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Matchings and cycles in $K_{1,3}$-free graphs

1. Introduction

This paper is an abridged version of [7] and [8]. We consider only finite undirected graphs without loops and multiple edges. Let G be a graph with vertex set V(G) and edge set E(G). A spanning subgraph of G will be called a factor of G; a k-regular factor of G will be shortly called a k-factor of G (for k=1, the term perfect matching is also used). We say that a graph (subgraph, component etc.) is odd or even according to whether it has odd or even number of vertices. A 2-matching of G is a factor of G every component of which is a path or a cycle. A 2-matching is called perfect if every its component is an edge or an odd cycle.

Hamiltonian cycle in G is a connected 2-factor, i.e., a spanning cycle. If G has a Hamiltonian cycle, then we say that G is Hamiltonian. Denote by $|M|$ the number of elements of a finite set M. We say that G is pancyclic if G contains a cycle of length k for every k, $3 \leq k \leq |V(G)|$. G is said to be panconnected if for every pair of distinct vertices x,y of G and every k, $d(x,y) \leq k \leq |V(G)| - 1$, there is a path in G with x and y as end-vertices (by d(x,y) we denote the distance of x, y).

Throughout the paper, we denote for $M \subset V(G)$ by $\langle M \rangle$ the induced subgraph on M and by $\Gamma(M)$ the set of all vertices in V(G) which are adjacent to at least one vertex in M. For a vertex $v \in V(G)$, the induced subgraph $N_1(v,G) = \langle \Gamma(v) \rangle$ will be called the neighbourhood of the first type of v in G. We say that an edge $xy \in E(G)$ is adjacent to v if $x \neq v \neq y$ and x or y (or both) is adjacent to v. The edge-induced subgraph on the set of all edges which are adjacent to v will be called the neighbourhood of the second type of v in G and denoted by $N_2(v,G)$. G is said to be locally connected if the neighbourhood $N_1(v,G)$ of every vertex $v \in V(G)$ is a connected graph. Analogously, we say that G is N_2-locally connected, if for every $v \in V(G)$ its second-type neighbourhood $N_2(v,G)$ is connected. Obviously, every locally connected graph is N_2-locally connected.

We say that a graph G is $K_{1,3}$-free, if G contains no copy of $K_{1,3}$ as an induced subgraph. Evidently, every induced subgraph of a $K_{1,3}$-free graph is also $K_{1,3}$-free. Finally, if H is a subgraph or a set of vertices of G, then by $G \setminus H$ we mean the induced subgraph on the set of all vertices which belong to G but not to H.

2. Matchings

In [10] Sumner proved that every connected $K_{1,3}$-free graph with even number of vertices has a perfect matching. Since every induced subgraph of a $K_{1,3}$-free graph is $K_{1,3}$-free, we easily see that if G is an odd connected $K_{1,3}$-free graph on at

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least three vertices, then for arbitrary $x \in V(G)$, for which $G \setminus x$ is connected, the even subgraph $G \setminus x$ has a perfect matching and hence G has an almost perfect matching, i.e., a factor with one vertex of degree 2 and all other vertices of degree 1. From this we see that every $K_{1,3}$-free graph on at least two vertices has a 2-matching. Nevertheless, the graphs on Fig. 1 show that a connected $K_{1,3}$-free graph with odd number of vertices need not have a perfect 2-matching.

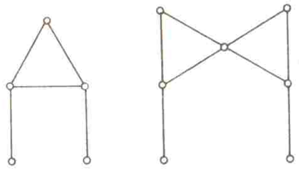

Fig. 1

Theorem 1. Let G be a connected $K_{1,3}$-free graph with odd number of vertices and suppose that F is a factor of G, each component of which is either a single edge or is odd. Then there exists a factor F' in G such that the only odd component of F' is identical to some of odd components of F and all the other components of F' are single edges.

Theorem 2. Let G be a connected $K_{1,3}$-free graph with odd number of vertices. Then the following conditions are equivalent:

(i) G has a perfect 2-matching,

(ii) G has a perfect 2-matching with exactly one odd cycle,

(iii) in G exists an odd cycle C such that each component of $G \setminus C$ is even.

Corollary. If in a connected $K_{1,3}$-free graph G with odd number of vertices exists an odd cycle C such that $G \setminus C$ is connected, then G has a perfect 2-matching.

Theorem 3. Let G be a connected $K_{1,3}$-free graph with odd number of vertices, let $|V(G)| \geq 3$. If G has at most one vertex of degree 1, then G has a perfect 2-matching.

3. Cycles

Oberly and Sumner [6] proved that every nontrivial connected, locally connected $K_{1,3}$-free graph is Hamiltonian. Clark [1] strengthened this result showing that under the same conditions, G is vertex pancyclic. Kanetkar and Rao [3] proved that every connected, locally 2-connected $K_{1,3}$-free graph is panconnected. Some other Hamiltonicity results in $K_{1,3}$-free graphs (not using local connectedness-type arguments) can be found in [2], [5], [9].

Theorem 4. If G is a connected, N_2-locally connected $K_{1,3}$-free graph with minimum degree $\delta(G) \geq 2$, then G has a 2-factor.

Example. The graph on Fig. 2 is a connected $K_{1,3}$-free graph with $\delta(G) = 2$ which has no 2-factor.

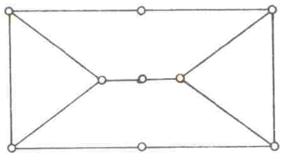

Fig. 2

Let us introduce the following condition.

Assumption (A): G does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Fig. 3) such that $N_1(x,G)$ of every vertex x of degree 4 in H is disconnected.

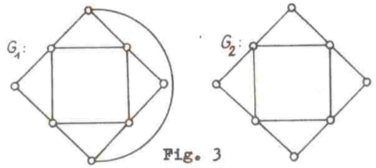

Fig. 3

Theorem 5. Let G be a 2-connected N_2-locally connected $K_{1,3}$-free graph which satisfies the assumption (A). Then G is Hamiltonian.

Example. The graphs on Fig. 4 and 5 are 2-connected, N_2-locally connected $K_{1,3}$-free graphs which are not Hamiltonian; the graph on Fig. 4 contains an induced G_1 but not G_2 while the graph on Fig. 5 contains an induced G_2 but not G_1.

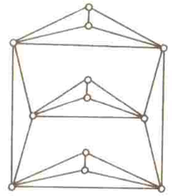

Fig. 4

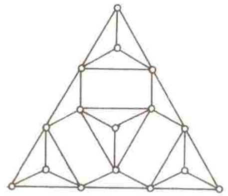

Fig. 5

Theorem 6. Let G be a 3-connected N_2-locally connected $K_{1,3}$-free graph which satisfies the assumption (A). Then G is pancyclic.

Example. The graph on Fig. 6 is a 2-connected N_2-locally connected $K_{1,3}$-free graph satisfying (A) which is Hamiltonian but not pancyclic.

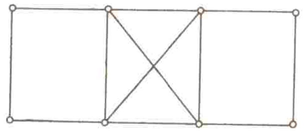

Fig. 6

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