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N_2-locally disconnected graphs

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Abstract

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The edge-induced subgraph on the set of all edges of a graph G that are adjacent to a given vertex x is called the neighbourhood of the second type of x in G and is denoted by $N_2(x, G)$ (an edge yz is said to be adjacent to x if $y \neq x \neq z$ and y or z is adjacent to x). A graph G is N_2-locally disconnected if $N_2(x, G)$ is disconnected for every vertex x of G. The main aim of the present paper is to find the maximum size of an N_2-locally disconnected graph of a given order.

1. Introduction

Graphs with prescribed properties of vertex neighbourhoods have been the subject of study of many papers in recent years. In the present paper we consider the maximum size of a graph of a given order in which all second-type neighbourhoods of vertices are disconnected. An exact value is found in the case of planar graphs and estimates are given in the general case.

We consider only finite undirected graphs without loops and multiple edges. The neighbourhood of a vertex $x \in V(G)$ of a graph $G=(V(G), E(G))$ defined in the obvious sense, i.e. as the induced subgraph on the set of all neighbouring vertices, will be referred to as the neighbourhood of the first type of x in G and denoted by $N_1(x, G)$. We say that an edge $yz \in E(G)$ is adjacent to a vertex $x \in V(G)$ if $y \neq x \neq z$ and y or z (or both) is adjacent to x. The edge-induced subgraph on the set of all edges that are adjacent to x will be called the neighbourhood of the second type of x in G and denoted by $N_2(x, G)$ (this concept was first introduced in [4] and studied e.g. in [1, 3, 5, 6]).

We say that G is locally disconnected (N_2-locally disconnected) if $N_1(x, G)$ ($N_2(x, G)$) is a disconnected graph for every $x \in V(G)$.

In [2] the authors showed that the asymptotic behaviour of the maximum size of a locally disconnected graph is the same as the size of the complete graph of the same

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order. In [7] the exact value of this maximum size is found in the special case of planar graphs. The main aim of the present paper is to study these questions in the sense of the second-type neighbourhood.

Clearly, the cycle C_n is an example of an N_2-locally disconnected graph for every $n \geq 5$; it is easy to see that there is no N_2-locally disconnected graph of order $n \leq 4$. For $n \geq 5$, we denote by $t(n)$ the maximum size of an N_2-locally disconnected graph of order n and by $t_p(n)$ the maximum size of a planar N_2-locally disconnected graph of order n. Since every N_2-locally disconnected graph is clearly also locally disconnected, the values of $t(n)$ and $t_p(n)$ are expected to be lower than those in [2, 7].

2. Planar graphs

Denote by $\lfloor a \rfloor$ the integer part of a real number a, i.e. the largest integer that is not greater than a.

Theorem 2.1. *Let n be an integer, $n \geq 5$. Then*

$$t_p(n) = \begin{cases} \frac{1}{5}(11n-36) & \text{for } n \equiv 1 \pmod 5, \\ \lfloor \frac{1}{5}(11n-30) \rfloor & \text{otherwise.} \end{cases}$$

Throughout the proof, we assume that for a planar graph G its fixed embedding into the plane is chosen. We denote by f the number of faces of G and by f_i the number of its i-gonal faces, $i = 3, \ldots$; by a large face we mean a face that is at least pentagonal. We say that G has a property P if every vertex $x \in V(G)$ is either incident at least twice to some large face of G (if x is a cutvertex of G) or is incident to at least two large faces of G. Clearly, every planar N_2-locally disconnected graph has the property P. We first prove the following auxiliary assertion.

Lemma 2.2. *Let G be a connected planar graph of order n and size m having the property P and such that $f_i = 0$ for $i = 4$ and for $i \geq 8$. Then*

$$m \leq \tfrac{1}{5}(11n-30) - \tfrac{3}{5}f_6 - \tfrac{7}{6}f_7.$$

Proof. From Euler's formula, $n - m + f = 2$, we have

$$m = n + f_3 + f_5 + f_6 + f_7 - 2.$$

Obviously, $2m = 3f_3 + 5f_5 + 6f_6 + 7f_7$ and therefore

$$m = 3n - 6 - 2f_5 - 3f_6 - 4f_7.$$

The property P yields

$$2n \leq 5f_5 + 6f_6 + 7f_7,$$

from which

$$m \leqslant \tfrac{11}{5}n - 6 - \tfrac{3}{5}f_6 - \tfrac{6}{7}f_7,$$

which proves the lemma. □

Proof of Theorem 2.1. (a) Let G be a planar N_2-locally disconnected graph of order n and the maximum size $m = t_p(n)$. Then G is connected (since it is maximal) and has the property P (since it is N_2-locally disconnected). Further, if there is a face in G that is at least octagonal then new edges can be introduced in such a way that the face splits into at most heptagonal large faces. The resulting graph has at least m edges and fulfils the assumptions of the lemma, from which we have

$$m \leqslant \tfrac{1}{5}(11n - 30).$$

(b) Let now $n \equiv 1 \pmod 5$, i.e. $n = 5k+1$ for some $k \geqslant 1$ and let G be a planar N_2-locally disconnected graph of order n and size $m = \lfloor \frac{1}{5}(11n-30) \rfloor = 11k-4$. Clearly, $f_i = 0$ for $i = 4$ and $i \geqslant 8$ since otherwise the same construction as in part (a) of the proof yields a contradiction. Thus, by the lemma,

$$m = 11k - 4 \leqslant \tfrac{1}{5}(11n - 30) - \tfrac{3}{5}f_6 - \tfrac{7}{6}f_7 = 11k - \tfrac{19}{5} - \tfrac{3}{5}f_6 - \tfrac{7}{6}f_7$$

from which necessarily $f_6 = f_7 = 0$. Euler's formula, $f = f_3 + f_5 = 2 - n + m$, then yields

$$f_3 + f_5 = 6k - 3,$$

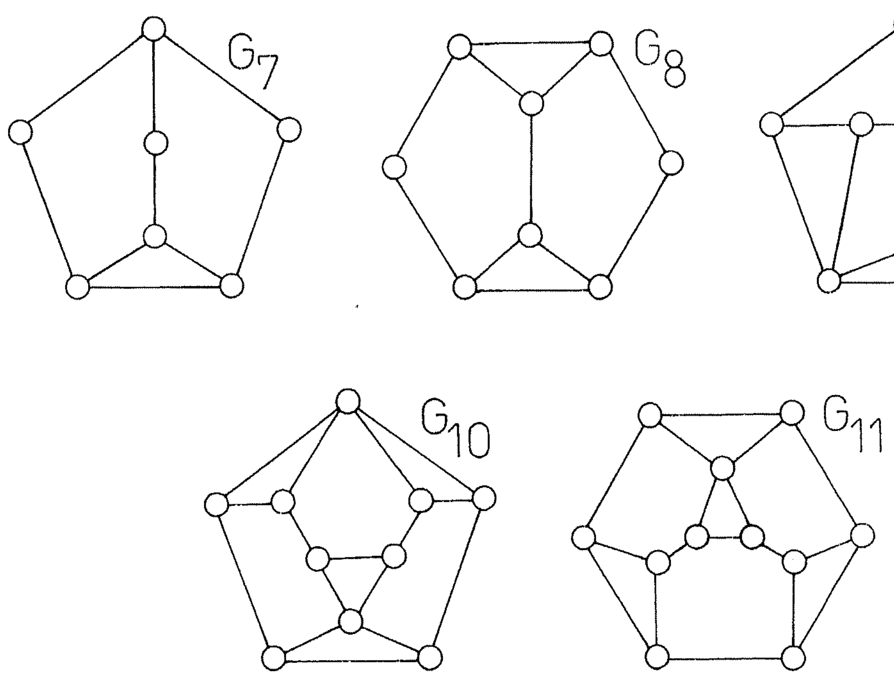

Fig. 1.

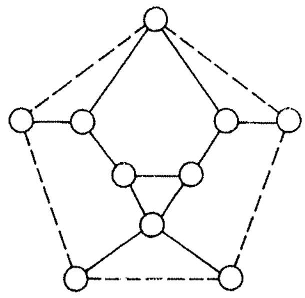

Fig. 2.

from which and from the obvious equality,

$$3f_3 + 5f_5 = 2m = 22k - 8,$$

we have

$$2f_5 = 4k + 1,$$

which contradicts the fact that f_5 is an integer.

Hence, we have proved that if $n \equiv 1 \pmod 5$ then every planar N_2-locally disconnected graph of order n has at most $11k - 5 = \frac{1}{5}(11n - 36)$ edges.

(c) It remains to prove that for every $n \geqslant 5$ there is a planar connected N_2-locally disconnected graph G_n of order n and a given maximum size. For $n = 5, 6$ we put $G_n = C_n$, for $n = 7, 8, 9, 10, 11$ the graphs G_n are shown in Fig. 1; for $n > 11$, we construct the graph G_n by recursively replacing one pentagonal face of G_{n-5} by the graph, which is shown in Fig. 2. It is easy to see that the resulting graphs have required properties. □

3. General case

Denote by $\Delta(G)$ the maximum degree of G.

Theorem 3.1. *Let G be an N_2-locally disconnected graph of order n. Then*

$$\Delta(G) < \frac{n}{2}.$$

Proof. Suppose, on the contrary, that there is a vertex $v \in V(G)$ of degree $d(V) \geqslant n/2$. Denote by M the set of all neighbouring vertices of v and put $\bar{M} = M \cup \{v\}$ and $N = V(G) - \bar{M}$. By the assumption, $|M| > |N|$. If $x \in M$ is an arbitrary vertex, then its neighbourhood $N_2(x, G)$ contains the edges vy for all $y \in M$, $y \neq x$ and from the disconnectedness of $N_2(x, G)$ it follows that there is a vertex $z \in N$ the only neighbour of which in $\bar{M}$ is x. Since x is arbitrary, we have $|N| \geqslant |M|$, which is a contradiction. □

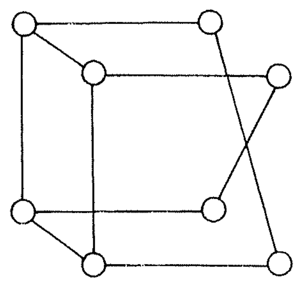

Fig. 3.

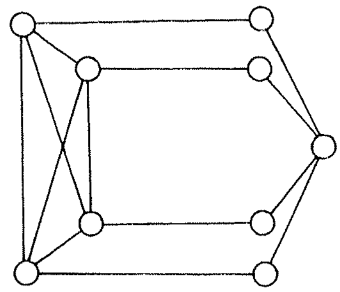

Fig. 4.

Theorem 3.2. *Let $n \geqslant 5$ be an integer. Then*

$$\tfrac{1}{8}(n-1)(n+5) \leqslant t(n) \leqslant \tfrac{1}{4}n(n-1) \quad \text{if } n \text{ is odd},$$

$$\tfrac{1}{8}n(n+2) \leqslant t(n) \leqslant \tfrac{1}{4}n(n-2) \quad \text{if } n \text{ is even}.$$

Proof. (a) Let G be an N_2-locally disconnected graph. From Theorem 3.1 we have $\Delta(G) \leqslant (n-1)/2$ if n is odd and $\Delta(G) \leqslant (n-2)/2$ if n is even. Hence,

$$m = \tfrac{1}{2} \sum_{v \in V(G)} d(v) \leqslant \tfrac{1}{2} n \cdot \Delta(G) \leqslant \begin{cases} \tfrac{1}{4}n(n-1) & \text{if } n \text{ is odd}, \\ \tfrac{1}{4}n(n-2) & \text{if } n \text{ is even}. \end{cases}$$

(b) To prove the lower bounds for $t(n)$, we construct a graph G_n of order n by the following construction.

(α) *n is even.* If we take the disjoint union of an arbitrary graph H of order $n/2$ and of its complement $\bar{H}$ such that both H and $\bar{H}$ have no vertices of degree 0 and join the pairs of corresponding vertices of H and $\bar{H}$ by $n/2$ new edges then the resulting graph of order n is N_2-locally disconnected and has $\frac{1}{8}n(n+2)$ edges (for $n = 8$ and $H = C_4$, see Fig. 3).

(β) *n is odd.* If we take the disjoint union of the complete graph $K_{(n-1)/2}$ and of the star $K_{1,(n-1)/2}$ and join every vertex of the complete graph with one vertex of degree 1 of the star then we obtain an N_2-locally disconnected graph of order n having $\frac{1}{8}(n-1)(n+5)$ edges (for $n=9$, see Fig. 4).

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