the system is its periodic continuation.

In [5] we have suggested a simple way of expressing the desired vector of initial values $\mathbf{x}_{p0}$ in the form of an infinite series. Now we shall show how the vector $\mathbf{x}_p(0)$ for the given mathematical model (1) of the physical system considered can be determined in a closed form.

It is well known from the theory of linear differential equations with constant coefficients that the general solution of the equation

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\,\mathbf{x}(t) \tag{2}$$

has the form

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0, \tag{3}$$

where $\mathbf{x}_0$ is an arbitrary constant vector. The general solution of the equation (1) is obtained by adding a particular solution of the system (1) to the general solution of the homogeneous system (2). One of these particular solutions is the function $\mathbf{x}_p(t)$. Thus the general solution of the system (1) may be written in the form

(4) $$\mathbf{x}(t) = \mathbf{x}_p(t) + e^{\mathbf{A}t}\mathbf{k},$$

where $\mathbf{k}$ is an arbitrary constant vector.

Let us consider the solution $\mathbf{x}^0(t)$ corresponding to the initial condition $\mathbf{x}^0(0) = \mathbf{0}$. According to the equation (4) this solution satisfies

(5) $$\text{for } t = 0: \quad \mathbf{x}^0(0) = \mathbf{x}_p(0) + e^{\mathbf{A}0}\mathbf{k} = \mathbf{0}$$

$$\text{for } t = T: \quad \mathbf{x}^0(T) = \mathbf{x}_p(T) + e^{\mathbf{A}T}\mathbf{k}$$

It follows from the periodicity of the function $\mathbf{x}_p(t)$ that

$$\mathbf{x}_p(0) = \mathbf{x}_p(T) = \mathbf{x}_{p0}$$

which together with the equations (5) yields

$$\mathbf{x}_{p0} + \mathbf{k} = \mathbf{0}$$

$$\mathbf{x}_{p0} + e^{\mathbf{A}T}\mathbf{k} = \mathbf{x}^0(T).$$

Further we obtain

$$\mathbf{x}_{p0} - e^{\mathbf{A}T}\mathbf{x}_{p0} = \mathbf{x}^0(T),$$

which means

6) $$(\mathbf{I} - e^{\mathbf{A}T})\,\mathbf{x}_{p0} = \mathbf{x}^0(T),$$

where $\mathbf{I}$ is the unit matrix of the type (s, s).

Let us remind that $\mathbf{x}^0(T)$ is the value at the moment $t = T$ of the function $\mathbf{x}^0(t)$ which is the solution of the equation (1) for the vector of initial values $\mathbf{x}_0 = \mathbf{0}$. This solution can be written in the form of a convolution integral

$$\mathbf{x}^0(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{f}(\tau)\,d\tau,$$

which yields in particular

$$\mathbf{x}^0(T) = \int_0^T e^{\mathbf{A}(T-\tau)}\,\mathbf{f}(\tau)\,d\tau.$$

Evaluating this integral either analytically or numerically we determine the right hand side of the relation (6). This relation then represents a system of a linear algebraic equations with s unknown components of the vector $\mathbf{x}_{p0}$. The matrix of this system is

(7) $$\mathbf{C} = \mathbf{I} - e^{\mathbf{A}T}.$$

This matrix can be viewed as a matrix of the form

$$\mathbf{C} = \lambda\mathbf{I} - \mathbf{B},$$

with

$$\mathbf{B} = e^{\mathbf{A}T} \quad \text{and} \quad \lambda = 1.$$

This means that the matrix $\mathbf{C}$, i.e. the matrix of the system (6), is regular if and only if the number $\lambda = 1$ is not a characteristic number of the matrix $\mathbf{B} = e^{\mathbf{A}T}$.

The following theorem holds (for its proof see e.g. [1]): Let φ be a function of a matrix. If λ is a characteristic number of a matrix $\mathbf{A}$ then $\varphi(\lambda)$ is a characteristic number of a matrix $\varphi(\mathbf{A})$.

Applying this theorem to the function $\varphi(\mathbf{A}) = e^{\mathbf{A}T}$ we conclude that if the matrix $\mathbf{A}$ (and hence also the matrix $\mathbf{A}T$) is singular, i.e. if zero is its characteristic number, then the matrix $e^{\mathbf{A}T}$ has a characteristic number $e^0 = 1$. However, since the matrices $\mathbf{A}$ and $e^{\mathbf{A}T}$ are of the same type (s, s) (and hence they have the same number of characteristic numbers) and since the function $\varphi(t) = e^t$ is strictly monotone, this assertion may be reversed. Thus if the matrix $\mathbf{A}$ is regular then zero is not its characteristic number and consequently, one is not a characteristic number of the matrix $e^{\mathbf{A}T}$. Therefore the matrix $\mathbf{C}$ is regular in virtue of the equation (7) and the system of algebraic equations (6) has exactly one solution $\mathbf{x}_{p0}$ for an arbitrary vector of the right hand sides $\mathbf{x}^0(T)$. In particular, the regularity of the matrix $\mathbf{A}$ is guaranteed if all characteristic numbers of the matrix $\mathbf{A}$ have negative real parts, i.e. if the physical system described by the system of differential equations (1) is asymptotically stable (for details see [4]).

3. ALGORITHM FOR THE CALCULATION OF THE VECTOR $\mathbf{x}_{p0}$

Let us now describe the way in which the analysis of the steady state of physical system described by the equation (1) proceeds.

1. We evaluate the matrix

(8) $$\mathbf{H}(t) = e^{\mathbf{A}t},$$

and, in particular, the matrix

(9) $$\mathbf{H}(T) = e^{\mathbf{A}T}.$$

A suitable method of determining the matrix $\mathbf{H}(t)$ based on the characteristic vectors of the matrix $\mathbf{A}$ and on the real Jordan form of the matrix $\mathbf{A}$, is presented in detail in paper [6].

2. We evaluate analytically or numerically the convolution itnegral

(10) $$\mathbf{x}^0(T) = \int_0^T e^{\mathbf{A}(T-\tau)}\,\mathbf{f}(\tau)\,d\tau.$$

3. By solving the system of algebraic equations

$$(\mathbf{I} - e^{\mathbf{A}T})\,\mathbf{x}_{p0} = \mathbf{x}^0(T) \tag{11}$$

we find the vector of initial values $\mathbf{x}_{p0}$.

4. We integrate numerically the differential equation (1) for the vector of initial values $\mathbf{x}_{p0}$ just found, using any suitable method, on the interval $t \in \langle 0, T \rangle$. The realation

$$\mathbf{x}_{p0} = \mathbf{x}(0) = \mathbf{x}(T) = \mathbf{x}(2T) = \mathbf{x}(3T) = \ldots \tag{12}$$

can be used to check the result.

4. NUMERICAL EXAMPLE

As an illustration of the process of computation let us determine the vector of initial values $\mathbf{x}_{p0}$ for steady state of an electric network whose equation has the form (see [5] p. 292)

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4{\cdot}3636\,.\,10^2; & -3{\cdot}6363\,.\,10^3 \\ 1{\cdot}8181; & -1{\cdot}8181\,.\,10^1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 400\,u_0(t) \\ 0 \end{bmatrix}, \tag{*}$$

where $u_0(t) = 100 \cos \omega t + 60 \cos 3\omega t + 30 \cos 5\omega t$; $\omega = 100\pi$ i.e. $T = 0{\cdot}02$ s.

1. We use the algorithm described in [6] to determine the matrix

$$\mathbf{H}(t) = e^{\mathbf{A}t} = \mathbf{D}_1 e^{\lambda_1 t} + \mathbf{D}_2 e^{\lambda_2 t},$$

where

$$\lambda_1 = -4{\cdot}199\,029\,.\,10^2, \quad \lambda_2 = -3{\cdot}463\,805\,.\,10^1,$$

$$\mathbf{D}_1 = \begin{bmatrix} 1{\cdot}042\,716\,.\,10^0; & 9{\cdot}438\,447\,.\,10^0 \\ -4{\cdot}719\,091\,.\,10^{-3}; & -4{\cdot}271\,622\,.\,10^{-2} \end{bmatrix},$$

$$\mathbf{D}_2 = \begin{bmatrix} -4{\cdot}271\,622\,.\,10^{-2}; & -9{\cdot}438\,447\,.\,10^0 \\ 4{\cdot}719\,091\,.\,10^{-3}; & 1{\cdot}042\,716\,.\,10^0 \end{bmatrix}.$$

2. The convolution integral (10) assumes the form

$$\int_0^T ([\mathbf{D}_1]\, e^{\lambda_1(T-\tau)} + [\mathbf{D}_2]\, e^{\lambda_2(T-\tau)})\,.\,400\,.$$

$$.\left(\begin{bmatrix} 100 \cos \omega\tau \\ 0 \end{bmatrix} + \begin{bmatrix} 60 \cos 3\omega\tau \\ 0 \end{bmatrix} + \begin{bmatrix} 30 \cos 5\omega\tau \\ 0 \end{bmatrix}\right) \mathrm{d}\tau = \mathbf{x}^0(T)\,.$$

This integral is relatively simple, hence we decide to evaluate it analytically. This leads to integrals of the type

$$c \int_0^T e^{a(T-\tau)} \cos b\tau \, \mathrm{d}\tau = \frac{ac}{a^2 + b^2}\,(e^{aT} - 1)\,,$$

where a, b, c are real constants. By integration we find

$$\mathbf{x}^0(T) = \begin{bmatrix} 7{\cdot}519\,045\,.\,10^1 \\ -3{\cdot}063\,643\,.\,10^{-1} \end{bmatrix}.$$

3. The system of algebraic equations (11) has the form

$$\begin{bmatrix} 1{\cdot}021\,131\,.\,10^0; & 4{\cdot}718\,919\,.\,10^0 \\ -2{\cdot}359\,394\,.\,10^{-3}; & 4{\cdot}784\,503\,.\,10^{-1} \end{bmatrix} . \begin{bmatrix} x_{p0_1} \\ x_{p0_2} \end{bmatrix} = \begin{bmatrix} 7{\cdot}519\,045\,.\,10^1 \\ -3{\cdot}063\,643\,.\,10^{-1} \end{bmatrix}.$$

Solving the system we find the vector of initial values

$$\mathbf{x}_{p0} = \begin{bmatrix} x_{p0_1} \\ x_{p0_2} \end{bmatrix} = \begin{bmatrix} 7{\cdot}488\,700\,.\,10^1 \\ -2{\cdot}710\,342\,.\,10^{-1} \end{bmatrix}.$$

This value is in good agreement with the result obtained in [5] by an approximative method.

The next step, i.e. a numerical integration of the equation (*) for $t \in \langle 0, 0{\cdot}02 \rangle$ yields practically the same results as in the quoted paper [5].

5. CONCLUSION

In the paper we have formulated a simple and well programmable algorithm of analysis of the steady state of a linear physical system with non-sinusoidal periodic input which is described by a differential equation (1).

Naturally, there are other methods of finding the periodic solution of the equation (1), for example the well known and seemingly advantageous method based on the use of Laplace transformation. This method uses the Laplace transforms of all components of the vector $\mathbf{f}(t)$. However, in many applications the time behavior of the inputs is relatively complicated and finding their Laplace transforms would be very difficult. This is the case for example with electric networks including thyristors. The effort to find a suitable method for analysis of the steady state of this type of electric networks served as a motivation for writing this paper as well as the previous one [5]. A number of calculated examples has shown both the methods to be very effective.

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In the present paper a method of evaluation of the vector of initial conditions $\mathbf{x}_{p0}$ for which the integration of the differential equation (1) yields its periodic solution, is suggested. The method makes it possible to find the vector $\mathbf{x}_{p0}$ in a closed form. In order to find the vector of initial conditions $\mathbf{x}_{p0}$ it is necessary to determine the matrix $e^{\mathbf{A}t}$; an algorithm for this is given in [6].

The described method of analysis of linear physical systems in the steady state is well programmable for digital computers. Its application is illustrated by a simple numerical example. This method as well as its alternative form [5] is very effective especially in the case of complicated time behavior of the input of the system, as e.g. for electric networks with thyristors.

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