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LOCALLY DISCONNECTED GRAPHS WITH LARGE NUMBERS OF EDGES

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Let G be a finite undirected graph, let v be its vertex. By the symbol $N_G(v)$ we denote the subgraph of G induced by the set of vertices which are adjacent to v; the graph $N_G(v)$ is called the neighbourhood graph of v in G.

If $N_G(v)$ is disconnected for each vertex v of G, the graph G is called locally disconnected [1].

At the Czechoslovak Conference on Graph Theory in Luhačovice in 1985 the second author has proposed the problem of finding the maximum number of edges of a locally disconnected graph with n vertices. In [1] it was shown that this number cannot be expressed as a linear function of n. Probably it could be expressed as a quadratic function of n, because so can the number of edges of a complete graph with n vertices.

In this paper we shall not find this maximum number, we shall only show that its asymptotical behaviour is the same as that of the number of edges of a complete graph with n vertices.

Theorem 1. *Let n be a square of an integer, $n \geqq 4$. Then there exists a locally disconnected graph with n vertices and* $\frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} + 3n - 2\sqrt{n}$ *edges.*

Proof. For $n = 4$ such a graph is a circuit of the length 4. Now let $n \geqq 9$. The vertex set of the required graph G consists of the vertices $u(i, j)$, where $1 \leqq i \leqq \sqrt{n}$, $1 \leqq j \leqq \sqrt{n}$. Two vertices $u(i_1, j_1)$, $u(i_2, j_2)$ are adjacent if and only if some of the following conditions is fulfilled:

(i) $i_1 \neq i_2, j_1 = j_2$;

(ii) $i_1 = i_2, j_1 \neq j_2, \min\{j_1, j_2\} = 1$;

(iii) $i_1 \neq i_2, j_1 \neq 1, j_2 \neq 1, j_1 \neq j_2$.

Evidently the number of pairs of vertices fulfilling (i) is $\sqrt{n}\binom{\sqrt{n}}{2}$, the number of pairs of vertices fulfilling (ii) is $\sqrt{n}(\sqrt{n} - 1)$ and the number of pairs fulfilling (iii) is $\binom{\sqrt{n}-1}{2}\sqrt{n}(\sqrt{n} - 1)$. By adding these three expressions we obtain $\frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} + 3n - 2\sqrt{n}$.

Now we shall investigate the graphs $N_G(u(i_0, j_0))$, where $1 \leqq i_0 \leqq \sqrt{n}$, $1 \leqq j_0 \leqq \sqrt{n}$. Fist suppose $j_0 = 1$. Then the vertex set of $N_G(u(i_0, j_0))$ is the union of disjoint sets $M_1 = \{u(i, j) | i \neq i_0 \quad j = 1\}$ and $M_2 = \{u(i, j) | i = i_0, \quad j \neq 1\}$. No vertex of M_1 is adjacent to a vertex of M_2 and both M_1, M_2 are non-empty, therefore $N_G(u(i_0, j_0))$ is disconnected. Now suppose $j_0 \neq 1$. Then the vertex set of $N_G(u(i_0, j_0))$ is the union of disjoint sets $M_3 = \{u(i, j) | i \neq i_0, \quad j = j_0\}$, $M_4 = \{u(i, j) | i \neq i_0 \quad j \neq 1\}$ and $M_5 = \{u(i_0, 1)\}$. The unique vertex $u(i_0, 1)$ of M_5 is adjacent to no vertex of $M_3 \cup M_4$, therefore $N_G(u(i_0, j_0))$ is again disconnected. The graph G is locally disconnected.

Note that

$$\lim_{n \to \infty} \left(\frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} + 3n - 2\sqrt{n}\right) \Big/ \left(\frac{1}{2}n^2 - \frac{1}{2}n\right) = 1.$$

The numerator of this fraction is the number from Theorem 1 and the denominator is the number of edges of a complete graph with n vertices. We see that a locally disconnected graph can have a number of edges which can be expressed by a function of n which behaves asymptotically the same as the number of edges of a complete graph with n vertices, i.e. the maximum number of edges of a graph with n vertices and without loops and multiple edges. We shall extend this result to the case when n is an arbitrary integer.

Theorem 2. *There exists a function $t(n)$ defined on the set of all positive integers with the following properties:*

(a) $\lim_{n \to \infty} t(n) \Big/ \left(\frac{1}{2}n^2 - \frac{1}{2}n\right) = 1;$

(b) *for each integer $n \geqq 4$ there exists a locally disconnected graph G with n vertices and $t(n)$ edges.*

Proof. Let n be an integer, $n \geqq 36$. By p we denote the upper integral part of $\sqrt{n}$, i.e. the least integer which is greater than or equal to $\sqrt{n}$. We construct a graph G. The vertex set V of G will be the union of pairwise disjoint sets $V_1, \ldots, V_p$. As $n \geqq 36$ and obviously $p \leqq \sqrt{n} + 1$, the inequalities $\frac{1}{2}p(p + 3) \leqq \frac{1}{2}(\sqrt{n} + 1)(\sqrt{n} + 4) \leqq n$ hold, which (together with $n \leqq p^2$) implies the existence of the integers $r_1, \ldots, r_p$ satisfying the conditions $r_1 = r_2 = r_3 = p$, $\frac{1}{2}p \leqq r_j \leqq p$ for $j = 4, \ldots, p$, $\sum_{j=1}^{p} r_j = n$. In G there is $|V_j| = r_j$ for $j = 1, \ldots, p$. The vertices of each V_j are denoted by $u(i, j)$ for $i = 1, \ldots, r_j$. Two vertices $u(i_1, j_1)$, $u(i_2, j_2)$ are adjacent if and only if some of the conditions (i), (ii), (iii) from the proof of Theorem 1 is fulfilled. Analogously to the proof of Theorem 1 we can prove that G is locally disconnected. We shall compute the number of edges of G. We start with the number of edges of the subgraph G_0 of G induced by the set $V - V_1$. We may consider G_0 as the graph obtained from a complete graph on $n - p$ vertices by deleting edges of p pairwise disjoint complete graphs, each of which has at most $p - 1$ vertices. Hence G_0 has at least $\frac{1}{2}(n-p)(n-p-1) - \frac{1}{2}p(p-1)(p-2)$ edges. As $\sqrt{n} \leqq p < \sqrt{n} + 1$, this number is greater than or equal to $\frac{1}{2}(n - \sqrt{n} - 1)(n - \sqrt{n} - 2) - \frac{1}{2}\sqrt{n}(\sqrt{n}+1)(\sqrt{n}-1) = \frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} - n + 2\sqrt{n} + 1$. Further the subgraph of G induced by V_1 is complete, therefore it has $\frac{1}{2}p(p-1)$ edges; this number is greater than or equal to $\frac{1}{2}\sqrt{n}(\sqrt{n}-1)$. The number of edges joining the vertices of V_1 with vertices of G_0 is at least $2p + \frac{1}{2}p(p-3) \geqq \frac{1}{2}n + \frac{1}{2}\sqrt{n}$. The whole graph G has at least $\frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} + 2\sqrt{n} + 1$ edges. By $t(n)$ for $n \geqq 36$ we denote the maximum number of edges of a graph G thus described; for n such that $4 \leqq n \leqq 35$ we may put $t(n) = n$, because every circuit of the length at least 4 is a locally disconnected graph. Thus for $n \geqq 36$ we have $t(n) \geqq \frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} + 2\sqrt{n} + 1$ and obviously $t(n) \leqq \frac{1}{2} - \frac{1}{2}n$, which is the number of edges of a complete graph with n vertices. As

$$\lim_{n\to\infty}\left(\frac{1}{2}n^2 - \frac{3}{2}n\sqrt{n} + 2\sqrt{n} + 1\right)\Big/\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) = 1,$$

we have also

$$\lim_{n\to\infty} t(n)/\left(\frac{1}{2}n^2 - \frac{1}{2}n\right) = 1.$$

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ЛОКАЛЬНО НЕСВЯЗНЫЕ ГРАФЫ С БОЛЬШИМИ ЧИСЛАМИ РЕБЕР

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Резюме

Символом $N_G(v)$ обозначается подграф графа G, порожденный множеством вершин, смежных с v. Если $N_G(v)$ несвязен для всех вершин v, граф G называется локально несвязным. Доказано, что максимальное число ребер локально несвязного графа с n вершинами имеет то же асимптотическое поведение, как и число ребер полного графа с n вершинами.