ON ASYMPTOTIC STABILITY OF LINEAR CIRCUITS WITH LUMPED PARAMETERS

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The criterion of asymptotic stability of a linear electric circuit which has been known till now requires each loop of the circuit to contain a resistor. This criterion represents a sufficient condition, however, not a necessary one. In the present paper a more general criterion of asymptotic stability for circuits which contain passive elements and independent voltage sources is formulated. This criterion admits that some loops contain no resistor. The proof of the assertion is sketched.

1. MOTIVATION OF THE PROBLEM

In order that a circuit may be a sufficiently adequate model of a real apparatus, we usually require it to be asymptotically stable, which means that each pair of its current or voltage responses $x_k(t)$ and $x_k^*(t)$ which correspond to different initial conditions satisfies

$$\lim_{t \to \infty} |x_k(t) - x_k^*(t)| = 0\,, \quad k = 1, 2, \ldots, r \tag{1}$$

where r stands for the number of branches (edges) of the circuit. If all branch responses in the equation (1) are voltages we shall speak about v-asymptotic stability of the circuit; if all branch responses are currents, then we shall speak about c-asymptotic stability of the circuit and finally, if the responses are voltages of some branches and currents of the others, we have the mixed asymptotic stability of the circuit.

It is well known that a mathematical model of a circuit with lumped parameters is given by a system of ordinary differential equations; a necessary and sufficient condition of its asymptotic stability is that all of its characteristic values have negative real parts. However, an application of this condition is computationally demanding and laborious: we should prefer if the asymptotic stability could be determined as simply as possible, without any computation, immediately from the circuit.

The following sufficient conditions were established (see e.g. [1], [3]):

(i) *If a circuit includes at least one tree such that each of its links contains a resistor (i.e., ech loop of the circuit is incident with at least one branch containing a resistor), then the circuit is c-asymptotically stable.*

(ii) *If a circuit includes at least one tree such that each of its branches contains a resistor (i.e. each cut set of the circuit is incident with at least one branch containing a resistor), then the circuit is v-asymptotically stable.*

However, the assumptions of these conditions of asymptotic stability are too restrictive. For example, the circuit in Fig. 1a does not satisfy the condition (i) while the circuit in Fig. 1b does not satisfy the condition (ii). Nevertheless, it can be easily seen that the former is c-asymptotically stable while the latter is v-asymptotically stable. Therefore, we will introduce more general conditions of asymptotic stability.

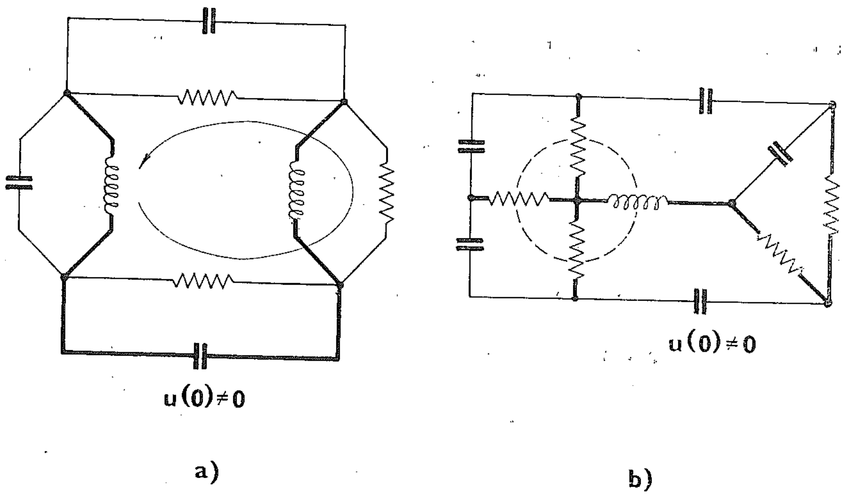

Fig. 1.

2. FORMULATION OF CONDITIONS OF ASYMPTOTIC STABILITY AND THEIR APPLICATION

We shall consider only circuits including passive elements and independent voltage sources.

Theorem 1. *A sufficient condition for a circuit to be c-asymptotically stable is that there exists a tree whose all links satisfy the following condition: If the link contains no resistor then it contains a capacitor and there exists a loop for which the capacitor is the single non-resistance element.*

We omit the proof of Theorem 1; we shall sketch later the proof of Theorem 3 which is a generalisation of Theorem 1.

Theorem 2. *A sufficient condition for a circuit to be v-asymptotically stable is that there exists a tree whose all branches satisfy the following condition: If the branch contains no resistor then it contains an inductor and there exists a cut set for which this inductor is the single non-resistance element.*

The proof of Theorem 2 is dual to that of Theorem 1.

As examples let us introduce the circuits in Figs. 2a and 2b which are (according to Theorems 1 and 2) c-asymptotically stable and v-asymptotically stable, respectively.

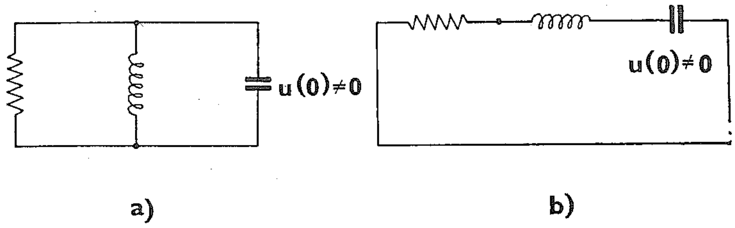

Fig. 2.

Theorems 1 and 2, which are evidently generalizations of the above mentioned conditions, may be further generalized as follows. A generalization of Theorem 1 is given in

Theorem 3. *A sufficient condition for a circuit to be c-asymptotically stable is that the following conditions be fulfilled:*

(i) *Each cycle of the circuit contains a resistor or a capacitor.*

(ii) *Each cycle of the circuit contains a resistor or an inductor.*

(iii) *If a cycle C_1 of the circuit contains no resistor, then there exists a cycle C_2 such that*

– *the cycles C_1 and C_2 have a common capacitor while*

– *they have no common inductor.*

These conditions of c-asymptotical stability may be formulated mathematically as follows (the concepts used are defined precisely e.g. in [1], [2], [4]): *A sufficient condition for the c-asymptotical stability of a circuit is that each cycle $\mathbf{c}^T\mathbf{v}$ of the circuit satisfies*

(i) $\mathbf{c}^T(\mathbf{R} + \mathbf{S})\,\mathbf{c} > 0$ *and simultaneously*

(ii) $\mathbf{c}^T(\mathbf{R} + \mathbf{L})\,\mathbf{c} > 0$,

(iii) *if* $\mathbf{c}^T\mathbf{R}\mathbf{c} = 0$ *then there exists a cycle* $\tilde{\mathbf{c}}^T\mathbf{v}$ *such that*

$$\tilde{\mathbf{c}}^T\mathbf{S}\mathbf{c} \neq 0 \quad \text{and} \quad \tilde{\mathbf{c}}^T\mathbf{L}\mathbf{c} = 0$$

where $\mathbf{v}$ is the vector of all branches of the graph of the circuit

$$\mathbf{v} = [v_1, v_2, \ldots, v_r]^T$$

$v_1, v_2, \ldots, v_r$ are the branches of the circuit, $\mathbf{c}$ is a vector of real numbers of the type $(r, 1)$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{L}$ are real square matrices of resistances, elastances (i.e., inverse values of capacitances) and inductances; the matrices $\mathbf{R}$ and $\mathbf{S}$ are diagonal ones. Within the class of circuits considered in the present paper, the matrices $\mathbf{R}$, $\mathbf{L}$, $\mathbf{S}$ are positive semidefinite.

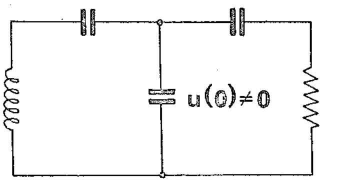

Fig. 3.

We could have formulated a sufficient condition of v-asymptotical stability by using duality; however, it is not our intention to do so in the present paper.

As an example, let us consider the circuit in Fig. 3: It does not satisfy the assumptions of Theorem 1, nevertheless, it satisfies the assumptions of Theorem 3 and hence it is c-asymptotically stable.

3. SKETCH OF PROOF OF THEOREM 3

Let us consider a linear circuit with a vector of voltages of sources $\mathbf{e}(t)$ and two its current solutions $\mathbf{i}_1(t)$, $\mathbf{i}_2(t)$ corresponding to two different initial conditions. Then the difference $\mathbf{i}_1(t) - \mathbf{i}_2(t)$ is obviously the solution of the circuit for $\mathbf{e}(t) = \mathbf{O}$ and

$$\lim_{t \to \infty} [\mathbf{i}_1(t) - \mathbf{i}_2(t)] = \mathbf{O}$$

holds if and only if its Laplace Transform $\mathbf{i}_1(p) - \mathbf{i}_2(p)$ has no poles in $\overline{\mathfrak{G}}$, where $\overline{\mathfrak{G}}$ stands for the closure of the set $\mathfrak{G}$ of all complex numbers with positive real parts. Consequently, the circuit is c-asymptotically stable provided that for each pair of real vectors of initial conditions $\mathbf{i}_0 = \mathbf{i}(O)$, $\mathbf{q}_0 = \mathbf{q}(O)$ there is a solution $\mathbf{i}$ corresponding to the vector $\mathbf{e} = \mathbf{O}$ and the Laplace Transform of this solution has no poles in $\overline{\mathfrak{G}}$.

It is known that under the assumptions of Theorem 3 the Laplace Transform of the solution $\mathbf{i}(p)$ exists and has the form

$$\mathbf{i}(p) = \mathbf{A}(p)\,(\mathbf{L}\mathbf{i}_0 - \mathbf{S}\mathbf{q}_0 p^{-1}), \tag{2}$$

where

$$\mathbf{A}(p) = \mathbf{X}\mathbf{W}^{-1}(p)\,\mathbf{X}^T = \mathbf{X}[\mathbf{X}^T(\mathbf{L}p + \mathbf{R} + \mathbf{S}p^{-1})\,\mathbf{X}]^{-1}\,\mathbf{X}^T$$

and $\mathbf{X}$ is a constant matrix whose columns form a basis of the space of all solutions of the equation

$$\mathbf{B}^T\mathbf{x} = \mathbf{O}, \tag{3}$$

where $\mathbf{B}$ is the node-branch incidence matrix of the graph of the circuit. It can be easily proved that the vector $\mathbf{x}$ is a solution of Eq. (3) if and only if $\mathbf{x}^T\mathbf{v}$ is a cycle.

The results from [2] imply that under our assumptions the real part of the matrix W^{-1} is positive definite in $\mathfrak{G}$, and hence W^{-1} has no poles in $\mathfrak{G}$; nevertheless, it might possibly have simple poles on the boundary of $\mathfrak{G}$ (i.e. some of its elements might have simple poles on the boundary of $\mathfrak{G}$).

a) We shall prove that the matrix W^{-1} has no poles on the imaginary axis except O and ∞. To this end, it suffices to prove that $\det W \neq O$ on the imaginary axis. Let there exist, to the contrary, a real $\omega_0 \neq O$ such that $\det W(j\omega_0) = O$. Then there exists a non-zero real vector w such that

$$W(j\omega_0) w = \mathbf{O} \tag{4}$$

and hence we have the following identity for the non-zero cycle $\mathbf{c}^T \mathbf{v}$ with $\mathbf{c} = Xw$:

$$\mathbf{c}^T R \mathbf{c} + j\left(\omega_0 \mathbf{c}^T L \mathbf{c} - \frac{1}{\omega_0} \mathbf{c}^T S \mathbf{c}\right) = O\,.$$

Hence $\mathbf{c}^T R \mathbf{c} = O$. According to the assumption (iii) of Theorem 3 there exists a cycle $\tilde{\mathbf{c}}^T \mathbf{v}$ such that $\tilde{\mathbf{c}}^T S \mathbf{c} \neq O$ and $\tilde{\mathbf{c}}^T L \mathbf{c} = O$. Thus according to the definition of X there exists a non-zero vector $\tilde{w}$ such that $\tilde{\mathbf{c}} = X\tilde{w}$. The relation (4) then implies $\tilde{w}^T W(j\omega_0) w = O$ and consequently

$$\omega_0^2 \tilde{\mathbf{c}}^T L \mathbf{c} = \tilde{\mathbf{c}}^T S \mathbf{c}$$

which is a contradiction.

α) Let O be a pole of the matrix W^{-1}. Then there are matrices H, K such that $W^{-1}(p) = Hp^{-1} + K(p)$; here H is a constant matrix and O is not a pole of the matrix $K(p)$. (These matrices may be obtained by decomposing the matrix W^{-1} into partial fractions). At the same time we have

$$W(p) = X^T S X p^{-1} + X^T (R + Lp) X\,.$$

The identity $I = WW^{-1}$ implies

$$X^T S X H = H^T X^T S X = \mathbf{O}\,.$$

Multiplying by p and passing to the limit $p \to O$ we obtain

$$X^T S X K_0 + X^T R X H = \mathbf{O}$$

where $K_0 = \lim_{p \to 0} K(p)$. These relations yield

$$H X^T (R + S) X H = \mathbf{O}\,.$$

Let the matrix H have a non-zero column h. Then it holds $\mathbf{c}^T (R + S)\, \mathbf{c} = O$ for a non-zero cycle $\mathbf{c}^T \mathbf{v} = (Xh)^T \mathbf{v}$ which contradicts the original assumption.

β) Let ∞ be a pole of the matrix W^{-1}. Then the relations

$$W^{-1}(p) = Hp + K(p)$$

and

$$W(p) = X^T L X p + X^T (R + S p^{-1}) X$$

yield similarly as above the identity

$$H^T X^T (R + L) X H = O$$

which again contradicts the original assumption.

b) We have proved the matrix W^{-1} has no poles in $\overline{\mathfrak{G}}$. In virtue of the relation (2), the only pole of $i(p)$ in $\overline{\mathfrak{G}}$ might be still O. Let O be a pole of the matrix $W^{-1}(p) X^T S p^{-1}$. Then there are matrices H, K such that

$$W^{-1} X^T S p^{-1} = H p^{-1} + K(p)$$

where H is constant and O is not a pole of $K(p)$, i.e. there exists $\lim_{p \to 0} K(p) = K_0$. This implies

$$X^T S p^{-1} = X^T L X K(p)\, p + X^T L X H + X^T R X K + \\ + (X^T R X H + X^T S X K)\, p^{-1} + X^T S X H p^{-2}\,. \tag{5}$$

This yields $X^T S X H = O$ and hence also

$$H^T X^T S X H = O\,. \tag{6}$$

By multiplying (5) by p, passing to the limit $p \to O$ and using the relation (6) we obtain

$$H^T X^T S = H^T X^T R X H\,. \tag{7}$$

Let the matrix H have a non-zero j-th column h. Then the identity

$$c^T S c = O \tag{8}$$

holds for a non-zero cycle $c^T v = (Xh)^T v$ in virtue of (6) and thus according to the assumption $c^T R c > O$. The matrix $H^T X^T R X H$ contains thus a non-zero element in its j-th row and j-th column. As the matrix S is diagonal and positive semidefinite, the relation (8) yields $c^T S = O$. Hence the j-th row of the matrix $H^T X^T S$ is zero which contradicts (7). This completes the proof that the matrix $W^{-1}(p)\, X^T S p^{-1}$ has no poles in $\overline{\mathfrak{G}}$, and in virtue of (2) the same holds for the matrix $i(p)$.

Remark. If the matrix L is assumed to be diagonal, i.e. the circuit considered has no magnetic couplings, it is easy to see that the asymptotic stability is guaranteed provided the assumptions of Theorem 3 are fulfilled for each loop of the circuit.

REFERENCES

[1] **Doležal, V. and Vorel, Z.**: Theory of Kirchhoff's Networks. Čas. pěst. mat., **87** (1962), No. 4, pp. 440—476.

[2] **Ryjáček, Z.**: On asymptotic stability of passive linear electrical networks. Aplikace matematiky, **24** (1979), No. 1, pp. 48—55.

[3] Knichal, Vl.: On Kirchhoff's Laws. Matem.-fyz. sborník Slov. Akad. vied a umení, 2 (1952), No. 3—4, pp. 13—27 (Czech).
[4] Čulík, K., Doležal, V. and Fiedler, M.: Combinatorial analysis in practice. SNTL, Prague 1967 (Czech).
[5] Mayer, D.: An introduction to the theory of electric circuits. SNTL, Prague 1978 (Czech).
[6] Mayer, D. and Ryjáček, Z.: A contribution to the modelling of electric systems by electric networks. Elektrotechnický časopis, 29 (1978), No. 1, pp. 4—20 (Czech).

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The introduction of the paper recalls the reasons why an electric circuit is required to be asymptotically stable and mentions sufficient conditions for asymptotic stability of linear circuits with lumped parameters witch are known from the literature. However, these conditions are too restrictive and thus in many cases inapplicable. The aim of the paper i sto established weaker conditions: more general sufficient conditions are presented, both for the current asymptotic stability and voltage asymptotic stability. Further, another sufficient condition in introduced, with is the most general one for the present. A sketch of its proof closes the paper.

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