On the independence number in $K_{1,r+1}$-free graphs

Zdeněk Ryjáček *
Department of Mathematics
University of West Bohemia
Americká 42
306 14 Plzeň
Czech Republic

Ingo Schiermeyer *
Lehrstuhl C für Mathematik
Rhein.-Westf. Techn. Hochschule
Templergraben 55
D-52062 Aachen
Germany

March 3, 1994

Abstract

In this paper we use the degree sequence, order, size and vertex connectivity of a $K_{1,r+1}$-free graph or of an almost claw-free graph to obtain several upper bounds on its independence number. We also discuss the sharpness of these results.

1. INTRODUCTION

In this paper, a *graph* will be a finite undirected graph without loops and multiple edges. For notation and terminology not defined here we refer to [1]. Throughout the paper, we denote by $n = |V(G)|$ the order, by $m = |E(G)|$ the size and by $\delta(G)$ (or simply δ) the minimum degree of G. For any $A, B \subset V(G)$ we put $e(A, B) = \{xy \in E(G)|\ x \in A,\ y \in B\}$, $N_A(B) = \{x \in A|\ xy \in E(G)$ for some $y \in B\}$ and, for $x \in V(G)$, $d_A(x) = |N_A(x)|$; $\langle A \rangle$ denotes the induced subgraph on A and $G \setminus A$ stands for $\langle V(G) \setminus A \rangle$.

A set $A \subset V(G)$ is *independent* if $xy \notin E(G)$ for any $x, y \in A$. The size of a maximum independent set in G is denoted by $\alpha(G)$ and referred to as the *independence number of* G. A set $B \subset V(G)$ is *dominating* if every vertex of G belongs to B or has a neighbour

*This research was done partly while the second author was visiting the University of West Bohemia and partly while the first author was visiting RWTH Aachen. Research supported by EC-grant No 927.

in B. The size of a minimum dominating set is called the *domination number of* G and denoted by $\gamma(G)$. If $\gamma(G) \leq k$, we say that G is *k-dominated.*

G is said to be $K_{1,r+1}$-free ($r \geq 2$) if G does not contain an induced subgraph which is isomorphic to the star $K_{1,r+1}$. In the special case $r = 2$ we say that G is *claw-free* and the star $K_{1,3}$ will be also called the *claw.*

In [6], the class of claw-free graphs was extended in the following way: we say that G is *almost claw-free* if there is an independent set $A \subset V(G)$ such that $\alpha(\langle N(x)\rangle) \leq 2$ for $x \in V(G) \setminus A$ and $\gamma(\langle N(x)\rangle) \leq 2 < \alpha(\langle N(x)\rangle)$ for $x \in A$. Equivalently, G is almost claw-free if the centres of induced claws are independent and their neighbourhoods are 2-dominated. Clearly, every claw-free graph is almost claw-free. It can be shown (see [6]) that every almost claw-free graph is $K_{1,5}$-free and $K_{1,1,3}$-free and that, for every $x \in A$, $\gamma(\langle N(x)\rangle) = 2$.

$K_{1,r+1}$-free and, especially, claw-free graphs are known to have many interesting properties. Gernert [3] proved that in 2-connected claw-free graphs, $\gamma(G) \leq \lceil n/3 \rceil$. Since for every claw-free graph G trivially $\alpha(G) \leq 2\gamma(G)$ (otherwise, if $\alpha(G) > 2\gamma(G)$, necessarily some vertex in the minimum dominating set dominates at least three independent vertices and we have a claw), we see that in 2-connected claw-free graphs $\alpha(G) \leq 2\lceil n/3 \rceil$. Li and Virlouvet [5] have shown that for every claw-free graph G, $\alpha(G) \leq 2n/(\delta+2)$. In [2] these results were extended to $K_{1,r+1}$-free graphs.

In the present paper we proceed with this work. We prove several upper bounds for the independence number of $K_{1,r+1}$-free and almost claw-free graphs and discuss their sharpness.

2. RESULTS

Our first theorem gives an upper bound on the independence number of a $K_{1,r+1}$-free graph in terms of its numbers of vertices and edges.

Theorem 1. Let G be a $K_{1,r+1}$-free graph ($r \geq 2$) having n vertices and m edges. Then

$$\alpha(G) = 1 \text{ if } m = \tbinom{n}{2},$$

$$\alpha(G) \leq \frac{1}{2}\left(2n + 2r - 1 - \sqrt{8m + (2r-1)^2}\right) \text{ if } 0 \leq m < \tbinom{n}{2},$$

and this bound is sharp.

Proof. (i) If $m = \binom{n}{2}$, then $G \simeq K_n$ and thus $\alpha(G) = 1$. Thus let $m < \binom{n}{2}$; then $\alpha(G) \geq 2$. Let $I \subset V(G)$ be an independent set of size α and let $R = V(G) \setminus I$. Then, since G is $K_{1,r+1}$-free, we have $1 \leq d_I(v) \leq r$ for every $v \in R$. Thus

$$m \leq |E(\langle R\rangle)| + |e(I,R)| + |E(\langle I\rangle)| \leq \tbinom{n-\alpha}{2} + r(n-\alpha) + 0 = \frac{1}{2}(n-\alpha)(n-\alpha+2r-1),$$

from which we have

$$(n-\alpha)(n-\alpha+2r-1)-2m \geq 0$$

or, equivalently,

$$\alpha^2-(2n+2r-1)\alpha+n^2+(2r-1)n-2m \geq 0.$$

As a solution of this quadratic inequality we obtain

$$\alpha \leq \frac{1}{2}\left(2n+2r-1-\sqrt{8m+(2r-1)^2)}\right).$$

(ii) To show the sharpness, choose arbitrary integers r, k, n such that $2 \leq r \leq k < n$, put $I = \{v_1, \ldots, v_k\}$ and $R = \{v_{k+1}, \ldots, v_n\}$, let I be independent and $\langle R \cup \{v_k\}\rangle$ be complete and join every vertex of R arbitrarily to some $r-1$ vertices in $I \setminus \{v_k\}$. Then the resulting graph G is $K_{1,r+1}$-free, has $|V(G)| = n$, $\alpha(G) = k$, $|E(G)| = m = \binom{n-k+1}{2} + (r-1)(n-k) = \frac{1}{2}(n-k)(n-k+2r-1)$ and Theorem 1 yields

$$\alpha(G) \leq \frac{1}{2}\left(2n+2r-1-\sqrt{4(n-k)(n-k+2r-1)+(2r-1)^2}\right) =$$

$$= \frac{1}{2}\left(2n+2r-1-\sqrt{(2n+2r-1-2k)^2}\right) = k.$$

Thus, Theorem 1 is sharp. ■

Next we turn our attention to conditions that give an upper bound on $\alpha(G)$ in terms of the degrees of the vertices of G.

Theorem 2. Let G be a $K_{1,r+1}$-free graph ($r \geq 2$) with degree-sequence $d_1 \leq d_2 \leq \ldots \leq d_n$. Then

$$\alpha(G) \leq \max\left\{k \mid k + \frac{1}{r}\sum_{i=1}^{k} d_i \leq n\right\}$$

and this bound is sharp.

Proof. (i) Let $I = \{v_1, \ldots, v_k\}$ ($k \geq 1$) be an independent set. Then every vertex $x \in R = V(G) \setminus I$ is adjacent to at most r vertices in I. Thus we have

$$\sum_{v_j \in I} d(v_j) \leq r(n-k)$$

and hence

$$\sum_{i=1}^{k} d_i \leq \sum_{j=1}^{k} d(v_j) \leq r(n-k)$$

from which we obtain

$$k + \frac{1}{r}\sum_{i=1}^{k} d_i \leq n.$$

(ii) To show the sharpness, we construct a graph G in the following way: Choose arbitrarily $\alpha \geq r \geq 2$ and $d_1 \leq d_2 \leq \ldots \leq d_{\alpha-(r-1)}$, set $t = \sum_{i=1}^{\alpha-(r-1)} d_i$, put $G_1 = \overline{K}_{\alpha-r+1}$, $G_2 = K_t$, $G_3 = \overline{K}_{r-1}$, take $G_2 + G_3$ and join the i-th vertex of G_1 to exactly d_i vertices of G_2 $(i = 1, \ldots, \alpha - r + 1)$ in such a way that no two vertices in G_1 have a common neighbour in G_2. Then $n = |V(G)| = \alpha + t$ and G has degree-sequence $d_1 \leq d_2 \leq \ldots \leq d_{\alpha-(r-1)} \leq t \leq \ldots \leq t + r - 1 \leq \ldots \leq t + r - 1$, from which $\sum_{i=1}^{\alpha} d_i = rt$ and hence $\alpha + \frac{1}{r}\sum_{i=1}^{\alpha} d_i = \alpha + t = n$. Thus, Theorem 2 is sharp. ∎

Corollary 3. [2] Let G be $K_{1,r+1}$-free $(r \geq 2)$ with minimum degree $\delta(G)$. Then

$$\alpha(G) \leq \frac{rn}{\delta + r} \; .$$

Proof. We proceed in the same way as above, i.e., $k\delta \leq \sum_{i=1}^{k} d_i \leq r(n-k)$ implying $k \leq \frac{rn}{\delta+r}$. ∎

Corollary 3 is a special case of the following more general result proved in [2].

Theorem A. If G is a $K_{1,r+1}$-free $(r \geq 2)$ graph of order n such that $\sigma_p = px$ for some p with $1 \leq p \leq \alpha$, then

$$\alpha(G) \leq \frac{rn}{x+r},$$

where

$$\sigma_p = \min\left\{\sum_{v_i \in I} d(v_i) | \; I = \{v_1, \ldots v_p\} \subset V(G) \text{ is an independent set}\right\}.$$

This condition also generalizes theorem 2; however, the condition of theorem 2 can be easily checked, whereas the computation of σ_p is more time-consuming.

Moreover, by the construction of the graph G in the example shown in the proof of Theorem 2, the sequence $\{\frac{1}{p}\sigma_p\}_{p=1}^{\alpha}$ is increasing for $\alpha > r$ and hence the upper bound given by Theorem A is sharp only for $p = \alpha$ (if we do not restrict to the integer parts of the derived bounds).

For example, if $r = 2,\ \alpha = 6,\ t = 5,\ n = 11$, then $\{\frac{1}{p}\sigma_p\}_{p=1}^{6} = \{1, 1, 1, 1, 1, \frac{10}{6}\}$ and for $p = 5$ we have

$$\frac{2n}{\frac{\sigma_5}{5} + 2} = \frac{22}{3} > 7 > 6 \ .$$

If G does not contain "too many claws" then the result of Corollary 3 can be strengthened in the following way. For $i \geq 3$ we denote $C_i = \{x \in V(G) |\ \alpha(\langle N(x) \rangle) = i\}$ (i.e., C_i is the set of all vertices of G which are centres of an induced $K_{1,i}$ but not of an induced $K_{1,i+1}$), and we put $c_i = |C_i|$.

Proposition 4. Let G be a graph on n vertices with minimum degree δ. Then

$$\alpha(G) \leq \frac{2n + c_3 + 2c_4 + 3c_5 + \ldots}{\delta + 2}$$

and this bound is sharp.

Proof. Let I and R be as in the proof of Theorem 2. Then, since $|N_I(x)| \leq i$ for $x \in R \cap C_i$ $(i \geq 3)$ and $N_I(x) \leq 2$ otherwise. Thus we have

$$\delta\alpha \leq |e(R, I)| \leq 3c_3 + 4c_4 + 5c_5 + \ldots + 2(n - \alpha - c_3 - c_4 - c_5 - \ldots) \ ,$$

from which

$$\delta\alpha \leq 2(n - \alpha) + c_3 + 2c_4 + 3c_5 + \ldots$$

and hence

$$\alpha \leq \frac{2n + c_3 + 2c_4 + 3c_5 + \ldots}{\delta + 2} \ .$$

It is easy to see that, using the idea of the proof, we can obtain sharp examples by a similar construction to that in the proof of Theorem 1. ■

Next we consider the case when G is almost claw-free. Since every almost claw-free graph is $K_{1,5}$-free, from Corollary 3 we have immediately $\alpha(G) \leq 4n/(\delta+4)$. We show that this bound can be improved.

Lemma 5. Let G be an almost claw-free graph with minimum degree δ. Then

$$\alpha(G) \leq \frac{4n}{\delta+5} .$$

Proof. Let I be an independent set of size α and $R = V(G) \setminus I$; denote $R_i = \{x \in R |\ d_I(x) = i\}$ and $r_i = |R_i|,\ i = 1,2,3,4$. Since every almost claw-free graph is $K_{1,5}$-free and I is a maximum independent set, $R_1 \cup R_2 \cup R_3 \cup R_4 = R$ and hence

$$I = N_I(R_1) \cup N_I(R_2) \cup (N_I(R_3) \setminus N_I(R_2)) \cup (N_I(R_4) \setminus N_I(R_2)) .$$

By definition, $|N_I(R_1)| \leq r_1$ and $|N_I(R_2)| \leq 2r_2$. Since every vertex $v \in R_3 \cup R_4$ is the centre of an induced $K_{1,3}$ and G is almost claw-free, $\langle N(v) \rangle$ is 2-dominated. Thus, for every $v \in R_3$ there is an $x \in R_2$ such that $|N_I(x) \cap N_I(v)| = 2$, and for every $v \in R_4$ there are $x_1, x_2 \in R_2$ such that $N_I(x_1) \cup N_I(x_2) = N_I(v)$. Hence we have $|N_I(R_3) \setminus N_I(R_2)| \leq r_3$ and $|N_I(R_4) \setminus N_I(R_2)| = 0$, from which

$$\alpha = |I| \leq r_1 + 2r_2 + r_3 .$$

Since obviously

$$\delta\alpha \leq |e(I,R)| = r_1 + 2r_2 + 3r_3 + 4r_4 ,$$

we obtain

$$(\delta+1)\alpha \leq 2r_1 + 4r_2 + 4r_3 + 4r_4 \leq 4(r_1 + r_2 + r_3 + r_4) = 4(n-\alpha) ,$$

from which

$$\alpha \leq \frac{4n}{\delta+5} .$$

■

Lemma 6. Let G be a $K_{1,5}$-free graph with minimum degree δ such that the set of centres of induced claws is independent. Then

$$\alpha(G) \leq \frac{2n}{\delta+1} .$$

Proof. Let I, R, R_i and r_i be as in the proof of Lemma 5. Again obviously

$$\alpha\delta \leq |e(I,R)| = r_1 + 2r_2 + 3r_3 + 4r_4 \ ,$$

from which, since $r_1 + r_2 + r_3 + r_4 = r - \alpha$,

$$\alpha\delta \leq 2(n-\alpha) - r_1 + r_3 + 2r_4 \ .$$

Since no two centres of claws can be adjacent, $R_3 \cup R_4$ is an independent set and $d_{R_3 \cup R_4}(v) \leq 2 \ \forall v \in I$. Thus,

$$3r_3 + 4r_4 \leq 2\alpha \ ,$$

or, equivalently,

$$\frac{3}{2}r_3 + 2r_4 \leq \alpha \ .$$

From this,

$$r_3 + 2r_4 \leq \frac{3}{2}r_3 + 2r_4 \leq \alpha \leq \alpha + r_1 \ ,$$

or, equivalently,

$$-r_1 + r_3 + 2r_4 \leq \alpha \ .$$

Hence we have

$$\alpha\delta \leq 2(n-\alpha) - r_1 + r_3 + 2r_4 \leq 2(n-\alpha) + \alpha \ ,$$

from which

$$\alpha \leq \frac{2n}{\delta + 1} \ .$$

■

Theorem 7. Let G be an almost claw-free graph with minimum degree δ. Then

$$\alpha(G) \le \begin{cases} \frac{2}{3}n & \text{for } \delta = 1 \ , \\ \frac{4}{7}n & \text{for } \delta = 2 \ , \\ \frac{2n}{\delta+1} & \text{for } \delta \ge 3 \ , \end{cases}$$

and this bound is sharp.

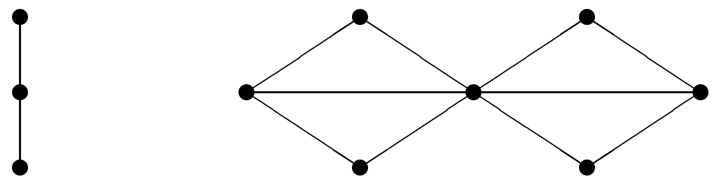
Proof. (i) The upper bound follows immediately from Lemma 5 for $\delta = 1, 2$ and from Lemma 6 for $\delta \ge 3$.

(ii) For $\delta = 1$, the graph tP_3 (i.e., t vertex-disjoint copies of the path on three vertices P_3) and for $\delta = 2$, the graph $t(2P_3 + K_1)$ (i.e., t vertex-disjoint copies of the butterfly $2P_3 + K_1$) achieve the upper bounds given by Theorem 7.

For $\delta \ge 3$ we construct the graph G by taking k vertex-disjoint copies $H_1, \ldots, H_k$ of $K_\delta \setminus e$ $(k \ge 2)$ and k additional vertices $x_1, \ldots, x_k$ and by joining each x_i to all the vertices of H_i and H_{i+1} for $i = 1, \ldots, k$ (mod k). The graph G is almost claw-free, has $n = k(\delta + 1)$ vertices, $\alpha(G) = 2k$ and Theorem 7 gives

$$\alpha(G) \le \frac{2n}{\delta + 1} = \frac{2k(\delta + 1)}{\delta + 1} = 2k \ .$$

■



P_3 *The butterfly* $2P_3 + K_1$

Figure 1

Next we prove an upper bound for $\alpha(G)$ using the vertex-conectivity of a $K_{1,r+1}$-free graph G.

Propositon 8. Let G be a $K_{1,r+1}$-free graph $(r \ge 2)$ with connectivity κ. Then

$$\alpha(G) \le \frac{(r-1)n - \kappa + 2}{r} \ .$$

Proof. We proceed by induction on the number of vertices in G.

1. If $n = \kappa + 1$ then

$$\frac{(r-1)n-\kappa+2}{r} = \frac{(r-1)(\kappa+1)-\kappa+2}{r} = \frac{\kappa(r-2)+r+1}{r} \geq \frac{r+1}{r} \geq 1$$

and, as $G \simeq K_n$, $\alpha(G) = 1$.

2. Suppose that the theorem is true for every $K_{1,r+1}$-free graph G on less than n vertices. Let $S \subset V(G)$ be a vertex cutset such that $|S| = \kappa$ and $I \subset V(G)$ be an independent set such that $|I| = \alpha(G)$. Denote $S_1 = S \cap I$; $k = |S_1|$; $G_1, \ldots, G_l$ the components of $G \setminus S$ and $n_i = |V(G_i)|$, $i = 1, \ldots, l$. Then, by the induction hypothesis, each of the subgraphs $\langle V(G_i) \cup S_1 \rangle$ has independence number at most $[(r-1)(n_i+k)+1]/r$ and hence

$$\alpha(G) \leq \frac{(r-1)(n_1+k)+1}{r} + \ldots + \frac{(r-1)(n_l+k)+1}{r} - (l-1)k =$$

$$\frac{(r-1)(n_1+...+n_l)+(r-1)lk+l+(1-l)rk}{r} =$$

$$\frac{(r-1)(n-\kappa)+(r-l)(k-1)+r}{r} .$$

Since $k \leq \kappa$ and $l \geq 2$, we further obtain

$$\alpha(G) \leq \frac{(r-1)(n-\kappa)+(r-2)(\kappa-1)+r}{r} = \frac{(r-1)n-\kappa+2}{r} .$$

■

In order to compare the bounds of Proposition 8 and of Corollary 3, we consider the following examples.

Example 9. For arbitrary $r > \kappa \geq 1$, the complete bipartite graph $G = K_{\kappa,r}$ is $K_{1,r+1}$-free with $n = |V(G)| = \kappa + r$, $\alpha(G) = r$, $\kappa(G) = \delta(G) = \kappa$ and Corollary 3 gives

$$\alpha(G) \leq \frac{rn}{\delta+r} = \frac{r(\kappa+r)}{\kappa+r} = r .$$

In the next example we construct for every integers $\delta > \kappa \geq 1$ a claw-free graph for which the bound in Corollary 3 is achieved.

Example 10. Choose arbitrary integers $\delta > \kappa \geq 1$ and let H be an arbitrary δ-regular κ-edge connected graph. Such graphs exist for all possible values of δ and κ except the case when δ is even and κ is odd. E.g., for $\kappa \geq 2$, one of the possible constructions is the following: for any $k > l \geq 2$ there is an l-regular graph $H_{l,k}$ on k vertices which has vertex connectivity l (cf. [1]) and we construct the graph H by taking a matching of κ edges $u_i w_i$ ($1 \leq i \leq \kappa$) and joining each vertex u_i to all vertices of a copy of $H_{\delta-\kappa,\delta-1}$ and each vertex w_i to all vertices of a second copy of $H_{\delta-\kappa,\delta-1}$. This graph has $2\kappa + 2(\delta - 1)$ vertices, is δ-regular and has edge-connectivity κ.

We construct the *middle graph* $G = M(H)$ of H (cf. [4]) by inserting a vertex x_i in the "middle" of each edge e_i, $1 \leq i \leq |E(H)|$ and adding the edge $x_i x_j$ for $1 \leq i < j \leq |E(H)|$ if only if e_i and e_j have a common vertex. Then G is claw-free with vertex connectivity κ, $\alpha(G) = \alpha = |V(H)|$ and $\delta(G) = \delta$ and G has $n = |V(H)| + |E(H)| = \alpha + \frac{1}{2}\alpha\delta = \alpha(\delta + 2)/2$ vertices. Corollary 3 thus gives

$$\alpha(G) \leq \frac{2n}{\delta + 2} = \frac{\alpha(\delta + 2)}{\delta + 2} = \alpha \ .$$

(If a δ-regular κ-edge connected graph does not exist, i.e., for δ even and κ odd, we take for H a graph with exactly two vertices of degree $\delta + 1$ and, by the same construction, we obtain $n = \alpha + \frac{1}{2}\alpha\delta + 1$ and $\alpha(G) \leq [\alpha(\delta+2)+2]/(\delta+2) = \alpha + 2/(\delta+2)$; as $2/(\delta+2) < 1$, the result is also sharp).

In [2] it is shown that the result of Corollary 3 is sharp for arbitrarily large δ, r and n; however, these graphs have connectivity $\kappa = \delta$. We next construct an infinite family of graphs with the same properties (i.e., with arbitrarily large δ, r and n) and with $\kappa < \delta$.

Example 11. Choose arbitrary integers $r \geq 2$, $s \geq 2$, $\delta \geq r$ and k such that $2r - 2 \leq k \leq \delta + r - 2$ and denote $t = rs$. Put $V(G) = A \cup A_1 \cup \ldots \cup A_t$, where A, $A_1, \ldots, A_t$ are pairwise disjoint sets such that $A = \{x_1, \ldots, x_t\}$ and

$$|A_i| = \begin{cases} \delta - k + r - 1 & \text{for } i \equiv r \pmod r, \\ k - 2r + 3 & \text{for } i \equiv r - 1 \pmod r, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\langle \{x_i\} \cup A_i \cup \ldots \cup A_{i+r-1} \rangle$ be complete for $i = 1, \ldots, t - r + 1$ and join x_i by an edge to every vertex of $A_i \cup \ldots \cup A_r \cup A_1 \cup \ldots \cup A_{i-t+r-1}$ for $i = t - r + 2, \ldots, t$. Then the graph G has $n = s\delta + t$ vertices, minimum degree δ, independence number $\alpha(G) = t$ and

$$\frac{rn}{\delta + r} = \frac{r(s\delta + sr)}{\delta + r} = sr = t;$$

moreover, G has connectivity $\kappa(G) = \min\{\delta, k\}$.

Examples 9,10 and 11 show that the bound of Corollary 3 is sharp for all pairs κ, δ such that $1 < \kappa \leq \delta$ and, therefore, Proposition 8 cannot be expected to give (for arbitrary n) a better bound than that expressed in terms of degrees in this case.

However, the following example shows that for $\delta = \kappa = 1$, the bound given by Proposition 8 is better and the difference between the two bounds can be arbitrarily large.

Example 12. Choose arbitrary integers $r \geq 2$ and $k \geq 2$, let H be a caterpillar with vertex set $V(H) = \{y_1, \ldots, y_k, x_1^1, \ldots, x_1^r, x_2^1, \ldots, x_2^r, \ldots, x_k^1, \ldots, x_k^r\}$ (where $\langle y_1, \ldots, y_k \rangle$ is a path, x_i^j have degree 1 and $y_i x_i^j \in E(H)$ for every $i = 1, \ldots, k$ and $j = 1, \ldots, r$), and denote by G the graph which is obtained from H by indentifying x_i^r with x_{i+1}^1 for all $i = 1, \ldots, k-1$ (for $k = 5$ and $r = 4$, see Fig.2).

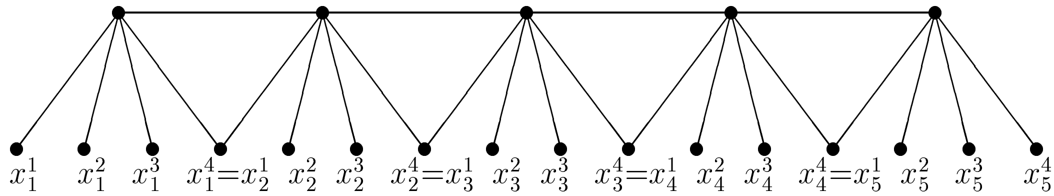


Figure 2

Then G is $K_{1,r+1}$-free with $\kappa(G) = \delta(G) = 1$, has $n = kr + 1$ vertices and independence number $\alpha(G) = k(r-1) + 1 = kr - k + 1$. Proposition 8 yields

$$\alpha(G) \leq \frac{(r-1)n + 1}{r} = \frac{(r-1)(kr+1) + 1}{r} = kr - k + 1 = \alpha(G) \ ,$$

while from Corollary 3 we obtain

$$\alpha(G) \leq \frac{rn}{r+1} = \frac{r(kr+1)}{r+1} = kr - k + 1 + \frac{k-1}{r+1} = \alpha(G) + \frac{k-1}{r+1} \ .$$

By the construction, $\frac{k-1}{r+1}$ can be arbitrarily large.

References

[1] Bondy, J.A.; Murty, U.S.R.: Graph theory with applications. Macmillan, London and Elsevier, New York, 1976.

[2] Faudree, R.J.; Gould, R.J.; Jacobson, M.S.; Lesniak, L.M.; Lindquester, T.E.: On independent generalized degrees and independence numbers in $K(1, m)$-free graphs. Discrete Mathematics 103(1992), 17-24.

[3] Gernert, D.: Forbidden and unavoidable supgraphs. Ars Combinatoria 27(1989), 165-176.

[4] Hamada, T.; Yoshimura, I.: Traversability and connectivity of the middle graph. Discrete Math. 14(1976), 247-255.

[5] Li, Hao; Virlouvet, C.: Neighborhood conditions for claw-free hamiltonian graphs. Ars Combinatoria 29A(1990), 109-116.

[6] Ryjáček, Z.: Almost claw-free graphs. Journal of Graph Theory (to appear).