Almost Claw-Free Graphs

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ABSTRACT

We say that G is almost claw-free if the vertices that are centers of induced claws ($K_{1,3}$) in G are independent and their neighborhoods are 2-dominated. Clearly, every claw-free graph is almost claw-free. It is shown that (i) every even connected almost claw-free graph has a perfect matching and (ii) every nontrivial locally connected $K_{1,4}$-free almost claw-free graph is fully cycle extendable. © 1994 John Wiley & Sons, Inc.

1. INTRODUCTION

Throughout the paper, a *graph* will be a finite, undirected graph $G = (V(G), E(G))$ without loops and multiple edges. We say that a graph G is *even* if it has even number of vertices; otherwise, we call it *odd*. If $M \subset V(G)$, then $\langle M \rangle$ denotes the induced subgraph on M, $G \backslash M$ stands for $\langle V(G) \backslash M \rangle$, and $c_0(G \backslash M)$ denotes the number of odd components of $G \backslash M$. The *square* G^2 of G has $V(G^2) = V(G)$ and $E(G^2) = \{uv \mid uv \in E(G)$ or $ux \in E(G)$ and $xv \in E(G)$ for some $x \in V(G)\}$. The three-edge star $K_{1,3}$ will be called the *claw* and the complete tripartite graph $K_{1,1,3}$ will be referred to as the *crown* (see Figure 1). If F is a graph, then we say that G is *F-free* if for every induced subgraph H of G we have $H \not\approx F$ (where $\approx$ denotes isomorphism).

A set $A \subset V(G)$ is *independent* if any $x, y \in A$ are nonadjacent. The size of a maximum independent set in G will be denoted by $\alpha(G)$ and referred to as the *independence number* of G. We say that a set $B \subset V(G)$ is a *dominating set* if every vertex of G belongs to B or has a neighbor in B. The size of a minimum dominating set of G will be called *domination number* of G and is denoted by $\gamma(G)$. If $\gamma(G) \leq k$, then we say that G is k-*dominated.* A *universal vertex* is a vertex that is adjacent to all the other vertices of G. Clearly, G is 1-dominated if and only if G has a universal vertex.

Journal of Graph Theory, Vol. 18, No. 5, 469–477 (1994)
 CCC 0364-9024/94/050469-09

The claw The crown

FIGURE 1

A 1-factor of G will be referred to as a *perfect matching*. We say that G is *hamiltonian* if G has a spanning cycle; G is *pancyclic* if for every m, $3 \leq m \leq |V(G)|$, there is a cycle of length m in G; G is *vertex pancyclic* if for any vertex $x \in V(G)$ and for every m, $3 \leq m \leq |V(G)|$, there is a cycle of length m containing x. Finally, G is said to be *fully cycle extendable* (see [3]) if every vertex of G lies on a triangle and for every nonhamiltonian cycle C in G there is a cycle C' in G such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$.

If $x \in V(G)$, then by the *neighborhood of x in G* (denoted by $N(x,G)$) we mean in this paper the induced subgraph on the set of all vertices that are adjacent to x. If $N(x,G)$ is connected (k-connected) for every $x \in V(G)$, then we say that G is *locally connected* (or *locally* k-*connected*). Similarly, G is said to be *locally claw-free* or *locally hamiltonian* if $N(x,G)$, for every $x \in V(G)$, is a claw-free or a hamiltonian graph, respectively; G is *locally* k-*dominated* if $\gamma(N(x,G)) \leq k$ for every $x \in V(G)$.

Claw-free graphs are known to have many interesting properties and have been subject of study of many authors in recent years. The following theorem appeared in [4] and [8].

Theorem A. Every even connected claw-free graph has a perfect matching.

In [5], Oberly and Sumner proved that every connected, locally connected claw-free graph G on at least three vertices is hamiltonian. Clark [2] proved that, under the same conditions, G is vertex pancyclic. Hendry [3] observed that Clark essentially proved the following stronger result.

Theorem B. If G is a connected, locally connected claw-free graph on at least three vertices, then G is fully cycle extendable.

Some further strengthenings of these results can be found in [6] and [7].

Our main goal is to extend Theorems A and B to a certain superclass of the class of claw-free graphs that admits some induced claws.

2. PROPERTIES OF ALMOST CLAW-FREE GRAPHS

It is easy to see that G is claw-free if, and only if, $\alpha(N(x,G)) \leq 2$ for every $x \in V(G)$. This fact gives a motivation for the following definition.

We say that a graph G is *almost claw-free* if there is a (possibly empty) independent set $A \subset V(G)$ such that $\alpha(N(x,G)) \leq 2$ for $x \notin A$ and $\gamma(N(x,G)) \leq 2 < \alpha(N(x,G))$ for $x \in A$. Equivalently, G is almost claw-free if G is locally 2-dominated and the set of all centers of induced claws is independent.

Since $\gamma(H) \leq \alpha(H)$ for every graph H, every claw-free graph is almost claw-free.

Proposition 1.

(i) A graph G is locally claw-free if and only if G is crown-free.
(ii) Every almost claw-free graph is locally claw-free.

Proof.

(i) If a vertex u centers and induced claw $\langle\{u,x,y,z\}\rangle$ in $N(v,G)$, then $\langle\{u,v,x,y,z\}\rangle$ is an induced crown in G. Conversely, for every induced crown in G, one of its vertices of degree 4 centers a claw in the neighborhood of the other one.

(ii) If G contains an induced crown, then its vertices of degree 4 are adjacent and both of them center an induced claw; consequently, G is not almost claw-free.

Example. The graphs in Figure 2 and Figure 5 are examples of locally claw-free graphs that are not almost claw-free.

Corollary 2. If G is almost claw-free, then $\gamma(N(x,G)) = 2$ for every $x \in A$.

Proof. Let $\gamma(N(x,G)) = 1$ for an $x \in A$ and let u be a universal vertex in $N(x,G)$. As $x \in A$, there is an induced claw centered at x, but then its vertices together with the vertex u induce a crown in G.

Corollary 3. Every almost claw-free graph is $K_{1,5}$-free.

Proof. If there is an induced $K_{1,5}$ centered at a vertex $x \in A$ then there is a neighbor of x that is adjacent to at least three of its endvertices (otherwise $N(x,G)$ cannot be 2-dominated) but then we again have an induced crown.

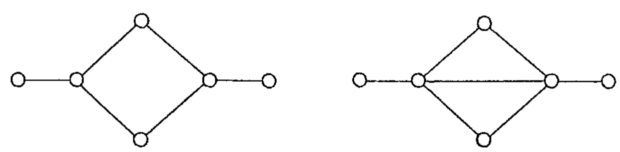

FIGURE 2

Example. The graph depicted in Figure 6 is an almost claw-free graph that is not $K_{1,4}$-free.

The following result appeared in [1].

Theorem C. If G is a k-connected claw-free graph ($k \geq 2$) with $\alpha(G^2) \leq k$, then G is hamiltonian.

From Theorem C we can easily deduce the following two assertions.

Corollary 4. If G is a k-connected claw-free graph ($k \geq 2$) with $\gamma(G) \leq k$, then G is hamiltonian.

Proof. If G is not hamiltonian, then we can choose a set $S \subset V(G)$, $|S| = k + 1$, which is independent in G^2. Let D be a minimum dominating set in G. Since $|D| \leq k$, there are vertices $u_1, u_2 \in S$ and $d \in D$ such that $du_1 \in E(G)$ and $du_2 \in E(G)$, which implies $u_1u_2 \in E(G^2)$, a contradiction.

Proposition 5. Every locally 2-connected almost claw-free graph is locally hamiltonian.

Proof. Follows immediately from Proposition 1 and from Corollary 4.

3. PERFECT MATCHINGS

The following theorem extends Theorem A.

Theorem 6. Every even connected almost claw-free graph has a perfect matching.

Proof. Let G be an even connected almost claw-free graph without any perfect matching. We make use of the following statement, which was proved in [9].

Theorem D. If G is an even connected graph that does not have a perfect matching, then there is a set $S \subset V(G)$ such that $c_0(G \backslash S) > |S|$ and every vertex of S is adjacent to vertices in at least three odd components of $G \backslash S$.

Let $S \subset V(G)$ have the properties given in Theorem D. Then every vertex of S centers an induced claw and, since G is almost claw-free, S is independent. Thus, for any $x \in S$, $N(x, G)$ has at least 3 components, which contradicts the fact that G is locally 2-dominated.

Examples. The graphs in Figure 2 show that Theorem 6 fails if G is only locally 3-dominated, the set A is not independent, or G is only locally claw-free ($\Leftrightarrow$ crown-free).

4. HAMILTONICITY

The following theorem extends Theorem B.

Theorem 7. Every connected, locally connected $K_{1,4}$-free almost claw-free graph on at least three vertices is fully cycle extendable.

Proof. Since every vertex of G lies on a triangle, it is sufficient to prove that for every cycle C of length $m \le |V(G)| - 1$ there is a cycle C' of length $m + 1$ such that $V(C) \subset V(C')$. Throughout the proof, we suppose that for every cycle $C \subset G$, one of its orientations is chosen, and for any $u \in V(C)$, we denote by u^- and u^+ the predecessor and successor of u on C, respectively. For $u, v \in V(C)$, uCv, or $u\overline{C}v$ denotes the u, v-arc of C with the same or opposite orientation with respect to the orientation of C; if $u = v$, then we define both uCv and $u\overline{C}v$ as a single vertex. Whenever vertices of an induced $K_{1,3}$ or $K_{1,4}$ are listed, its center is always the first vertex of the list.

The proof proceeds in a series of steps.

1. We show that for every cycle $C \subset G$ there are vertices $w \in V(C)\backslash A$ and $x \notin V(C)$ such that $xw \in E(G)$. Indeed, by the connectedness of G, there are $v \in V(C)$ and $x \notin V(C)$ such that $xv \in E(G)$. Since G is locally connected, we can find a shortest path Q in $N(v, G)$ joining x to one of v^-, v^+. Let v_1 be the vertex consecutive to x on Q. Then $v_1 \in V(C)$ and $v_1 v \in E(G)$; we denote by w that of the vertices v, v_1 that is not in A.

2. Let a cycle $C \subset G$ and the vertices x, w be chosen in such a way that, among all cycles with vertex set $V(C)$, the path Q that joins x in $N(w, G)$ to one of w^-, w^+ (say, w^+) is shortest possible and suppose that C cannot be extended through x. As $xw^- \notin E(G)$ and $xw^+ \notin E(G)$ (otherwise we can extend C) and w cannot center an induced claw, we have $w^-w^+ \in E(G)$.

Denote by $x = x_0, x_1, \ldots, x_k, x_{k+1} = w^+$ the vertices of Q. By the minimality of Q, $x_i \in V(C)$ for $1 \le i \le k$ and $x_i x_j \notin E(G)$ for $|i - j| \ge 2$. Considering induced claws centered at w, we have $k \le 2$; on the other hand, trivially $k \ge 1$.

3. Suppose first that $k = 2$. Obviously $xw^- \notin E(G)$, and by the minimality of Q, $xx_2 \notin E(G)$; as $\langle w, x, w^-, x_2\rangle \not\simeq K_{1,3}$, we have $w^-x_2 \in E(G)$. Thus, by the symmetry, we can suppose without loss of generality that $x_1 \in x_2^+ C w^-$ (see Figure 3).

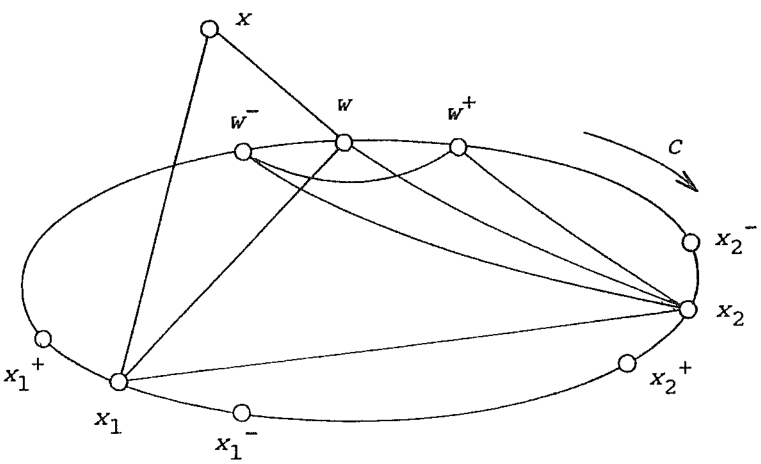

FIGURE 3

We consider the following cases.

Case	*Cycle* C_1
$w^+ = x_2^-$	$wx_2Cw^-w^+w$
$x_2^-x_2^+ \in E(G)$	$wx_2w^+Cx_2^-x_2^+Cw$
$x_2^-w \in E(G)$	$wx_2Cw^-w^+Cx_2^-w$
$x_2^+w \in E(G)$	$wx_2^+Cw^-w^+Cx_2w$

In each of these cases, x_2 and w are consecutive on C_1 and the path $Q_1 = \langle x, x_1, x_2\rangle$ is a x, C-path in $N(w, G)$ with $|V(Q_1)| < |V(Q)|$, which contradicts the minimality of Q. Consequently, neither of these possibilities can occur and hence $\langle x_2, x_2^-, x_2^+, w\rangle \approx K_{1,3}$, which implies $x_2 \in A$. Since A is independent, we have $x_1 \notin A$ and hence obviously $x_1^-x_1^+ \in E(G)$. Now we can easily see that $x_2^+ \neq x_1^-$ and $x_1^+ \neq w^-$ since otherwise the cycles $wCx_2w^-\overline{C}x_1^+x_2^+x_1xw$ and $wCx_1^-x_1^+x_1xw$ extend C. We now consider $\langle x_2, x_2^-, x_2^+, x_1, w^-\rangle$.

Case	*Cycle* C' *of Length* $|V(C)| + 1$
$x_1w^- \in E(G)$	$wCx_1^-x_1^+Cw^-x_1xw$
$x_1x_2^- \in E(G)$	$wxx_1x_2^-\overline{C}w^+x_2Cx_1^-x_1^+Cw$
$x_1x_2^+ \in E(G)$	$wxx_1x_2^+Cx_1^-x_1^+Cw^-x_2\overline{C}w$
$w^-x_2^- \in E(G)$	$wxx_1x_2Cx_1^-x_1^+Cw^-x_2^-\overline{C}w$

Since also $x_2^-x_2^+ \notin E(G)$ and $\langle x_2, x_2^-, x_2^+, x_1, w^-\rangle \not\approx K_{1,4}$, we have $w^-x_2^+ \in E(G)$. But then, as obviously $xx_1^+ \notin E(G)$, $xx_2 \notin E(G)$, $x_1 \notin A$, and $\langle x_1, x, x_1^+, x_2\rangle \not\approx K_{1,3}$, necessarily $x_2x_1^+ \in E(G)$ and the cycle $wCx_2x_1^+Cw^-x_2^+Cx_1xw$ again extends C. This contradiction proves that $k \neq 2$ and, hence, by (2), $k = 1$.

4. We easily observe that $x_1^+ \neq w^-$, $x_1^- \neq w^+$, $xx_1^- \notin E(G)$ and $xx_1^+ \notin E(G)$. If $x_1^- x_1^+ \in E(G)$, then the cycle $wxx_1w^+Cx_1^-x_1^+Cw$ extends C; therefore also $x_1^- x_1^+ \notin E(G)$, which implies that $\langle x_1, x_1^-, x_1^+, x\rangle \approx K_{1,3}$, and consequently, $x_1 \in A$. Moreover, as obviously $xw^+ \notin E(G)$ and $x_1^+ w^+ \notin E(G)$ (otherwise the cycle $wxx_1\overline{C}w^+x_1^+Cw$ extends C) and since $\langle x_1, x_1^-, x_1^+, w^+, x\rangle \not\approx K_{1,4}$, we have $x_1^- w^+ \in E(G)$ (see Figure 4).

5. We show that the vertices x_1^- and x_1^+ have a common neighbor $d \in N(x_1, G)$. If, on the contrary, no such vertex exists, then, since $N(x_1, G)$ is 2-dominated and $x \in N(x_1, G)$, there is a vertex $u \in N(x_1, G)$ that is adjacent to x and to one of x_1^+, x_1^- (say, x_1^+; the second case is similar). If $u \notin V(C)$ then we can extend C replacing the edge $x_1x_1^+$ by the path $x_1ux_1^+$; thus $u \in V(C)$. As $ux_1 \in E(G)$ and $x_1 \in A$, u cannot center a claw and, consequently, $u^-u^+ \in E(G)$; but then, since obviously $x_1^- \neq u \neq x_1^+$, the replacement of u^-uu^+ and $x_1x_1^+$ by u^-u^+ and $x_1xux_1^+$ extends C. Hence x_1^- and x_1^+ have a common neighbor $d \in N(x_1, G)$ and, obviously, $d \in V(C)$.

6. Suppose that $d \in w^+Cx_1^-$ and consider $\langle d, d^-, d^+, x_1\rangle$.

Case	*Cycle C' of Length $\lvert V(C)\rvert + 1$*
$d^-d^+ \in E(G)$	$wxx_1w^+Cd^-d^+Cx_1^-dx_1^+Cw$
$d^-x_1 \in E(G)$	$wxx_1d^-\overline{C}w^+x_1^-\overline{C}dx_1^+Cw$
$d^+x_1 \in E(G)$	$wxx_1d^+Cx_1^-w^+Cdx_1^+Cw$

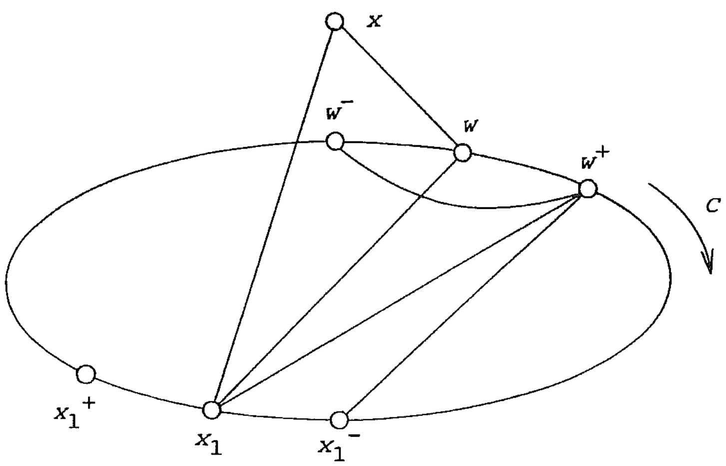

FIGURE 4

As each of these cases yields a contradiction and $\langle d, d^-, d^+, x_1 \rangle \not\approx K_{1,3}$, necessarily $d \in x_1^+ C w^-$.

7. We now consider $\langle w^+, x_1, w^-, w^{++} \rangle$. As clearly $w^{++} \neq x_1^-$, $w^- w^{++} \notin E(G)$ (otherwise $w x x_1 C w^- w^{++} C x_1^- w^+ w$ extends C) and $\langle w^+, x_1, w^-, w^{++} \rangle \not\approx K_{1,3}$, we see that $x_1 w^- \in E(G)$ or $x_1 w^{++} \in E(G)$.

If $x_1 w^- \in E(G)$ then, since $\langle x_1, x_1^-, x_1^+, w^-, x \rangle \not\approx K_{1,4}$, we have $x_1^+ w^- \in E(G)$ and, observing $\langle d, d^-, d^+, x_1 \rangle$, we have the following possibilities:

Case	*Cycle C' of Length* $\lvert V(C) \rvert + 1$
$d^- d^+ \in E(G)$	$x w C x_1^- d x_1^+ C d^- d^+ C w^- x_1 x$
$d^- x_1 \in E(G)$	$x w C x_1^- d C w^- x_1^+ C d^- x_1 x$
$d^+ x_1 \in E(G)$	$x w C x_1^- d \overline{C} x_1^+ w^- \overline{C} d^+ x_1 x$.

Thus, $\langle d, d^-, d^+, x_1 \rangle \approx K_{1,3}$, which is a contradiction. Hence we have $x_1 w^- \notin E(G)$ and, consequently, $x_1 w^{++} \in E(G)$, which implies $w^{++} \in N(x_1, G)$.

8. Similarly as in (5) we can show that x and w^{++} have no common neighbor in $N(x_1, G)$ and hence, as $N(x_1, G)$ is 2-dominated, we can assume without loss of generality that $w^{++} d \in E(G)$. We observe $\langle d, d^-, d^+, x_1^- \rangle$.

Case	*Cycle C' of Length* $\lvert V(C) \rvert + 1$
$d^- d^+ \in E(G)$	$w x x_1 w^+ C x_1^- d x_1^+ C d^- d^+ C w$
$d^+ x_1^- \in E(G)$	$w x x_1 C d w^{++} C x_1^- d^+ C w^- w^+ w$
$d^- x_1^- \in E(G)$	$w x x_1 w^+ C x_1^- d^- \overline{C} x_1^+ d w$

Thus, $\langle d, d^-, d^+, x_1^- \rangle \approx K_{1,3}$. This contradiction completes the proof.

Examples. The graphs in Figure 5 show that Theorem 7 fails if G is only locally 3-dominated, the set A is not independent, or G is only locally claw-free ($\Leftrightarrow$ crown-free). The graph in Figure 6 shows that Theorem 7 fails if G is locally connected and almost claw-free but not $K_{1,4}$-free.

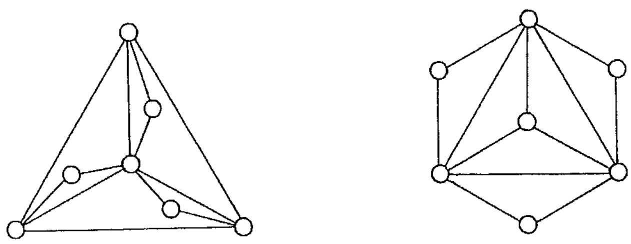

FIGURE 5

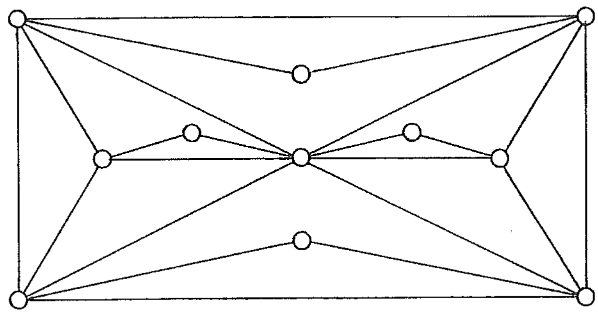

FIGURE 6

ACKNOWLEDGMENT

The author is grateful to one of the referees whose suggestion led to a strengthening of Theorem 7.

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