Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 112 (1987), No. 4, 390–394

Persistent URL: http://dml.cz/dmlcz/108555

GRAPHS WITH NON-ISOMORPHIC VERTEX NEIGHBOURHOODS OF THE FIRST AND SECOND TYPES

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(Received August 8, 1985)

Summary. The paper is devoted to the relation between the classes $\mathfrak{G}_1$, $\mathfrak{G}_2$ of graphs with non-isomorphic vertex neighbourhoods of the first and second types; the main theorem of the paper implies that each of the classes $\mathfrak{G}_1 - \mathfrak{G}_2$, $\mathfrak{G}_2 - \mathfrak{G}_1$, $\mathfrak{G}_1 \cap \mathfrak{G}_2$ is infinite.

Keywords: Neighbourhood of a vertex, local properties of graphs, asymmetrical graphs.

AMS Classification: 05C99.

INTRODUCTION

Let $G = (V(G), E(G))$ be a finite undirected graph without loops and multiple edges, $u \in V(G)$ its vertex. The neighbourhood of u (defined in the obvious sense, i.e., as the induced subgraph on the set of all vertices which are adjacent to u in G) will be referred to as the *neighbourhood of the first type of u* and denoted by $N_1(u, G)$. We say that an edge $vw \in E(G)$ is adjacent to u if $v \neq u \neq w$ and either v or w is adjacent to u. According to [3], [5], [2] we define the "line-version" of $N_1(u, G)$ as follows: *The neighbourhood of the second type of u* (denoted by $N_2(u, G)$) is the edge-induced subgraph (see e.g. [1], [6]) on the set of all edges which are adjacent to u. (More precisely: the edge set of $N_2(u, G)$ contains all the edges $vw \in E(G)$ for which $\min \{\varrho(v, u), \varrho(w, u)\} = 1$, $\varrho(x, y)$ denoting the distance of vertices x, y).

J. Sedláček [3], [5] introduced the following classes $\mathfrak{G}_1$, $\mathfrak{G}_2$ of asymmetrical graphs: $\mathfrak{G}_i$ contains all graphs G such that for every pair of distinct vertices $u,v \in V(G)$ the neighbourhoods of the i-th type $N_i(u, G)$, $N_i(v, G)$ are non-isomorphic, $i = 1, 2$.

In [3] it is shown that for every integer $n \geqq 6$ there exists a graph $G_n \in \mathfrak{G}_1$ with n vertices; the corresponding graph G_6 (with the minimum number of vertices) is shown in Fig. 1. The analogous question for the class $\mathfrak{G}_2$ is solved in [2]: A graph $G_n \in \mathfrak{G}_2$ with n vetices exists if and only if $n \geqq 7$; the corresponding minimal graph G_7 with 7 vertices is shown in Fig. 2.

As shown in [5], the graph in Fig. 1 belongs, in fact, to $\mathfrak{G}_1 - \mathfrak{G}_2$, and hence $\mathfrak{G}_1 - \mathfrak{G}_2 \neq \emptyset$; analogously, the graph in Fig. 2 belongs to $\mathfrak{G}_2 - \mathfrak{G}_1$, and hence

$\mathfrak{G}_2 - \mathfrak{G}_1 \neq \emptyset$. Further, an example is given in [5] of a graph with 8 vertices which belongs to $\mathfrak{G}_1 \cap \mathfrak{G}_2$; hence $\mathfrak{G}_1 \cap \mathfrak{G}_2 \neq \emptyset$. In the present paper we shall show that each of the classes $\mathfrak{G}_1 - \mathfrak{G}_2$, $\mathfrak{G}_2 - \mathfrak{G}_1$, $\mathfrak{G}_1 \cap \mathfrak{G}_2$ is infinite, and we shall find the minimal member in the last of them.

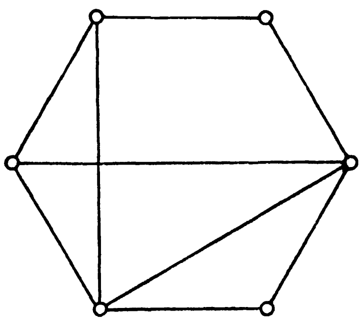

Fig. 1

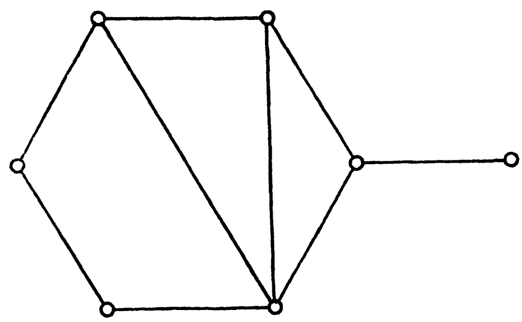

Fig. 2

MAIN THEOREM

Theorem. *Let n be an integer. Then there exists a graph G_n with n vertices which belongs to the class*

a) $\mathfrak{G}_1 - \mathfrak{G}_2$ *if and only if* $n \geqq 6$,

b) $\mathfrak{G}_2 - \mathfrak{G}_1$ *if and only if* $n \geqq 7$,

c) $\mathfrak{G}_1 \cap \mathfrak{G}_2$ *if and only if* $n \geqq 7$.

Corollary. *Each of the classes* $\mathfrak{G}_1 - \mathfrak{G}_2$, $\mathfrak{G}_2 - \mathfrak{G}_1$, $\mathfrak{G}_1 \cap \mathfrak{G}_2$ *is infinite.*

We shall first prove some auxiliary assertions. We say that a vertex $u \in V(G)$ is *universal* if it is adjacent to all the other vertices of G.

Lemma 1. *Let $n \geqq 6$ be an integer; suppose that G_n is a connected graph having n vertices $u_1, \ldots, u_n$, and that none of them is universal. Let us construct the graph G_{n+1} with $n + 1$ vertices by adding a new vertex u_{n+1} to G_n and making it universal in G_{n+1}. Then*

a) $G_n \in \mathfrak{G}_1 \Leftrightarrow G_{n+1} \in \mathfrak{G}_1$,

b) $G_n \in \mathfrak{G}_2 \Leftrightarrow G_{n+1} \in \mathfrak{G}_2$.

Proof. 1. Let $i = 1$ or $i = 2$ and $G_n \in \mathfrak{G}_i$; suppose $G_{n+1} \notin \mathfrak{G}_i$, i.e., for some distinct vertices $u_\alpha, u_\beta \in V(G_{n+1})$ there exists an isomorphism $f: N_i(u_\alpha, G_{n+1}) \to N_i(u_\beta, G_{n+1})$. Since u_{n+1} is universal in $N_i(u_j, G_{n+1})$ for $1 \leqq j \leqq n$ while $N_i(u_{n+1}, G_{n+1}) \simeq G_n$ has no universal vertex, necessarily $\alpha \leqq n$ and $\beta \leqq n$ ($\simeq$ denotes isomorphism).

Hence either $f(u_{n+1}) = u_{n+1}$ and then the partial mapping $f|_{V(N_i(u_\alpha, G_n))}$ is an isomorphism $N_i(u_\alpha, G_n)$ onto $N_i(u_\beta, G_n)$, which is impossible, or $f(u_{n+1})$ is another universal vertex u_γ in $N_i(u_\beta, G_{n+1})$, and in this case interchanging the universal vertices u_γ, u_{n+1} we again obtain a contradiction.

2. If, conversely, $G_n \notin \mathfrak{G}_i$ for $i = 1$ or $i = 2$, then we have an isomorphism $f: N_i(u_\alpha, G_n) \to N_i(u_\beta, G_n)$; defining $f(u_{n+1}) = u_{n+1}$ we obtain an isomorphism $f: N_i(u_\alpha, G_{n+1}) \to N_i(u_\beta, G_{n+1})$ and hence $G_{n+1} \notin \mathfrak{G}_i$.

Lemma 2. *Let $n \geqq 6$ be an integer; suppose that G_n is a graph with n vertices $u_1, \ldots, u_n$ such that the only universal vertex in G_n is u_n and that the minimum degree of G_n is at least 2. Let us construct the graph G_{n+1} with $n + 1$ vertices by adding a new vertex u_{n+1} to G_n and joining it to u_n by an edge. Then*

a) $G_n \in \mathfrak{G}_1 \Leftrightarrow G_{n+1} \in \mathfrak{G}_1$,

b) $G_n \in \mathfrak{G}_2 \Leftrightarrow G_{n+1} \in \mathfrak{G}_2$.

Proof. a) 1. Let $G_n \in \mathfrak{G}_1$. Evidently $N_1(u_i, G_n) = N_1(u_i, G_{n+1})$ for $1 \leqq i \leqq n - 1$; moreover, u_n is the only vertex of degree n in G_{n+1} and u_{n+1} is the only vertex of degree 1 in G_{n+1}. Hence $G_{n+1} \in \mathfrak{G}_1$.

2. Suppose conversely that $G_n \notin \mathfrak{G}_1$, i.e., some distinct vertices $u_\alpha, u_\beta \in V(G_n)$ have isomorphic neighbourhoods. Since u_n is the only universal vertex in G_n, necessarily $\alpha \neq n \neq \beta$; hence

$$N_1(u_\alpha, G_{n+1}) = N_1(u_\alpha, G_n) \simeq N_1(u_\beta, G_n) = N_1(u_\beta, G_{n+1})$$

and therefore $G_{n+1} \notin \mathfrak{G}_1$.

b) 1. Let $G_n \in \mathfrak{G}_2$ and suppose that $G_{n+1} \notin \mathfrak{G}_2$, i.e., there exists an isomorphism $f: N_2(u_\alpha, G_{n+1}) \to N_2(u_\beta, G_{n+1})$ for some $u_\alpha, u_\beta \in V(G_{n+1})$, $u_\alpha \neq u_\beta$. First observe that the neighbourhoods of u_i for $i \neq n$ have n vertices while $N_2(u_n, G_{n+1})$ has $n - 1$ vertices; hence $\alpha \neq n \neq \beta$. Further, evidently $N_2(u_{n+1}, G_{n+1}) \simeq K_{1,n-1}$. If $\alpha = n + 1$ then $N_2(u_\beta, G_{n+1}) \simeq K_{1,n-1}$ and $1 \leqq \beta \leqq n - 1$; considering neighbourhoods of the neighbouring vertices of u_β we obtain a contradiction. Hence $\alpha \neq n + 1$; similarly $\beta \neq n + 1$ and therefore $1 \leqq \alpha, \beta \leqq n - 1$. The vertex u_{n+1} has degree 1 both in $N_2(u_\alpha, G_{n+1})$ and in $N_2(u_\beta, G_{n+1})$; hence either $f(u_{n+1}) = u_{n+1}$ and then the partial mapping $f|_{V(N_2(u_\alpha, G_n))}$ is an isomorphism $N_2(u_\alpha, G_n)$ onto $N_2(u_\beta, G_n)$, which is impossible, or $f(u_{n+1})$ is another vertex u_γ of degree 1 in $N_2(u_\beta, G_n)$ and in this case by interchanging the vertices u_{n+1}, u_γ we again obtain a contradiction.

2. Suppose conversely that $G_n \notin \mathfrak{G}_2$, i.e., we have an isomorphism $f: N_2(u_\alpha, G_n) \to N_2(u_\beta, G_n)$ for some $u_\alpha, u_\beta \in V(G_n)$, $\alpha \neq \beta$. Necessarily $\alpha \neq n \neq \beta$ since u_n is universal in $N_2(u_i, G_n)$ for $1 \leqq i \leqq n - 1$ while $N_2(u_n, G_n)$ has no universal vertex. Further, u_n is the only vertex of degree $n - 1$ both in $N_2(u_\alpha, G_n)$ and in $N_2(u_\beta, G_n)$, and hence $f(u_n) = u_n$. Therefore, if we define $f(u_{n+1}) = u_{n+1}$, we obtain an isomorphism $N_2(u_\alpha, G_{n+1})$ onto $N_2(u_\beta, G_{n+1})$, i.e. $G_{n+1} \notin \mathfrak{G}_2$.

Lemma 3. *Let $n \geqq 6$ be an integer; suppose that G_n is a graph with n vertices $u_1, \ldots, u_n$ such that the only universal vertex in G_n is u_{n-1} and the only vertex of degree 1 in G_n is u_n. Let us construct the graph G_{n+1} with $n + 1$ vertices by adding a new vertex u_{n+1} to G_n and joining it to u_n by an edge. Then*

a) $G_n \in \mathfrak{G}_1 \Leftrightarrow G_{n+1} \in \mathfrak{G}_1$,

b) $G_n \in \mathfrak{G}_2 \Leftrightarrow G_{n+1} \in \mathfrak{G}_2$.

Proof. a) 1. If $G_n \in \mathfrak{G}_1$, then, since $N_1(u_i, G_n) = N_1(u_i, G_{n+1})$ for $1 \leqq i \leqq n - 1$, $N_1(u_{n+1}, G_{n+1})$ is the graph which consists of an isolated vertex and $N_1(u_n, G_{n+1})$ consists of two isolated vertices, evidently $G_{n+1} \in \mathfrak{G}_1$.

2. If, conversely, $G_n \notin \mathfrak{G}_1$, then there exist vertices u_α, u_β, $\alpha \neq \beta$, such that $N_1(u_\alpha, G_n) \simeq N_1(u_\beta, G_n)$. Evidently $1 \leqq \alpha, \beta \leqq n - 1$ and hence $N_1(u_\alpha, G_{n+1}) = N_1(u_\alpha, G_n) \simeq N_1(u_\beta, G_n) = N_1(u_\beta, G_{n+1})$, i.e. $G_{n+1} \notin \mathfrak{G}_1$.

b) 1. If $G_n \in \mathfrak{G}_2$, then evidently $G_{n+1} \in \mathfrak{G}_2$, since $N_2(u_i, G_{n+1}) = N_2(u_i, G_n)$ for $1 \leqq i \leqq n$, $i \neq n - 1$, and these neighbourhoods have $n - 1$ vertices and are connected, while $N_2(u_{n-1}, G_{n+1})$ is disconnected and $N_2(u_{n+1}, G_{n+1})$ has exactly two vertices.

2. If, conversely, $G_n \notin \mathfrak{G}_2$, then $N_2(u_\alpha, G_n) \simeq N_2(u_\beta, G_n)$ for some $\alpha \neq \beta$. One can easily observe that necessarily $\alpha \neq n - 1 \neq \beta$ and hence evidently $G_{n+1} \notin \mathfrak{G}_2$.

Proof of the theorem. The assertion concerning the non-existence of the graph $G_n \in \mathfrak{G}_1 - \mathfrak{G}_2$ with n vertices for $n \leqq 5$ is contained in [3], the non-existence of the graph G_n on n vertices which belongs either to $\mathfrak{G}_2 - \mathfrak{G}_1$ or to $\mathfrak{G}_1 \cap \mathfrak{G}_2$ follows for $n \leqq 6$ from [2], Theorem 2.1.

a) For $n \geqq 6$ define the graph $G_n \in \mathfrak{G}_1 - \mathfrak{G}_2$ by using the following construction:
- for $n = 6$ see the graph G_6 in Fig. 1;
- having obtained G_n, construct G_{n+1} using
  Lemma 1 for $n \equiv 0 \pmod 3$,
  Lemma 2 for $n \equiv 1 \pmod 3$,
  Lemma 3 for $n \equiv 2 \pmod 3$.

b) For $n \geqq 7$ define the graph $G_n \in \mathfrak{G}_2 - \mathfrak{G}_1$ by using the following construction:
- for $n = 7$ see the graph G_7 in Fig. 2;
- having obtained G_n, construct G_{n+1} using
  Lemma 1 for $n \equiv 1 \pmod 3$,
  Lemma 2 for $n \equiv 2 \pmod 3$,
  Lemma 3 for $n \equiv 0 \pmod 3$.

c) For $n \geqq 7$ define the graph $G_n \in \mathfrak{G}_1 \cap \mathfrak{G}_2$ by using the following construction:
- for $n = 7$ see the graph G_7 in Fig. 3; one can easily observe that $G_7 \in \mathfrak{G}_1 \cap \mathfrak{G}_2$;
- having obtained G_n, construct G_{n+1} using
  Lemma 1 for $n \equiv 1 \pmod 3$,
  Lemma 2 for $n \equiv 2 \pmod 3$,
  Lemma 3 for $n \equiv 0 \pmod 3$.

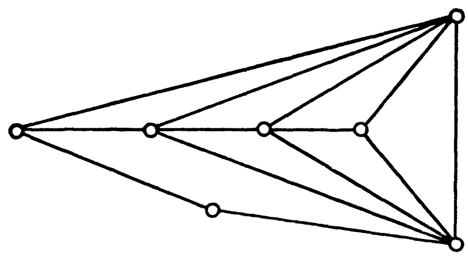

Fig. 3

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Souhrn

GRAFY S NEIZOMORFNÍMI OKOLÍMI UZLŮ 1. A 2. DRUHU

ZDENĚK RYJÁČEK

V článku se zkoumá vzájemný vztah tříd $\mathfrak{G}_1$, $\mathfrak{G}_2$ grafů s neizomorfními okolími uzlů prvního, resp. druhého druhu; z hlavní věty článku jako důsledek vyplývá, že každá z tříd $\mathfrak{G}_1 - \mathfrak{G}_2$, $\mathfrak{G}_2 - \mathfrak{G}_1$, $\mathfrak{G}_1 \cap \mathfrak{G}_2$ je nekonečná.

Резюме

ГРАФЫ С НЕИЗОМОРФНЫМИ ОКРУЖЕНИЯМИ ВЕРШИН ПЕРВОГО И ВТОРОГО ТИПОВ

ZDENĚK RYJÁČEK

В статье изучается взаимоотношение классов $\mathfrak{G}_1$, $\mathfrak{G}_2$ графов с неизоморфными окружениями вершин первого и второго типа. Из главной теоремы в качестве следствия вытекает, что каждый из классов $\mathfrak{G}_1 - \mathfrak{G}_2$, $\mathfrak{G}_2 - \mathfrak{G}_1$, $\mathfrak{G}_1 \cap \mathfrak{G}_2$ бесконечен.

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