Hamiltonian Circuits in N_2-Locally Connected $K_{1,3}$-Free Graphs

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ABSTRACT

There are many results concerned with the hamiltonicity of $K_{1,3}$-free graphs. In the paper we show that one of the sufficient conditions for the $K_{1,3}$-free graph to be Hamiltonian can be improved using the concept of second-type vertex neighborhood. The paper is concluded with a conjecture.

1. INTRODUCTION

In this paper we deal with finite simple graphs. For a vertex v of a graph $G = (V(G), E(G))$, the neighborhood of v, defined in the obvious sense, i.e., as the induced subgraph on the set of all vertices that are adjacent to v, will be called the neighborhood of the first type of v in G and denoted by $N_1(v, G)$, or briefly, $N_1(v)$. We say that an edge $xy \in E(G)$ is adjacent to v if $x \neq v \neq y$ and x or y is adjacent to v. We define the neighborhood of the second type of v in G (denoted by $N_2(v, G)$ or, briefly, $N_2(v)$) as the edge-induced subgraph on the set of all edges that are adjacent to v (this concept was first introduced in [10]). Many questions that have been investigated for the neighborhoods of the first type can also be studied for the second type neighborhoods (see, e.g., [11], [12], [7], [8]).

We say that G is locally connected if for every $v \in V(G)$, the neighborhood $N_1(v)$ is a connected graph. Analogously, G is N_2-locally connected, if the second-type neighborhood $N_2(v)$ of every vertex $v \in v(G)$ is connected. Obviously, every locally connected graph is N_2-locally connected.

Denote by $d(v)$ the degree of a vertex $v \in V(G)$, by $\delta(G)$ the minimum degree of G, and by n the number of vertices of G. The following theorem gives sufficient conditions for G to be locally and N_2-locally connected (proofs can be found in [1] and [7]):

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Theorem. (a) If for every pair of vertices x, y of G,

$$d(x) + d(y) > \frac{4}{3}(n - 1),$$

then G is locally connected.

(b) If for every pair of nonadjacent vertices x, y of G,

$$d(x) + d(y) \geqq n,$$

then G is N_2-locally connected.

Corollary. (a) G is locally connected provided $\delta(G) > 2/3(n - 1)$.

(b) G is N_2-locally connected provided $\delta(G) \geqq n/2$.

We say that G is $K_{1,3}$-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph. Many results show that $K_{1,3}$-free graphs have, under some additional conditions, some Hamiltonian properties (see, e.g., [2], [3], [4], [6]). Oberly and Sumner [6] proved that every connected, locally connected $K_{1,3}$-free graph on at least three vertices is Hamiltonian. Clark [2] showed that under the same conditions G is vertex pancyclic. In the present paper we find sufficient conditions for G to be Hamiltonian that are weaker than those in [6]. Some further results concerning factors and pancyclicity of $K_{1,3}$-free graphs can be found in [9].

2. MAIN RESULT

Lemma. Every non-Hamiltonian connected N_2-locally connected $K_{1,3}$-free graph with $\delta(G) \geqq 2$ contains an induced subgraph H such that H is isomorphic to either G_1 or G_2 (see Figure 1) and the first-type neighborhood $N_1(v, G)$ of every vertex $v \in V(H)$ of degree 4 in H is disconnected.

Proof. Let G be a connected N_2-locally connected $K_{1,3}$-free graph without vertices of degree 1 that is not Hamiltonian.

(1) For every longest circuit C in G, an edge x_0u can be found such that u is on C while x_0 is not on C. Denote by u_1, u_2 the vertices neighboring u on C. Since G is N_2-locally connected, we can find a shortest path P in $N_2(u)$ from x_0 to one of u_1 or u_2; we may assume without loss of generality that P is a path from x_0 to u_1 and that $u_2 \notin V(P)$. Let the longest circuit C and the edge x_0u be chosen so that the path P is shortest possible and let $x_0, x_1, \ldots, x_k = u_1$ be its vertices. From the minimality of P it follows that no x_i, x_j can be adjacent for $|i - j| > 1$.

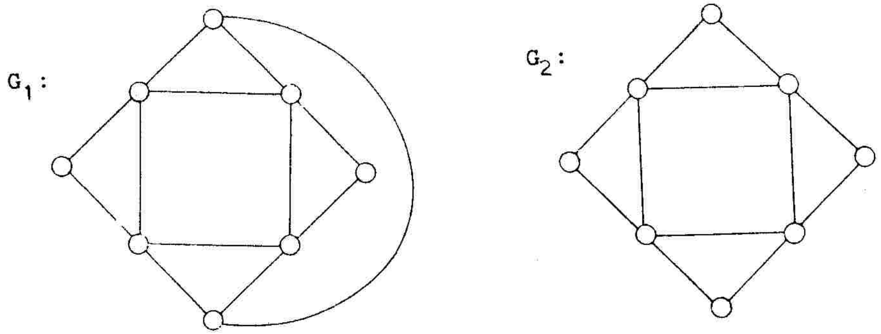

FIGURE 1

(2) At least one of x_j's ($j = 1, \ldots, k - 1$) is on C since otherwise deleting from C the edge u_1u and adding P and the edge x_0u we could obtain a circuit that is longer than C. Specifically, we have $k \geqq 2$.

(3) If u_1, u_2 were nonadjacent, then since $\{u_1, u_2, x_0, u\}$ cannot induce $K_{1,3}$, necessarily $x_0u_1 \in E(G)$ or $x_0u_2 \in E(G)$, which is a contradiction with (2) and with the choice of P. Hence $u_1u_2 \in E(G)$.

(4) We prove that x_{k-1} is not adjacent to u: suppose, on the contrary, that $x_{k-1}u \in E(G)$. If x_{k-1} were not on C, then, replacing the edge u_1u by the path $u_1x_{k-1}u$, we should obtain a circuit longer than C; therefore x_{k-1} is on C.

For every vertex x on C, $x \neq u$, we denote by x', x'' the vertices neighboring x on C. If $ux'_{k-1} \in E(G)$, then replacing in C the edge $x'_{k-1}x_{k-1}$ by the path $x'_{k-1}ux_{k-1}$ and the path u_1uu_2 by the edge u_1u_2 we should obtain a circuit C' of the same (maximal) length as C, and such that if we denoted $u'_1 = x_{k-1}$ then u'_1 should be a neighbor of u on C' and in $N_2(u)$ should exist a path from x_0 to u'_1 shorter than P; but this is a contradiction with the choice of C and P. Similarly, we show that x''_{k-1} and u cannot be adjacent. Since $\{x'_{k-1}, x''_{k-1}, u, x_{k-1}\}$ cannot induce $K_{1,3}$, necessarily $x'_{k-1}x''_{k-1} \in E(G)$; replacing in C the path $x'_{k-1}x_{k-1}x''_{k-1}$ by the edge $x'_{k-1}x''_{k-1}$ and the edge u_1u by the path $u_1x_{k-1}u$ we again obtain a contradiction.

(5) The only vertex of P that is nonadjacent to u is x_{k-1}: if another x_j ($1 \leqq j < k - 1$) were nonadjacent to u, then necessarily $j \leqq k - 3$ (since otherwise the edge $x_{k-2}x_{k-1}$ should not be in $N_2(u)$), and by (1), $\{x_{j-1}, x_{k-2}, x_k, u\}$ should induce $K_{1,3}$.

(6) $k = 3$. If $k > 3$, then according to (1) and (5), necessarily $\{x_0, x_2, u_1, u\}$ should induce $K_{1,3}$. Since by (2) $k \geqq 2$, it remains to prove that $k \neq 2$.

Suppose that $k = 2$. Evidently $x_1'x_0 \notin E(G)$, since otherwise replacing in C the edge $x_1'x_1$ by the path $x_1'x_0x_1$ lengthens C; similarly, $x_1''x_0 \notin E(G)$ (we do not exclude the case when x_1' or x_1'' is identical with one of u_1, u_2). Since $\{x_1', x_1'', x_0, x_1\}$ cannot induce $K_{1,3}$, necessarily $x_1'x_1'' \in E(G)$. This again enables us to make C longer. Hence we have $V(P) = \{x_0, x_1, x_2, u_1\}$, $x_1u \in E(G)$, and $x_2u \notin E(G)$ (see Figure 2).

(7) x_2 is on C, for if $x_2 \notin V(C)$, then by (2) $x_1 \in V(C)$; since x_0 is adjacent to neither x_1' nor x_1'' (the proof is analogous to that in (6)) and $\{x_0, x_1', x_1'', x_1\}$ cannot induce $K_{1,3}$, necessarily $x_1'x_1'' \in E(G)$ and then deleting from C the path $x_1'x_1x_1''$ and the edge u_1u and adding the edge $x_1'x_1''$ and the path $u_1x_2x_1x_0u$ we obtain a circuit that is longer than C.

Similarly, $x_1 \in V(C)$, for otherwise necessarily $x_1x_2' \notin E(G)$ (since otherwise replacing in C the edge x_2x_2' by the path $x_2x_1x_2'$ can C be made longer) and analogously $x_1x_2'' \notin E(G)$; since $\{x_1, x_2', x_2'', x_2\}$ cannot induce $K_{1,3}$, necessarily $x_2'x_2'' \in E(G)$, but this again enables us to make C longer.

(8) We can assume without loss of generality that $x_1x_2 \in E(C)$ (i.e., $x_1'' = x_2$ and $x_2'' = x_1$). Suppose, on the contrary, that $x_1x_2 \notin E(C)$. Then, since x_0 is adjacent neither to x_1' nor to x_1'' (proof is similar to that in (6)), necessarily $x_1'x_1'' \in E(G)$. If $x_2'x_2'' \in E(G)$, then replacing in C the path $x_1'x_1x_1''$ by the edge $x_1'x_1''$, the path $x_2'x_2x_2''$ by the edge $x_2'x_2''$ and the edge u_1u by the path $u_1x_2x_1x_0u$, a circuit should arise that is longer than C; therefore $x_2'x_2'' \notin E(G)$. Since $\{x_1, x_2', x_2'', x_2\}$ cannot induce $K_{1,3}$, x_1 is adjacent either to x_2' or to x_2''. In the first case replacing in C the path $x_1'x_1x_1''$ by the edge $x_1'x_1''$ and the edge $x_2'x_2$ by the path $x_2'x_1x_2$ we obtain a circuit C' of the same length as C and such that $x_1x_2 \in E(C')$; the second case is similar. Hence one of x_1', x_1'' (say, x_1'') is identical to x_2 and one of x_2', x_2'' (say, x_2''), is identical to x_1.

(9) Evidently $x_1'x_2 \in E(G)$ since x_1' can be adjacent neither to x_0 (by (6)) nor to u (proof is easy) and $\{x_1', u, x_2, x_1\}$ cannot induce $K_{1,3}$; from this it easily follows that $x_2' \neq u_1$ and $x_1' \neq u_2 \neq x_2'$ (see Figure 3).

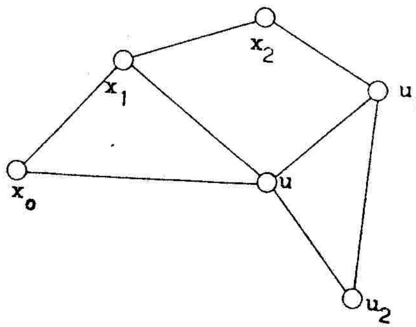

FIGURE 2

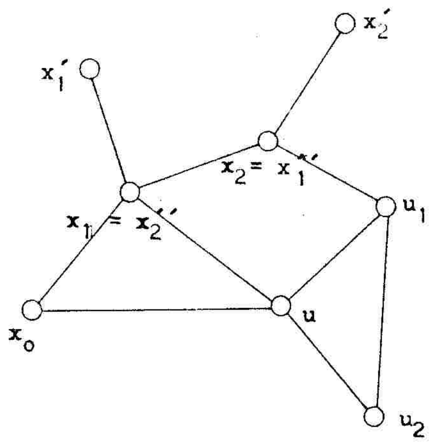

FIGURE 3

(10) $x_0x_2' \notin E(G)$, since otherwise replacing in C the path $x_1'x_1x_2x_2'$ by the path $x_1'x_2x_1x_0x_2'$, a circuit is obtained that is longer than C. Similarly, $x_1x_2' \notin E(G)$, since in this case $x_1'x_2' \in E(G)$ (since $\{x_0, x_1', x_2', x_1\}$ cannot induce $K_{1,3}$ and x_0 is adjacent to neither x_1' nor x_2') and again a circuit is easily constructed that is longer than C.

(11) $x_2'u_1 \in E(G)$ since x_1 can be adjacent neither to x_2' nor to u_1 and $\{x_1, x_2', u_1, x_2\}$ cannot induce $K_{1,3}$. Therefore, G contains the subgraph that is shown in Figure 4.

(12) Deleting from C the paths $x_1'x_1x_2x_2'$ and u_1uu_2 the circuit C splits into two paths P_1, P_2 in two possible ways:

(i) one of the paths (say, P_1) joins x_1' with u_2 and the other one joins x_2' with u_1,
(ii) one of the paths (say, P_1) joins x_1' with u_1 and the other one joins x_2' with u_2.

In the second case a circuit can be easily constructed which is longer than C (see Figure 5) and hence the only possible case is case (i)—see Figure 6.

(13) $x_2u_2 \notin E(G)$ since otherwise joining the end point x_1' of P_1 with the end point u_1 of P_2 by the path $x_1'x_1x_0uu_1$ and the end point x_2' of P_2 with the end point u_2 of P_1 by the path $x_2'x_2u_2$ we again make C longer.

Similarly, we can show that none of the edges x_1u_2, $x_1'x_2'$, $x_1'u_1$, $x_1'u$, $x_2'u$, and $x_2'u_2$ can be an edge of G. Further, by (1) $x_0x_2 \notin E(G)$ and $x_1u_1 \notin E(G)$, by (2) $x_0u_1 \notin E(G)$ and $x_0u_2 \notin E(G)$, by (4) $x_2u \notin E(G)$, by (6) $x_0x_1' \notin E(G)$. From this and from (10) it follows that the induced subgraph of G on the set $\{x_0, x_1, x_2, x_1', x_2', u_1, u, u_2\}$ is isomorphic either to G_1 or to G_2. It remains

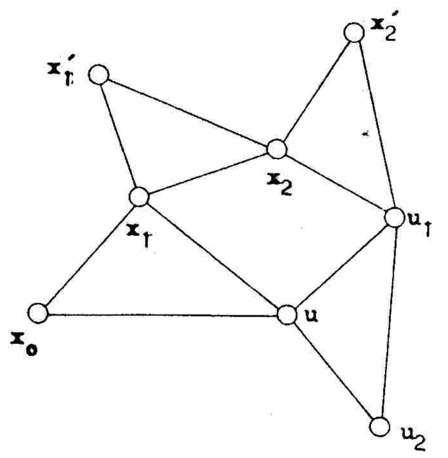

FIGURE 4

to prove that the first-type neighborhoods of the vertices u, x_1, x_2, and u_1 are disconnected.

(14) $N_1(u)$ is disconnected since if it were connected then we could show step by step in the same way as in the proof of the main theorem of [6] that C is not maximal.

(15) The disconnectedness of $N_1(x_1)$ can be verified in the same way as in (14) considering x_1 instead of u and x_1', x_2 instead of u_1, u_2.

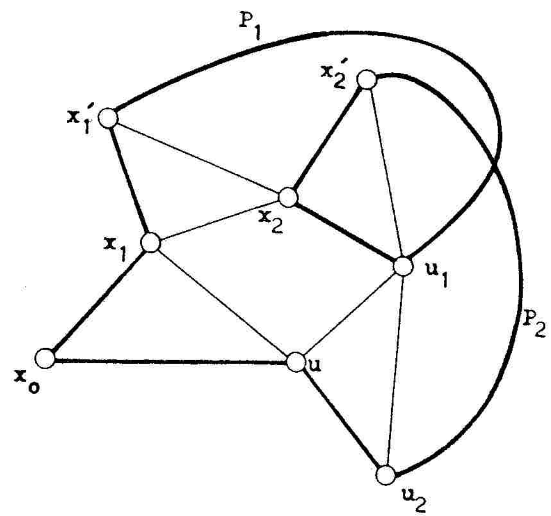

FIGURE 5

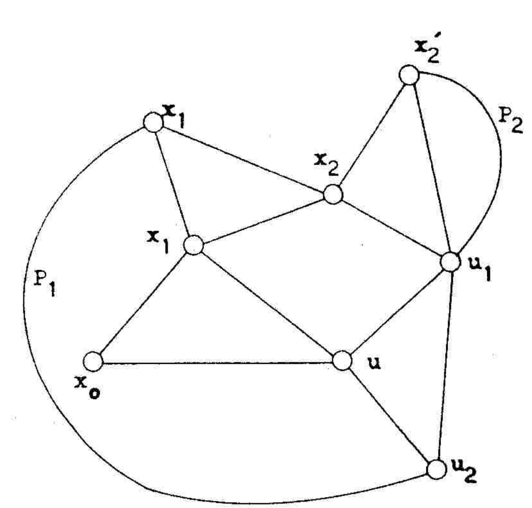

FIGURE 6

(16) We prove the following two assertions:

(A) If a vertex $y \in V(P_1)$ is adjacent to both x_2' and u_1 then the vertices y', y'' neighboring y on C are adjacent.
(B) If a vertex $y \in V(P_2)$ is adjacent to both x_1' and x_1 then the vertices y', y'' neighboring y on C are adjacent.

Proof of (A). From (13) it follows that y cannot be x_1' or u_2; hence y divides P_1 into two subpaths P_1^1 (containing u_2) and P_1^2 (containing x_1'). We can easily see that each of these paths must have at least two edges. Let $y' \in V(P_1^1)$, $y'' \in V(P_1^2)$ be the neighbors of y on C. The vertex y' cannot be adjacent to x_2' since otherwise deleting from C the edges $y'y$, $x_1'x_1$, $x_2'x_2$, and u_1u, and adding the edges $y'x_2'$, yu_1, $x_1'x_2$, and the path x_1x_0u we would obtain a circuit that is longer than C. In the same way we show that y'' cannot be adjacent to x_2' and since $\{y', y'', x_2, y\}$ cannot induce $K_{1,3}$, necessarily $y'y'' \in E(G)$.

The proof of (B) is similar and is left to the reader.

(17) We prove that $N_1(x_2)$ is disconnected; suppose, on the contrary, that $N_1(x_2)$ is connected. Since both x_1x_1' and $x_2'u_1$ are in $N_1(x_2)$, there is a path in $N_1(x_2)$ that joins one of x_1, x_1' with one of x_2', u_1. Let Q be the shortest path in $N_1(x_2)$ from x_1 or x_1' to x_2' or u_1 and denote by $y_0, y_1, \ldots, y_p$ its vertices (i.e., $y_0 = x_1$ or $y_0 = x_1'$ and $y_p = x_2'$ or $y_p = u_1$). From the minimality of Q it follows that no y_i, y_j are adjacent for $|i - j| > 1$ and hence $p \leqq 3$ (otherwise $\{y_0, y_2, y_p, x_2\}$ should induce $K_{1,3}$). On the other hand, $p \geqq 2$, since by (13) none of x_1, x_1' can be adjacent to any of x_2', u_1. Hence either $p = 2$ or $p = 3$.

(17a) Consider the case $p = 2$. Evidently $y_1 \in V(C)$ since otherwise C could be extended through y_1; similarly, the neighbors y_1', y_1'' of y_1 on C cannot be adjacent. Further, if y_1 were nonadjacent to any of x_1, x_1' (say, x_1) and simultaneously to any of x_2', u_1 (say, u_1), then $\{x_1, y_1, u_1, x_2\}$ should induce $K_{1,3}$. Hence y_1 is adjacent either to both x_1 and x_1', or to both u_1 and x_2'. Since y_1', y_1'' cannot be adjacent and $y_1 \notin M$, we see (using (16) (A) and (B)) that in the first case $y_1 \in V(P_1)$ while in the second case $y_1 \in V(P_2)$. Hence we have the following two possibilities:

(i) y_1 is on P_1 and is adjacent to both x_1 and x_1'.
(ii) y_1 is on P_2 and is adjacent to both x_2' and u_1.

Case (i). Since $y_1 \in V(P_1)$, it divides P_1 into two subpaths P_1^1 (containing u_2) and P_1^2 (containing x_1'), each of them having evidently at least two edges. Let $y_1' \in V(P_1^1)$, $y_1'' \in V(P_1^2)$ be the neighbors of y_1 on C. Since $\{y_1', y_1'', x_2, y_1\}$ cannot induce $K_{1,3}$ and y_1', y_1'' cannot be adjacent, necessarily $y_1'x_2 \in E(G)$ or $y_1''x_2 \in E(G)$; simultaneously y_1 is adjacent to either u_1 or x_2'. In each of these four cases we can easily construct a circuit that is longer than C (for $y_1'x_2 \in E(G)$ and $y_1u_1 \in E(G)$—see Figure 7).

Case (ii). This implies a contradiction in the same way as the preceding one (details are left to the reader) and hence $p \neq 2$.

(17b) Suppose that $p = 3$, i.e., the vertices of Q are y_0, y_1, y_2, y_3 (y_0 is one of x_1; x_1' and y_3 is one of u_1, x_2'). Then y_1 is adjacent to both x_1 and x_1', and y_2 is adjacent to both x_2' and u_1 since if, e.g., $y_1x_1 \notin E(G)$, then $\{x_1, y_1, u_1, x_2\}$ would

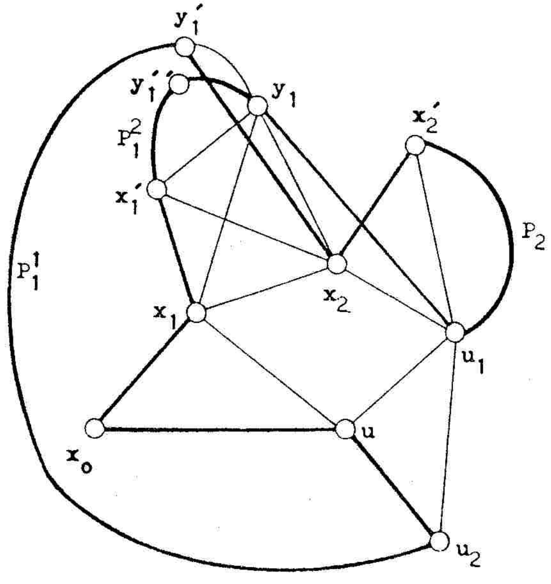

FIGURE 7

induce $K_{1,3}$; other cases are similar. We can therefore assume without loss of generality that $y_0 = x_1'$ and $y_3 = x_2'$. Now, obviously, $y_1 \in V(C)$, since otherwise C could be extended through y_1; similarly, $y_2 \in V(C)$. The neighbors y_1', y_1'' of y_1 on C cannot be adjacent since otherwise we could replace in C the path $y_1'y_1y_1''$ by th edge $y_1'y_1''$ and the edge $x_1'x_1$ by the path $x_1'y_1x_1$, and would have obtained the (impossible) case $p = 2$. Similarly, $y_2'y_2'' \notin E(G)$. From this and from (16) (A) and (B) we conclude that $y_1 \in V(P_1)$ and $y_2 \in V(P_2)$.

Denote again by P_1^1, P_1^2 the subpaths of P_1 determined by y_1, and by y_1', y_1'' the neighbors of y_1 on them; analogously, define the subpaths P_2^1 and P_2^2 of P_2 and the vertices y_2', y_2'' on them. Excluding the cases $y_1'' = x_1'$, $y_2'' = x_2'$, and $y_2' = u_1$ and observing induced $K_{1,3}$ on $\{y_1', y_1'', x_2, y_1\}$ and on $\{y_2', y_2'', x_2, y_2\}$ we conclude after some considerations that are similar to those in (17a) that each of these cases leads to the contradiction. Therefore, $N_1(x_2)$ is disconnected.

(18) The disconnectedness of $N_1(u_1)$ can be verified analogously using u_1 instead of x_2 and the edges x_2x_2' and uu_2 instead of x_1x_1' and u_1x_2'.

Theorem. Let G be a connected, N_2-locally connected $K_{1,3}$-free graph without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure 1) such that $N_1(x, G)$ of every vertex x of degree 4 in H is disconnected. Then G is Hamiltonian.

The proof follows immediately from the lemma.

We say that G is locally quasiconnected if, for each $xy \in E(G)$, either $N_1(x)$ or $N_1(y)$ is connected (this concept was introduced by Nebeský [5]).

Corollary. Let G be a connected, N_2-locally connected $K_{1,3}$-free graph without vertices of degree 1, which satisfies one of the following conditions:

(i) the induced subgraph of G on the set of all vertices $v \in V(G)$ with disconnected $N_1(v, G)$ has no induced circuit of length 4,
(ii) G is locally quasiconnected.

Then G is Hamiltonian.

Examples. The graphs in Figures 8 and 9 are connected, N_2-locally connected $K_{1,3}$-free graphs with $\delta(G) = 3$ that are not Hamiltonian; the graph in Figure 8 contains an induced G_1 but not G_2, while the graph in Figure 9 contains an induced G_2 but not G_1. The graph in Figure 10 satisfies the assumptions of the theorem and is Hamiltonian but not vertex pancyclic. Also the graph of Figure 10 fails to satisfy either condition of the above corollary.

We conclude the paper with a conjecture.

Conjecture. Every 3-connected N_2-locally connected $K_{1,3}$-free graph is Hamiltonian.

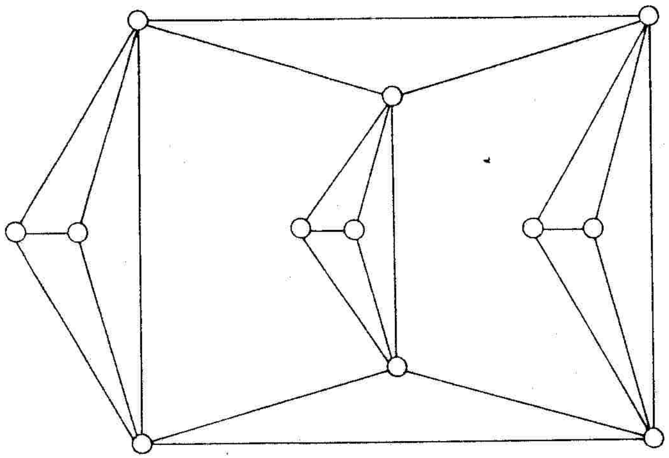

FIGURE 8

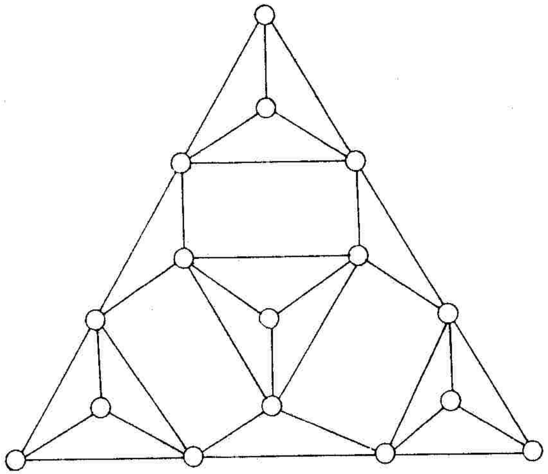

FIGURE 9

Remark. From [4], Theorem 7, it follows that the conjecture is true for graphs on fewer than 20 vertices.

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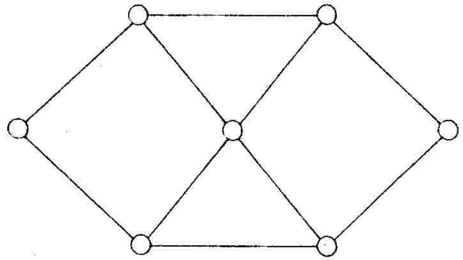

FIGURE 10

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