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Hamiltonicity in claw-free graphs through induced bulls

Zdeněk Ryjáček

Department of Mathematics, University of West Bohemia, 306 14 Plzeň, Czech Republic

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Abstract

In the present paper we show that if G is a 2-connected claw-free graph such that the vertices of degree 1 of every induced bull have a common neighbour in G then G is hamiltonian. This statement was originally conjectured by H.J. Broersma and H.J. Veldman

1. Introduction

Throughout this paper, a *graph* will be a finite undirected graph without loops and multiple edges, $V(G)$ and $E(G)$ its vertex and edge sets, respectively. For $M \subset V(G)$, $\langle M \rangle$ stands for the induced subgraph on M; for $G_1 \subset G$ we denote $G - G_1 = \langle V(G) - V(G_1) \rangle$. A graph G is said to be *hamiltonian* if G contains a cycle of length $|V(G)|$. A complete subgraph (not necessarily maximal) of G will be referred to as a *clique*. The *claw* is the three-edge star $K_{1,3}$ and the *bull* is the only graph B with degree sequence 3, 3, 2, 1, 1 (see Fig. 1).

An induced subgraph H of G that is isomorphic to the claw or to the bull will be called an *induced claw* or *induced bull*; in this case we write $H \approx K_{1,3}$ or $H \approx B$, respectively. A graph is said to be *claw-free* if it contains no induced claw. For a set $M \subset V(G)$ we denote $N(M) = \{y \in V(G) - M \,|\, xy \in E(G) \text{ for some } x \in M\}$. Finally, we say that vertices $x, y \in V(G)$ *have a common neighbour* if $N(x) \cap N(y) \neq \emptyset$.

2. Main result

In this section we prove the following theorem that was conjectured in [1] (see also [2, p. 136]).

Theorem. *Let G be a 2-connected claw-free graph. If for every induced bull B in G the vertices of degree 1 in B have a common neighbour in G, then G is hamiltonian.*

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the claw $K_{1,3}$ the bull B

Fig. 1.

Proof. Suppose that G satisfies the hypothesis of the theorem and is not hamiltonian and choose a longest cycle C in G with a fixed orientation. Throughout the proof we denote by v^- and v^+ the predecessor and successor of a vertex $v \in V(C)$. For $u, v \in V(C)$ we denote by uCv (or $u\overleftarrow{C}v$) the u,v-segment of C with the same (opposite) orientation with respect to the orientation of C. For $u = v$ we define both uCv and $u\overleftarrow{C}v$ as a single vertex. Whenever vertices of an induced claw or bull are listed, they are ordered to form a nonincreasing degree sequence; thus, for an induced claw, its centre is the first and, for an induced bull, the 'tips of its horns' are the last vertices of the list.

We first prove the following auxiliary assertion.

Claim 1. *Let v, x be such that $v \in V(C)$, $x \notin V(C)$ and $xv \in E(G)$. Denote $Y = \{y \in V(G) | yv \in E(G), y = x$ and $yx \notin E(G)\}$, $Z = \{z \in V(G) - (\{v\} \cup Y) | N(z) \cap Y \neq \emptyset\}$. Then*

(i) $\{v^-, v^+\} \subset Y$ *and* $v^-v^+ \in E(G)$,

(ii) $\langle \{v\} \cup Y \rangle$ *is a clique and* $Y \subset V(C)$,

(iii) *if* $y \in Y$ *and* $y \notin \{v^-, v^+\}$, *then* $y^-y^+ \in E(G)$,

(iv) (a) $N(x) \cap N(Y) = \{v\}$;

(b) *there is no path* $vxu_1 \cdots u_k y$ $(k \geqslant 1)$ *such that* $y \in Y$ *and* $u_i \notin V(C)$ *for* $i = 1, \ldots, k-1$,

(v) (a) $N(x) \cap N(Z) = \emptyset$;

(b) *there is no path* $vxu_1 \cdots u_k z$ $(k \geqslant 1)$ *such that* $z \in Z$ and $u_i \notin V(C)$ *for* $i = 1, \cdots, k-1$,

(vi) $\langle Y \cup Z \rangle$ *is a clique.*

Proof. (i) By the maximality of C, obviously $xv^- \notin E(G)$ and $xv^+ \notin E(G)$, thus $\{v^-, v^+\} \subset Y$. Since $\langle v, x, v^-, v^+ \rangle \not\approx K_{1,3}$, we have $v^-v^+ \in E(G)$.

(ii) If $y_1y_2 \notin E(G)$ for some $y_1, y_2 \in Y$ then $\langle v, x, y_1, y_2 \rangle \approx K_{1,3}$, a contradiction. But then since v^- and $v^+ \in Y$, $Y \subset V(C)$ by the maximality of C.

(iii) Let $y \in Y$, $y \notin \{v^-, v^+\}$, and suppose that $y^-y^+ \notin E(G)$. As $\langle y, y^-, y^+, v \rangle \not\approx K_{1,3}$, v is adjacent to at least one of the vertices y^-, y^+; on the other hand, if v is adjacent to both y^- and y^+, then, as obviously $xy^- \notin E(G)$ (otherwise the cycle $vyCv^-v^+Cy^-xv$ contradicts the maximality of C) and, similarly, $xy^+ \notin E(G)$, $\langle v, x, y^-, y^+ \rangle \approx K_{1,3}$ — a contradiction. Hence, by symmetry, we can assume that

$vy^- \in E(G)$ and $vy^+ \notin E(G)$ (not excluding the possible case $y^- = v^+$). We now consider $\langle v, y, y^-, y^+, x\rangle$. As $\{y, y^-, y^+\} \cap N(x) = \emptyset$ and, by the assumption, $y^-y^+ \notin E(G)$ and $vy^+ \notin E(G)$, $\langle v, y, y^-, y^+, x\rangle \approx B$. By the assumption of the theorem, x and y^+ have a common neighbour u. If $u \notin V(C)$ then the cycle $vxuy^+Cv^-v^+Cyv$ contradicts the maximality of C. Thus $u \in V(C)$. But then, by (i), $u^-u^+ \in E(G)$, which again yields a contradiction using the cycle $vxuy^+Cu^-u^+Cv^-v^+Cyv$ if $u \in V(y^+Cv^-)$, or $vxuy^+Cv^-v^+Cu^-u^+Cyv$ if $u \in V(v^+Cy^-)$, respectively. Hence we have $y^-y^+ \in E(G)$.

(iv) (a) Suppose, on the contrary, that $u \in N(x) \cap N(y)$, $u \neq v$, for some $y \in Y$. If $u \notin V(C)$ then clearly $y \notin \{v^-, v^+\}$ (otherwise we find a cycle longer than C replacing the edge vy by the path $vxuy$) and hence, by (iii), $y^-y^+ \in E(G)$. But then again, as clearly $yv^+ \in E(G)$, the cycle $vxuyv^+Cy^-y^+Cv$ contradicts the maximality of C; thus $u \in V(C)$. (Remember: $v^+ \in Y$ and by (ii), $\{v\} \cup Y$ is a clique. Thus y is adjacent to v^+.) Obviously $u \notin \{v^-, v^+\}$. It is also easy to see that $u \neq y^-$ (otherwise cycle $vyCv^-v^+Cuxv$ contradicts the maximality of C) and similarly $u \neq y^+$. If $y = v^-$, then cycle $vxuy\overleftarrow{C}u^+u^-\overleftarrow{C}v$ is longer than C and if $y = v^+$, then cycle $vxuyCu^-u^+Cv$ is longer than C, thus $y \notin \{v^-, v^+\}$. By (i) and (iii), we have $u^-u^+ \in E(G)$ and $y^-y^+ \in E(G)$ which again yields a contradiction using the cycle $vxuyv^+Cu^-u^+Cy^-y^+Cv$ (if $u \in V(v^+Cy^-)$) or $vxuyv^+Cy^-y^+Cu^-u^+Cv$ (if $u \in V(y^+Cv^-)$).

(iv) (b) can be proved by the same argument.

(v) (a) Let, on the contrary, $u \in N(x) \cap N(z)$ for some $z \in Z$ and choose a $y \in Y$ such that $yz \in E(G)$. Since by (iv)(a), $z \notin Y$, necessarily $u \neq v$. By the definition of Y, $u \notin Y$. By (iv)(b) (for $k = 2$, $u_1 = u$ and $u_2 = z$), necessarily $u \in V(C)$. Now, if $z \notin V(C)$, then, since cleary $u^-x \notin E(G)$, $u^-z \notin E(G)$ and $\langle u, u^-, x, z\rangle \not\approx K_{1,3}$, we have $xz \in E(G)$ which contradicts (iv). Hence both u and z are on C.

Suppose first that $y = v^+$. If z and u are not consecutive on C, then, by (i), $u^-u^+ \in E(G)$ and, by (iii) (applied to the vertex u), $z^-z^+ \in E(G)$. But then we can find a cycle longer than C replacing vy, z^-zz^+ and u^-uu^+ by $vxuzy$, z^-z^+ and u^-u^+. If $z = u^+$ then obviously $vxu\overleftarrow{C}yzCv$ is longer than C; hence we have $z = u^-$. We consider $\langle v, y, v^-, x, y^+\rangle$. Obviously $xv^- \notin E(G)$, $xy \notin E(G)$ and, by (iv)(a), $xy^+ \notin E(G)$. In the case $vy^+ \in E(G)$ the cycle $vxuCv^-yz\overleftarrow{C}y^+v$ and in the case $v^-y^+ \in E(G)$ the cycle $vxuCv^-y^+Czyv$ is longer than C and hence $\langle v, y, v^-, x, y^+\rangle \approx B$. By the assumption of the theorem, x and y^+ have a common neighbour a. But then, by the maximality of C, $a \in V(C)$ and, by (i), $a^-a^+ \in E(G)$, so again we can get a cycle longer than C replacing v^-v, yy^+ and a^-aa^+ by v^-y, $vxay^+$ and a^-a^+.

Thus we have $y \neq v^+$. Similarly we can show that $y \neq v^-$ and hence, by (iii), we have $y^-y^+ \in E(G)$.

Now, as $u, z \notin Y$, neither u and v nor z and v can be consecutive on C. By (iv), $uy \notin E(G)$ and hence also u and y are not consecutive on C. If u and z are consecutive on C then we can find a cycle longer than C by removing uz, v^-vv^+ and y^-yy^+ and by adding $uxvyz$, v^-v^+ and y^-y^+ (not excluding the possible cases $z = y^-$ or $z = y^+$). Thus $z \notin \{u^-, u^+\}$. By (i), we have $u^-u^+ \in E(G)$ and, by (iii), $z^-z^+ \in E(G)$. It remains to consider (up to symmetry) the following cases.

Case	Cycle of length $\|V(C)\|+1$
$z=y^+, u\in V(z^+Cv^-)$	$vxuzCu^-u^+Cv^-v^+Cyv$
$z=y^+, u\in V(v^+Cy^-)$	$vxuzCv^-v^+Cu^-u^+Cyv$
$z\notin\{y^-,y^+\}, u\in V(v^+Cy^-), z\in V(y^+Cv^-)$	$vxuzyv^+Cu^-u^+Cy^-y^+Cz^-z^+Cv$
$z\notin\{y^-,y^+\}, u\in V(y^+Cv^-), z\in V(y^+Cu^-)$	$vxuzyv^+Cy^-y^+Cz^-z^+Cu^-u^+Cv$
$z\notin\{y^-,y^+\}, u\in V(y^+Cv^-), z\in V(u^+Cv^-)$	$vxuzyv^+Cy^-y^+Cu^-u^+Cz^-z^+Cv$

Thus, (v)(a) is proved.

(v) (b) Suppose that there is such a path $vxu_1\ldots u_kz$ for some $z\in Z$ and let again $y\in Y$ be such that $yz\in E(G)$. As $z\notin Y$, we have $u_k\neq v$ and, by (iv)(b), $u_k\notin Y$ and $u_k\in V(C)$. If $z\notin V(C)$, then from $u_k^-u_{k-1}\notin E(G), u_k^-z\notin E(G)$ and $\langle u_k,u_k^-,u_{k-1},z\rangle\not\approx K_{1,3}$ we have $u_kz\in E(G)$ which again contradicts (iv)(b). Hence both u_k and z are on C. The remainder of the proof of (v)(b) is quite analogous to that of the part (v)(a) (replacing the edge xu by the path $xu_1\ldots u_k$) and is therefore omitted.

(vi) We prove that $\langle Y\cup Z\rangle$ is a clique.
Suppose first that there is a $z\in Z$ and a $y\in Y$ which are nonadjacent. Choose a $y'\in N(z)\cap Y$ and consider $\langle v,y',y,x,z\rangle$: by the construction, $xy'\notin E(G)$ and $xy\notin E(G)$, by (iv)(a), $xz\notin E(G)$, by (v)(a), $vz\notin E(G)$ and hence $\langle v,y',y,x,z\rangle\approx B$, but then, by the assumption of the theorem, x and z have a common neighbour which contradicts (v). Thus $N(z)\supset Y$ for every $z\in Z$. Now, if there are $z_1,z_2\in Z$ such that $z_1z_2\notin E(G)$ then, for any $y\in Y$, $\langle y,v,z_1,z_2\rangle\approx K_{1,3}$ by (v)(a). Hence $\langle Y\cup Z\rangle$ is a clique. This completes the proof of Claim 1. □

Since G is connected, we can choose vertices x,w such that $x\notin V(C)$, $w\in V(C)$ and $xw\in E(G)$; by part (i) of Claim 1, $w^-w^+\in E(G)$. We claim the following.

Claim 2. *There is a system of cliques $K_0,K_1,\ldots,K_k$ in G such that*

(i) $K_0=\{w\}$, $V(K_k)\neq\emptyset$ *and* $|V(K_i)|\geqslant 2$ *for* $1\leqslant i\leqslant k-1$,

(ii) $V(K_i)\cap V(K_j)=\emptyset$ *for* $i\neq j$,

(iii) *for every* i, $1\leqslant i\leqslant k-1$,

$$x_i\in V(K_i)\Rightarrow N(x_i)=V(K_{i-1})\cup V(K_i)\cup V(K_{i+1})-\{x_i\},$$

(iv) $V(K_0)\cup V(K_1)\cup\cdots\cup V(K_k)=V(C)$.

Proof. We construct the cliques $K_0,K_1,\ldots,K_k$ by induction.

(1) Put $K_0=\{w\}$ and $K_1=\langle Z_1\rangle$, where $Z_1=\{z\in V(G)-V(K_0)|zw\in E(G)$ and $zx\notin E(G)\}$. (Note that $\{w^+,w^-\}\subset K_1$.) Then, by part (ii) of Claim 1 (where w replaces v and $Z_1=Y$), we have that $\langle V(K_0)\cup V(K_1)\rangle$ is a clique and

$V(K_0) \cup V(K_1) \subset V(C)$. If equality holds then we are done; otherwise $Z_2 = \{z \in V(G) - [V(K_0) \cup V(K_1)] \mid N(z) \cap V(K_1) \neq \emptyset\} \neq \emptyset$ and we can put $K_2 = \langle Z_2 \rangle$. By part (vi) of Claim 1, $\langle V(K_1) \cup V(K_2) \rangle$ is a clique. By the construction, the cliques K_0, K_1, K_2 satisfy (i)–(iii) of Claim 2 and, by the maximality of C, $V(K_0) \cup V(K_1) \cup V(K_2) \subset V(C)$.

(2) Suppose that we have already obtained cliques $K_0, K_2, \ldots, K_n$ ($n \geqslant 2$) satisfying the conditions (i)–(iii). By the maximality of C, $V(K_0) \cup \cdots \cup V(K_n) \subset V(C)$. If equality holds then we are done, hence we can suppose that $V(C) - [V(K_0) \cup \cdots \cup V(K_n)] \neq \emptyset$. (Note that this implies that $|V(K_n)| \geqslant 2$.) Let $Z_{n+1} = \{z \in V(G) - [V(K_0) \cup \cdots \cup V(K_n)] \mid N(z) \cap V(K_n) \neq \emptyset\}$. Before showing that $\langle V(K_n) \cup Z_{n+1} \rangle$ is a clique, we first prove the following assertion concerning the case $n = 2$.

If $n = 2$ then $N(x) \cap N(Z_3) = \emptyset$.

Suppose $n = 2$ and, on the contrary, $u \in N(x) \cap N(z)$ for some $z \in Z_3$. By part (v)(b) of Claim 1, necessarily $u \in V(C)$ and, since $\langle u, u^-, z, x \rangle \not\simeq K_{1,3}$, we have also $z \in V(C)$ (otherwise $zx \in E(G)$ which contradicts (v) of Claim 1). Note that by part (i) of Claim 1, $u^-u^+ \in E(G)$. Choose a vertex $v \in N(z) \cap V(K_2)$.

We first treat the special cases when some of the vertices u, z, v, w^+, w^-, w are consecutive on C (except the obvious consecutive pairs ww^+ and ww^-). Obviously, $uw \notin E(C)$ and $zw \notin E(C)$. The case $uv \in E(C)$ contradicts (v)(a) and the cases $uw^+ \in E(C)$ and $uw^- \in E(C)$ contradict (iv)(a) of Claim 1. The cases $zw^+ \in E(C)$ and $zw^- \in E(C)$ imply $z \in V(K_2)$ and the case $vw \in E(C)$ implies $v \in V(K_1)$, which is impossible. Thus, the only possible special cases are $uz \in E(C), zv \in E(C), vw^+ \in E(C)$ or $vw^- \in E(C)$.

Suppose first that u, z are consecutive on C. If, e.g., $z = u^+$, i.e., $vu^+ \in E(G)$, then, by part (vi) of Claim 1 (in which u replaces v), we have also $vu^- \in E(G)$; thus the cases $z = u^-$ and $z = u^+$ are equivalent. By the symmetry, we can assume without loss of generality that $v \in V(u^+Cw^-)$. Now, we have $v^+ \neq w^-$ since otherwise the cycle $wxuCvu^-\overleftarrow{C}w^+w^-w$ extends C. We consider $\langle v, v^-, v^+, w^+ \rangle$:

Case	Cycle of length $\lvert V(C)\rvert + 1$
$v^-w^+ \in E(G)$	$wxuCv^-w^+Cu^-vCw$
$v^+w^+ \in E(G)$	$wxuCvu^-\overleftarrow{C}w^+v^+Cw$
$v^-v^+ \in E(G)$	$wxu\overleftarrow{C}w^+vu^+Cv^-v^+Cw$

Hence u and z cannot be consecutive on C.

By part (iii) of Claim 1, we now have $z^-z^+ \in E(G)$ and, by part (ii) of Claim 1, z is adjacent to both u^- and u^+. Now, z, v cannot be consecutive, for otherwise, if, e.g., $zv \in E(u^+Cw^-)$ then we can find a cycle longer than C by removing ww^+, u^-u and vz and by adding wxu, vw^+ and zu^-; the second case is similar. Finally, neither

$vw^- \in E(C)$ nor $vw^+ \in E(C)$:

Case	Cycle of length $\lvert V(C)\rvert + 1$
$v^+ = w^-, z \in V(u^+Cv^-)$	$wxuCz^-z^+Cvzu^-\overleftarrow{C}w^+w^-w$
$v^+ = w^-, z \in V(w^+Cu^-)$	$wxuCvzu^-\overleftarrow{C}z^+z^-\overleftarrow{C}w^+w^-w$
$v^- = w^+, z \in V(u^+Cw^-)$	$wxu\overleftarrow{C}vzu^+Cz^-z^+Cw^-w^+w$
$v^- = w^+, z \in V(v^+Cu^-)$	$wxu\overleftarrow{C}z^+z^-\overleftarrow{C}vzu^+Cw^-w^+w$

Thus, no two of the vertices u, z, v, w^-, w^+ are consecutive on C, but then we can again find a cycle longer than C by removing $u^-uu^+, v^-vv^+, z^-zz^+$ and ww^+ and by adding u^-u^+, v^-v^+, z^-z^+ and $wxuzvw^+$. This contradiction proves that $N(x) \cap N(Z_3) = \emptyset$.

We now show, that, for any $n \geqslant 2$, $\langle V(K_n) \cup Z_{n+1} \rangle$ is a clique. Suppose first that, on the contrary, $zy \notin E(G)$ for some $z \in Z_{n+1}$ and $y \in V(K_n)$, choose vertices $y_1 \in V(K_n) \cap N(z)$, $v_1 \in V(K_{n-1})$ and $v_2 \in V(K_{n-2})$ and consider $\langle v_1, y_1, y, z, v_2 \rangle$. The induction hypothesis (iii) implies $v_2y_1 \notin E(G)$, $v_2y \notin E(G)$ and $v_1z \notin E(G)$ for $n \geqslant 2$ and $v_2z \notin E(G)$ for $n \geqslant 3$. By part (v)(a) of Claim 1, $v_2z \notin E(G)$ for $n = 2$. Hence $\langle v_1, y_1, y, z, v_2 \rangle \approx B$ and, by the assumption of the theorem, there is a vertex $u \in N(v_2) \cap N(z)$. But then, if $n = 2$, we have $v_2 = w$ and since $wu \in E(G)$, by the definition of K_1, necessarily $u = x$ or $ux \in E(G)$ which contradicts part (v)(a) of Claim 1 or the assertion $N(x) \cap N(Z_3) = \emptyset$, respectively. If $n = 3$ then, by part (iii) of the induction hypothesis, the only possibility is $u = w$ which yields $zx \in E(G)$, contradicting again the assertion $N(z) \cap N(Z_3) = \emptyset$. For $n \geqslant 4$, since $v_2 \in V(K_{n-2})$, by part (iii) of the induction hypothesis we have $u \in V(K_{n-3}) \cup V(K_{n-2}) \cup V(K_{n-1})$ which contradicts the fact that $uz \in E(G)$ and $z \in Z_{n+1}$. Hence $N(z) \supset V(K_n)$ for every $z \in Z_{n+1}$. Now, considering $\langle y, v, z_1, z_2 \rangle$ for any $z_1, z_2 \in Z_{n+1}, y \in V(K_n)$ and $v \in V(K_{n-1})$, we have $z_1z_2 \in E(G)$. Consequently, $\langle V(K_n) \cup Z_{n+1} \rangle$ is a clique.

By the construction, the cliques $K_0, K_1, \ldots, K_{n+1}$ satisfy the conditions (i)–(iii) and, by the maximality of C, $V(K_0) \cup \cdots \cup V(K_{n+1}) \subset V(C)$. Since $V(C)$ is finite, the construction yields after a finite number of steps a system of cliques satisfying all the conditions (i)–(iv) of Claim 2.

The graph G is 2-connected and hence there is a vertex $w' \in V(C)$ and an x, w'-path P such that $w \notin V(P)$ and no vertex of P except w' is on C. By (iii) of Claim 2, $w' \in V(K_k)$, but then, by the properties of the cliques $K_0, K_1, \ldots, K_k$, there is a w, w-path P' such that $V(P') = V(K_0) \cup \cdots \cup V(K_k) = V(C)$ and the paths P and P' together with the edge wx yield a cycle that is longer than C. This contradiction proves the theorem. □

Remark. The graph depicted in Fig. 2 shows that the assumptions of the theorem do not imply pancyclicity of G.

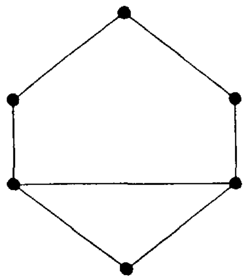

Fig. 2.

Acknowledgements

The author was recently informed that during the refereeing process of this paper Zhiquan Hu proved a stronger version of the conjecture (Conjecture 2 of [1]).

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