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FACTORS AND CIRCUITS IN $K_{1,3}$-FREE GRAPHS

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In the paper sufficient conditions are given for the existence of a perfect 2-matching, for the existence of a 2-factor and for the pancyclicity of a connected $K_{1,3}$-free graph.

1. Introduction

In this paper we consider only finite undirected graphs without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A spanning subgraph of G will be called a *factor* of G; a k-regular factor of G will be shortly called a *k-factor* of G (for $k = 1$, the term *perfect matching* is also used). We say that a graph (subgraph, component etc.) is *odd* or *even* according as it has an odd or even number of vertices. A *2-matching* of G is a factor of G whose every component is a path or a circuit. A 2-matching is called *perfect* if its every component is an edge or an odd circuit.

A *Hamiltonian circuit* in G is a connected 2-factor, i.e., a spanning circuit. If G has a Hamiltonian circuit, then we say that G is *Hamiltonian*. Denote by $|M|$ the number of elements of a finite set M. We say that G is *pancyclic* if G contains a circuit of length k for every k, $3 \leqslant k \leqslant |V(G)|$. G is said to be *panconnected* if for every pair of distinct vertices x, y of G and every k, $d(x, y) \leqslant k \leqslant |V(G)|-1$, there is a path of length k in G with x and y as end-vertices (by $d(x, y)$ we denote the distance of x, y).

Throughout the paper, for $M \subset V(G)$, we denote by $\langle M \rangle$ the induced subgraph on M and by $\Gamma(M)$ the set of all vertices in $V(G)$ which are adjacent to at least one vertex in M. For a vertex $v \in V(G)$, the induced subgraph $N_1(v, G) = \langle \Gamma(v) \rangle$ will be called the *neighbourhood of the first type* of v in G. We say that an edge $xy \in E(G)$ is *adjacent* to v if $x \neq v \neq y$ and x or y (or both) is adjacent to v. The edge-induced subgraph on the set of all edges which are adjacent to v will be called the *neighbourhood of the second type* of v in G and denoted by $N_2(v, G)$. G is said to be *locally connected* if the neighbourhood $N_1(v, G)$ of every vertex $v \in V(G)$ is a connected graph. Analogously, we say that G is *N_2-locally connected* if for every $v \in V(G)$ its second-type neighbourhood $N_2(v, G)$ is connected. Obviously, every locally connected graph is N_2-locally connected.

We say that a graph G is *$K_{1,3}$-free* if G contains no copy of $K_{1,3}$ as an induced subgraph. Evidently, every induced subgraph of a $K_{1,3}$-free graph is also $K_{1,3}$-free. Finally, if H is a subgraph or a set of vertices of G, then by $G \backslash H$ we mean the induced subgraph on the set of all vertices which belong to G but not to H.

2. Matchings

In [11] Sumner proved that every connected $K_{1,3}$-free graph with an even number of vertices has a perfect matching. Since every induced subgraph of a $K_{1,3}$-free graph is also $K_{1,3}$-free, we easily see that if G is an odd connected $K_{1,3}$-free graph on at least three vertices, then for any $x \in V(G)$ for which $G \backslash x$ is connected, the even subgraph $G \backslash x$ has a perfect matching and hence G has an almost perfect matching, i.e., a factor with one vertex of degree 2 and all other vertices of degree 1. Hence every connected $K_{1,3}$-free graph on at least two vertices has a 2-matching. Nevertheless, the graphs in Fig. 1 show that a connected $K_{1,3}$-free graph with an odd number of vertices need not have a perfect 2-matching.

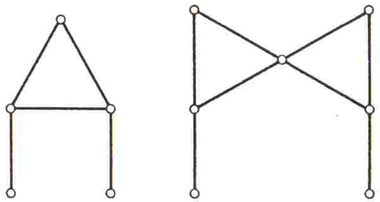


Fig. 1

LEMMA 1. *Let G be a connected $K_{1,3}$-free graph with an odd number of vertices and suppose that F is a factor in G whose each component is either a single edge or is odd. Then there exists a factor F' in G such that the only odd component of F' is identical to some odd component of F and all the other components of F' are single edges.*

Proof. Suppose that such a factor F' does not exist and let F'' be a factor of G such that (i) every odd component of F'' is identical to some component of F, (ii) every even component of F'' is a single edge, and (iii) F'' has a minimum

number of odd components. Since $|V(G)|$ is odd, F'' has at least three odd components. Let P be a path in G such that the end-vertices of P are in different odd components H_1, H_2 of F'' and no other vertex of P is a vertex of an odd component of F'' (existence of such H_1, H_2 and P follows from the connectedness of G).

Denote by N the set of all vertices x for which there exists a vertex y on P such that $\langle x, y\rangle$ is a component of F'' and let $M = V(H_1) \cup V(H_2) \cup V(P) \cup N$. Then evidently every component of F'' either is a subgraph of $\langle M\rangle$ or is disjoint from $\langle M\rangle$. Since $|M|$ is even and $\langle M\rangle$ is a connected induced subgraph of G, $\langle M\rangle$ has a perfect matching, which contradicts (iii). ∎

THEOREM 1. *Let G be a connected $K_{1,3}$-free graph with an odd number of vertices. Then the following conditions are equivalent:*

(i) *G has a perfect 2-matching.*

(ii) *G has a perfect 2-matching with exactly one odd circuit.*

(iii) *In G there exists an odd circuit C such that each component of $G \backslash C$ is even.*

Proof. (i)⇒(ii) follows from Lemma 1.

(ii)⇒(iii). If C is the only odd circuit of a perfect 2-matching, then $G \backslash C$ has a perfect matching and thus cannot have an odd component.

(iii)⇒(i). Choosing a perfect matching in each component of $G \backslash C$ and adding C we obtain a perfect 2-matching in G. ∎

Another sufficient condition is given by the following assertion.

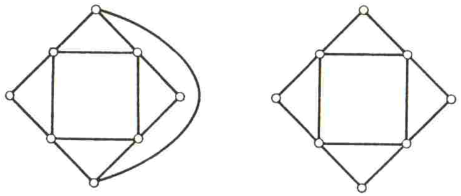
THEOREM 2. *Let G be a connected $K_{1,3}$-free graph with an odd number of vertices, and let $|V(G)| \geq 3$. If G has at most one vertex of degree 1, then G has a perfect 2-matching.*

Proof. By Tutte's theorem (see, e.g., [4], Corollary 6.5.1), G has a perfect 2-matching if and only if $|\Gamma(A)| \geq |A|$ for every independent set of vertices A. Thus, in a connected $K_{1,3}$-free graph G with no perfect 2-matching there exists an independent set A such that $|\Gamma(A)| < |A|$. Since G is $K_{1,3}$-free and A is independent, every vertex in $\Gamma(A)$ is adjacent to at most two vertices in A. Hence the vertices in A are contained in at most $2|\Gamma(A)| \leq 2(|A|-1) = 2|A|-2$ edges and since no vertex has degree 0, necessarily at least two vertices have degree 1. ∎

3. 2-Factors and pancyclicity

Oberly and Sumner [6] proved that every nontrivial connected, locally connected $K_{1,3}$-free graph is Hamiltonian. Clark [1] strengthened this result by showing that under the same conditions, G is vertex pancyclic. Kanetkar and Rao [3] proved that every connected, locally 2-connected $K_{1,3}$-free graph is panconnected. Some other hamiltonicity results in $K_{1,3}$-free graphs (not using local connectedness-type arguments) can be found in [2], [5], [8].

In [7], the sufficient condition for hamiltonicity from [6] is weakened: it is shown that a connected, N_2-locally connected $K_{1,3}$-free graph without vertices of degree 1 is Hamiltonian if it satisfies the following condition:



G_1 G_2

Fig. 2

ASSUMPTION (A). G does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Fig. 2) such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected.

Further, examples are given in [7] showing that a 2-connected, N_2-locally connected $K_{1,3}$-free graph need not be Hamiltonian. In this section we show that (i) every such graph has a 2-factor, and (ii) if G satisfies (A) and is, moreover, 3-connected, then G is pancyclic.

THEOREM 3. *If G is a connected, N_2-locally connected $K_{1,3}$-free graph with minimum degree $\delta(G) \geq 2$, then G has a 2-factor.*

Proof. Suppose G has no 2-factor and let C be a 2-regular subgraph with a maximum number of vertices. For each $x \in V(C)$ denote by C_x the only component of C containing x, and by x', x'' the vertices neighbouring x on C_x. Since G is connected, an edge x_0u can be found such that $u \in V(C)$ while $x_0 \notin V(C)$. Since G is N_2-locally connected and $\delta(G) \geq 2$, we can find a shortest path P in $N_2(u, G)$ from x_0 to one of u', u''; we may assume without loss of generality that P is a path from x_0 to u' and that $u'' \notin V(P)$.

Let the largest 2-regular subgraph C and the edge x_0u be chosen so that C has the minimum number of components and, among all such 2-regular subgraphs, the path P is the shortest possible. Let $x_0, x_1, \ldots, x_k = u'$ be the vertices of P. By the choice of P, no x_i, x_j are adjacent for $|i-j| > 1$. Obviously x_0 is adjacent neither to u' nor to u''; since $\{x_0, u', u'', u\}$ cannot induce $K_{1,3}$, necessarily $u'u'' \in E(G)$. Similarly we see that $k \geq 2$ and at least one of x_i $(1 \leq i \leq k-1)$ is in $V(C)$.

CLAIM 1. *At most one vertex on P is nonadjacent to u. If $x_{k-1} \in V(C_u)$, then $x_{k-1}u \notin E(G)$.*

If two vertices x_i, x_j of P are nonadjacent to u, then for $|i-j| = 1$ the edge x_ix_j does not belong to $N_2(u, G)$ and for $|i-j| \geq 2$, $\{x_{i-1}, x_{i+1}, x_{j+1}, u\}$ induces $K_{1,3}$.

Let $x_{k-1} \in V(C_u)$ and $x_{k-1}u \in E(G)$. Then $x'_{k-1}u \notin E(G)$, since otherwise, replacing in C_u the path $u'uu''$ by the edge $u'u$ and the edge $x'_{k-1}x_{k-1}$ by the path $x'_{k-1}ux_{k-1}$, the path P can be made shorter; similarly $x''_{k-1}u \notin E(G)$. Since $\{x'_{k-1}, x''_{k-1}, x_{k-1}, u\}$ cannot induce $K_{1,3}$, necessarily $x'_{k-1}x''_{k-1} \in E(G)$, but then, replacing in C_u the path $x'_{k-1}x_{k-1}x''_{k-1}$ by the edge $x'_{k-1}x''_{k-1}$ and the edge $u'u$ by the path $u'x_{k-1}u$ again makes P shorter.

CLAIM 2. $x_1 \notin V(C_u)$.

Let, on the contrary, $x_1 \in V(C_u)$. Evidently x_0 is adjacent to neither x'_1 nor x''_1 (since otherwise C_u can be extended through x_0) and since $\{x'_1, x''_1, x_0, x_1\}$ cannot induce $K_{1,3}$, necessarily $x'_1x''_1 \in E(G)$.

Suppose that x_1 is adjacent to u. If $|V(C_u)| = 4$ (i.e., $x'_1x''_1 = u'u''$), then, deleting from C_u the edges x_1u'' and uu' and adding the path x_1x_0u and the edge $u'u''$, C is extended. Thus the length of C_u is at least 5, but then, replacing in C_u the path $x'_1x_1x''_1$ by the edge $x'_1x''_1$, the path $u'uu''$ by the edge $u'u''$ and adding to C a new component $\langle x_0, x_1, u\rangle$, we again have a contradiction. Hence $x_1u \notin E(G)$ and necessarily $x_2u \in E(G)$.

If $x_2 \notin V(C)$, then from $\langle x_0, x'_1, x_2, x_1\rangle$ we see that $x'_1x_2 \in E(G)$ and C_u can be extended through x_2; hence $x_2 \in V(C)$. Similarly, if $x_1x_2 \in E(C)$, then replacing $u'uu''$ and x_1x_2 by $u'u''$ and $x_1x_0ux_2$ gives a contradiction; hence $x_1x_2 \notin E(C)$.

Consider C_{x_2} (not excluding the case $C_{x_2} = C_u$). At least one of x'_2, x''_2 (say, x'_2) is not on P. Since $x_1u \notin E(G)$ and $\{x_1, x'_2, u, x_2\}$ cannot induce $K_{1,3}$, necessarily x'_2 is adjacent to x_1 or to u, but in both cases C_{x_2} can be extended through x_0.

CLAIM 3. $k \leq 3$.

If $k \geq 5$, then $\{x_0, x_2, u', u\}$ or $\{x_0, x_3, u', u\}$ induces $K_{1,3}$; thus $k < 5$. Let $k = 4$. Then, considering $\langle x_0, x_2, u', u\rangle$, we have obviously $x_2u \notin E(G)$ and hence both x_1 and x_3 are adjacent to u. By Claim 1, $x_3 \notin V(C_u)$ and since evidently $x_3 \in V(C)$, necessarily $C_{x_3} \neq C_u$. If $x'_3u \in E(G)$, then replacing in C x'_3x_3 and $u'u$ by x'_3u and x_3u', the number of components of C is decreased; thus $x'_3u \notin E(G)$. Similarly $x''_3u \notin E(G)$ and since $\{x'_3, x''_3, u, x_3\}$ cannot induce $K_{1,3}$, necessarily $x'_3x''_3 \in E(G)$. If the length of C_{x_3} is at least 4, then the replacement of $x'_3x_3x''_3$ by $x'_3x''_3$ in C_{x_3} and of $u'u$ by $u'x_3u$ in C_u contradicts the choice of P. Thus C_{x_3} is a triangle and considering $\langle x_2, x'_3, u', x_3\rangle$ and $\langle x_2, x''_3, u', x_3\rangle$ we easily see that, if $x'_3 \neq x_2 \neq x''_3$, then both x'_3 and x''_3 must be adjacent to x_2.

By Claim 2, $x_1 \notin V(C_u)$ and since $x_1u \in E(G)$, by the choice of P we have $x_1 \in V(C)$, i.e., $C_{x_1} \neq C_u$; since C_{x_3} is a triangle, also $C_{x_1} \neq C_{x_3}$. If one of x'_1,

x_1'' (say, x_1') is on P (i.e., $x_1'' = x_2$), then deleting from C $u'u$, x_3x_3' and x_1x_2 and adding x_3u', x_2x_3' and x_1u, the number of components of C is decreased; hence both x_1' and x_1'' are not on P. Considering $\langle x_0, x_1', x_1'', x_1\rangle$ we see that x_1' and x_1'' are adjacent; from $\langle x_0, x_1', x_2, x_1\rangle$ and $\langle x_0, x_1'', x_2, x_1\rangle$ we further deduce that both x_1' and x_1'' are adjacent to x_2.

Evidently x_2 is on C. In the case $x_3' = x_2$ (or, analogously, $x_3'' = x_2$), one can easily obtain a contradiction; thus both x_2' and x_2'' are not on P and from $\langle x_1, x_2', x_3, x_2\rangle$ we see that x_2' is adjacent to x_3 or to x_1. In the first case we replace in C the edges x_2x_2' and x_3x_3' by x_2x_3' and $x_2'x_3$, while in the second case we replace x_1x_1' and x_2x_2' by x_1x_2' and $x_1'x_2$ for $C_{x_1} \neq C_{x_2}$ and $u'u$, $x_1'x_1x_1''$, x_2x_2' and x_3x_3' by $x_1'x_1''$, x_1x_2', x_2x_3', x_3u' and x_1u for $C_{x_1} = C_{x_2}$. In each case the number of components of C is decreased, a contradiction.

CLAIM 4. $k \leqslant 2$.

Let, on the contrary, $k = 3$. Evidently at least one of x_1, x_2 is on C. If $x_2 \notin V(C)$, then, by Claim 2, $C_{x_1} \neq C_u$; since obviously x_1' cannot be adjacent to x_0, we see, considering $\langle x_0, x_1', x_2, x_1\rangle$, that x_1' is adjacent to x_2, but then, replacing x_1x_1' by $x_1x_2x_1'$ gives a contradiction. Similarly, if x_1 is not on C, then, by the choice of P, $x_1u \notin E(G)$ and, by Claim 1, $x_2u \in E(G)$ and $C_{x_2} \neq C_u$. Since obviously x_2' cannot be adjacent to x_1 and $\{x_1, x_2', u', x_2\}$ cannot induce $K_{1,3}$, we have $x_2'u' \in E(G)$, but then the number of components of C can be decreased joining together C_{x_2} and C_u. Thus both x_1 and x_2 are on C and $C_{x_1} \neq C_u$. Considering $\langle x_0, x_1', x_1'', x_1\rangle$ we see that $x_1'x_1'' \in E(G)$ and, similarly, each of x_1', x_1'' which is not on P is adjacent to x_2.

Suppose that x_2 is on C_{x_1}. Then x_1x_2 cannot belong to $E(C)$ (since otherwise replacing in C $u'u$ and x_1x_2 by $u'x_2$ and x_1x_0u, C is extended) and hence both x_2' and x_2'' are not on P. Since x_2' is not adjacent to u' (otherwise replace $u'u$, x_2x_2' and $x_1'x_1x_1''$ by $u'x_2'$, $x_1'x_1''$ and $x_2x_1x_0u$) and $\{x_1, x_2', u', x_2\}$ cannot induce $K_{1,3}$, necessarily $x_1x_2' \in E(G)$, but then, replacing $u'u$, x_2x_2' and $x_1'x_1x_1''$ by $x_1'x_1''$, $u'x_2$ and $x_2'x_1x_0u$, we again have a contradiction.

Suppose that x_2 is on C_u. By Claim 1, x_2 is not adjacent to u and hence $x_1u \in E(G)$. Clearly C_u has length at least 5 (otherwise replace $u'u$, x_2u'' and x_1x_1' by $x_1'x_2$, $u'u''$ and x_1x_0u). If $x_2'x_1 \in E(G)$, then, replacing x_2x_2' and x_1x_1' by x_1x_2' and $x_1'x_2$, the number of components of C is decreased. Thus $x_1x_2' \notin E(G)$, and, similarly, $x_1x_2'' \notin E(G)$ (not excluding the cases $x_2' = u'$ or $x_2'' = u'$). From this, considering $\langle x_1, x_2', x_2'', x_2\rangle$, we have $x_2'x_2'' \in E(G)$, but then, replacing in C $x_2'x_2x_2''$, $u'uu''$ and $x_1'x_1x_1''$ by $x_2'x_2''u'u''$ and $x_1'x_2x_1''$ and adding to C a new component $\langle x_0, x_1, u\rangle$, C is extended.

Thus $C_{x_1} \neq C_{x_2} \neq C_u$; from $\langle x_1, x_2', u', x_2\rangle$ it then follows that x_2' is adjacent to x_1 or to u', but in the first case, replacing x_1x_1' and x_2x_2' by x_1x_2' and $x_1'x_2$, the number of components of C is decreased, while in the second case, replacing $u'u$, x_2x_2' and x_1x_1' by $u'x_2'$, $x_1'x_2$ and x_1x_0u, C is extended through x_0. Thus, Claim 4 is proved.

Now, since x_0 cannot be adjacent to u', i.e., $k \geqslant 2$, by Claim 4, $k = 2$. By Claim 2, $x_1 \notin V(C_u)$, and since evidently $x_1 \in V(C)$, we have $C_{x_1} \neq C_u$. Consider $\langle x_0, x_1', u', x_1\rangle$: x_0 can be adjacent to neither x_1' nor u' and hence $x_1'u' \in E(G)$, but then again C can be extended through x_0 by replacing $u'u$ and x_1x_1' by $x_1'u'$ and x_1x_0u. This contradiction completes the proof. ∎

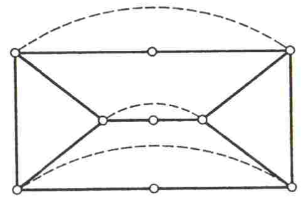


Fig. 3

EXAMPLE. The graph in Fig. 3 is a connected $K_{1,3}$-free N_2-locally connected graph which is maximally non-Hamiltonian (see [9], [10]). Deleting intermittent edges gives a connected $K_{1,3}$-free graph with $\delta(G) \geqslant 2$ and without any 2-factor.

THEOREM 4. *Let G be a 3-connected N_2-locally connected $K_{1,3}$-free graph which satisfies the assumption* (A). *Then G is pancyclic.*

Proof. (1) Let r be the smallest integer such that in G there is a circuit of length r, but none of length $r+1$; suppose that $r < |V(G)|$. Then for every circuit C of length r there exists an edge x_0u such that $u \in V(C)$ and $x_0 \notin V(C)$. Denote by u_1, u_2 the neighbours of u on C. Since G is N_2-locally connected, we can find a shortest path in $N_2(u, G)$ from x_0 to one of u_1, u_2; we may assume without loss of generality that P is a path from x_0 to u_1 and that $u_2 \notin V(P)$. Let the circuit C of length r and the edge x_0u be chosen so that the path P is the shortest possible and let $x_0, x_1, \ldots, x_k = u_1$ be its vertices. From the minimality of P we have $x_ix_j \notin E(G)$ for $|i-j| > 1$.

(2) At least one vertex x_j ($1 \leqslant j \leqslant k-1$) is on C. Suppose, on the contrary, that the only vertex of P lying on C is u_1. If x_{k-1} is adjacent to u, then replacing in C u_1u by $u_1x_{k-1}u$ we extend C; hence $x_{k-1}u \notin E(G)$ and thus $x_{k-2}u \in E(G)$. Since G is 3-connected, an edge vw can be found such that $u_1 \neq v \neq u$, v is on C and w is not on C (otherwise C is a bicomponent of G with biarticulation $\{u, u_1\}$). Let v', v'' be the neighbours of v on C. If $wv' \in E(G)$, then, replacing in C $v'v$ by $v'wv$, C is extended; thus $wv' \notin E(G)$ and similarly $wv'' \notin E(G)$. Since $\{v', w, v'', v\}$ cannot induce $K_{1,3}$, necessarily $v'v'' \in E(G)$, but then, replacing $v'vv''$ by $v'v''$ and u_1u by $u_1x_{k-1}x_{k-2}u$, we have a contradiction.

(2a) By the minimality of P, every vertex of P which is adjacent to u is on C.

(3)–(18): the rest of the proof is quite analogous to the proof of the main lemma of [7] (in part (14), use (2a)), and is therefore omitted. ∎

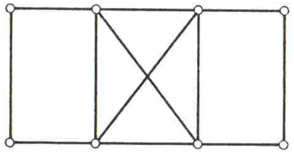


Fig. 4

EXAMPLE. The graph in Fig. 4 is a 2-connected N_2-locally connected $K_{1,3}$-free graph satisfying (A), which is Hamiltonian, but not pancyclic.

Summarizing the obtained and some other recent results, we have the following table.

Let G be a connected $K_{1,3}$-free graph on at least three vertices.

If	Then	References
$\lvert V(G)\rvert$ is even $\lvert V(G)\rvert$ is odd	G has a perfect matching G has an almost perfect matching	[11]
G has at most 1 vertex of degree 1	G has a perfect 2-matching	
G is N_2-locally connected, $\delta(G) \geqslant 2$	G has a 2-factor	
G is N_2-locally connected, $\delta(G) \geqslant 2$, (A)	G is Hamiltonian	[7]
G is N_2-locally connected, 3-connected, (A)	G is pancyclic	
G is locally connected	G is vertex pancyclic	[1]
G is locally 2-connected	G is panconnected	[3]

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