FORBIDDEN SUBGRAPHS AND CYCLE EXTENDABILITY

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Abstract

A graph G on n vertices is *pancyclic* if G contains cycles of all lengths ℓ for $3 \leq \ell \leq n$ and G is *cycle extendable* if for every nonhamiltonian cycle $C \subset G$ there is a cycle $C' \subset G$ such that $V(C) \subset V(C')$ and $|V(C') \setminus V(C)| = 1$. We prove that

(i) every 2-connected $K_{1,3}$-free graph is pancyclic, if G is P_5-free and $n \geq 6$, if G is P_6-free and $n \geq 10$, or if G is P_7-free, deer-free and $n \geq 14$, and

(ii) every 2-connected $K_{1,3}$-free and Z_2-free graph on $n \geq 10$ vertices is cycle extendable using at most two chords of the cycle.

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1. Introduction

We only consider simple, undirected graphs and refer to [2] for terminology and notation not defined here. A graph G with $n \geq 3$ vertices is *hamiltonian* if G contains a cycle of length n, and *pancyclic* if G contains a cycle C_k of length k for each k with $3 \leq k \leq n$. A graph G is *cycle extendable* if for every nonhamiltonian cycle $C \subset G$ there is a cycle $C' \subset G$ such that $V(C) \subset V(C')$ and $|V(C') \setminus V(C)| = 1$. If C_m is a cycle with m vertices labeled $v_1, v_2, \ldots, v_m$ such that $\{v_i v_{i+1} \mid 1 \leq i \leq m-1\} \cup \{v_m v_1\} \subset E(G)$ and $v_j v_{j+k} \in E(G)$ for some j, k (modulo m), then the edge $v_j v_{j+k}$ is called a *k-chord* of C_m. Clearly, this k-chord can be used to construct a cycle of length $m - k + 1$ from the given cycle C_m. We say that a graph G is *cycle $k-$extendable* if each nonhamiltonian cycle C can be extended to a cycle C' that has one additional vertex and uses at most k chords of C. Finally, a graph G has a *$k-$pancyclic ordering* if the vertices of G can be ordered such that the graph induced by the first j vertices ($j \geq k$) is hamiltonian. Thus a graph with a 3-pancyclic ordering has a pancyclic ordering.

If G and G' are graphs, then we say that G is *G'-free* if G contains no induced subgraph isomorphic to G'. Specifically, we denote by C the claw $K_{1,3}$, by D the deer, by H the hourglass, by P_k the path with k vertices (i.e. of length $k-1$) and by Z_1 and Z_2 the graphs obtained by identifying a vertex of K_3 with an end-vertex of P_2 and P_3, respectively (see Figure 1).

Probably the first sufficient condition for hamiltonicity of a graph in terms of forbidden subgraphs is due to Goodman and Hedetniemi [4].

Theorem A [4]. *If G is a 2-connected CZ_1-free graph, then G is hamiltonian.*

Gould and Jacobson [5] extended this result to CZ_2-free graphs.

Theorem B [5]. *If G is a 2-connected CZ_2-free graph then G is a cycle or is pancyclic.*

Hendry [6] further extended this result showing the following.

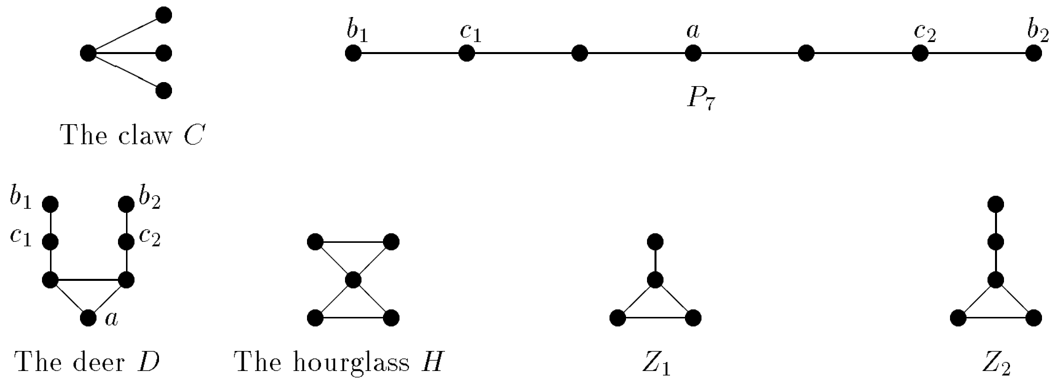


Figure 1

Theorem C [6]. *If G is a 2-connected, CZ_2-free graph on $n \geq 10$ vertices, then G is cycle extendable.*

The graph G_1 in Figure 2 shows that C and Z_3, as forbidden subgraphs, are not sufficient to guarantee even hamiltonicity. Also, the triangles in G_1 that contain a vertex of degree 2 can be replaced by an arbitrary K_r for $r \geq 3$ without changing the conclusion, so there is an infinite family of CZ_3-free graphs that are not hamiltonian.

A result similar to Theorem B was proved for CP_5-free graphs by Bedrossian [1].

Theorem D [1]. *Let G be a 2-connected CP_5-free graph. Then G is either pancyclic or a cycle.*

The graph G_2 in Figure 2 (given in [1]) shows that, to guarantee pancyclicity, P_5 cannot be replaced by the forbidden subgraph P_6 in the hypothesis of Theorem D.

However, to obtain hamiltonicity, the following result of Broersma and Veldman ([3]) can be used to weaken the hypothesis of Theorem D. If G' is a subgraph of G and $u, v \in V(G')$, then G' is said to satisfy property $\Phi(u, v)$ if $(N(u) \cap N(v)) - V(G') \neq \emptyset$, where $N(x)$ denotes

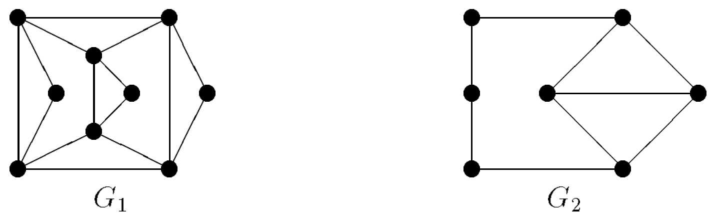


Figure 2

the neighborhood of the vertex x in G. The symbols $\wedge$ and $\vee$ are used here to denote "and" and "or", respectively.

Theorem E [3]. *Let G be a 2-connected, C-free graph. If every induced subgraph of G isomorphic to D or P_7 (see Figure 1) satisfies $\Phi(a,b_1)\vee\Phi(a,b_2)\vee(\Phi(a,c_1)\vee\Phi(a,c_2))$, then G is hamiltonian.*

This result has the following immediate consequence.

Corollary F [3]. *Let G be a 2-connected C-free graph. If G is P_6-free or DP_7-free, then G is hamiltonian.*

In this paper we will show that

(i) every CDP_7-free (and thus also CP_6-free or even CP_5-free) graph is pancyclic or belongs to a finite family of exceptional graphs, and

(ii) every CZ_2-free graph is either 2-chord extendable or belongs to a finite family of exceptional graphs.

These families of exceptional graphs are fully described.

2. Results

Proposition 1 (Reduction Procedure RP). *Let G be a CDP_7-free graph on $n \geq k \geq 9$ vertices. If G contains a C_k, then G also contains a C_{k-1}.*

We first introduce some additional notation which will be useful in the proofs that follow. Let C be a cycle in a graph G. If an orientation of C is fixed and $u, v \in V(C)$, then by $u \overrightarrow{C} v$ we denote the consecutive vertices on C from u to v in the direction specified by the orientation of C. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. If C is a cycle of G with a fixed orienation and $u \in V(D)$, then u^+ denotes the successor of u on C and u^- its predecessor, respectively.

In the proofs the following four statements for C-free graphs can easily be verified, and will be frequently used and just referred by the indicated label.

(A) Let C_m be a cycle with $m \geq 2k+2 \geq 6$ vertices labeled $v_1, v_2, \ldots, v_m$ and a k-chord v_jv_{j+k}. If there are no i-chords for $2 \leq i \leq k-1$, then $v_{j-1}v_{j+k}, v_jv_{j+k+1} \in E(G)$.

(B) If, moreover, $v_{j-1}v_{j+k-1} \notin E(G)$ or $v_{j+1}v_{j+k+1} \notin E(G)$, then $v_{j-1}v_{j+k+1} \in E(G)$.

(C) Let v_jv_{j+i} be an i-chord with $3 \leq i \leq \frac{k}{2}$ in a cycle C_k without 2-chords. If $v_jv_{j+i-1} \notin E(G)$, then $v_jv_{j+i+1} \in E(G)$, and likewise if $v_{j+1}v_{j+i} \notin E(G)$, then $v_{j-1}v_{j+i} \in E(G)$.

(D) Let v_jv_{j+i} be an i-chord in a cycle C_k. If $i \geq 2$ and $v_{j+1}v_{j+i+2} \in E(G)$ or if $i \geq 3$ and $v_{j+2}v_{j+i+1} \in E(G)$, then $v_jv_{j+i} \overleftarrow{C} v_{j+1}v_{j+i+2} \overrightarrow{C} v_j$ or $v_jv_{j+i} \overleftarrow{C} v_{j+2}v_{j+i+1} \overrightarrow{C} v_j$ is a C_{k-1}, respectively.

Proof (of Propositon 1). Let $v_1, \ldots, v_k$ be the vertices of C_k. Since G is P_7-free, the cycle C_k contains a chord. Let i $(2 \leq i \leq \frac{k}{2})$ be smallest integer such that G has an i-chord. Among all chords of C_k choose such a minimal i-chord $(2 \leq i \leq \frac{k}{2})$. Choose a labeling $v_1, v_2, \ldots, v_k$ of the vertices of C_k such that $(\{v_jv_{j+1} \mid 1 \leq j \leq k-1\} \cup \{v_kv_1, v_1v_{i+1}\}) \subset E(G)$. We then distinguish the following five cases.

Case 1. i=2

Then $v_1v_3v_4 \ldots v_kv_1$ is a C_{k-1}.

Case 2. i=3

By (A) we have $v_1v_5, v_kv_4 \in E(G)$. If $v_2v_5 \in E(G)$, then we obtain a C_{k-1} by (D). Hence, we may assume that $v_2v_5 \notin E(G)$ and so $v_kv_5 \in E(G)$ by (B). If $v_2v_6 \in E(G)$, $v_2v_7 \in E(G)$, or $v_3v_6 \in E(G)$ then we obtain a C_{k-1} by (D). Hence we may assume that v_2v_6, v_2v_7, and $v_3v_6 \notin E(G)$. If $v_3v_7 \in E(G)$, then $v_kv_5v_4v_1v_2v_3v_7 \overrightarrow{C} v_k$ is a C_{k-1}. Therefore, $v_3v_7 \notin E(G)$ and thus $v_4v_7 \notin E(G)$ by (A). Suppose now that $v_5v_8 \in E(G)$. Then $v_4v_8, v_5v_9 \in E(G)$ by (A) and $v_4v_9 \in E(G)$ by (B), since $v_4v_7 \notin E(G)$ by (D). Now if $v_6v_k \in E(G)$ or $v_7v_k \in E(G)$, then $v_kv_6v_5v_1v_2v_3v_4v_8 \overrightarrow{C} v_k$ or $v_kv_7v_6v_5v_1v_2v_3v_4v_9 \overrightarrow{C} v_k$ is a C_{k-1}. If $v_6v_k, v_7v_k \notin E(G)$, then $G[\{v_2, v_3, v_4, v_k, v_5, v_6, v_7\}]$ is an induced deer, a contradiction. Hence we may assume that $v_5v_8 \notin E(G)$. If $v_4v_8 \in E(G)$, then $v_3v_8, v_4v_9 \in E(G)$ by (D) and $v_3v_9 \in E(G)$ by (B) and we obtain the same contradiction. Hence we may assume that $v_4v_8 \notin E(G)$. Analogously, if $v_3v_8 \in E(G)$, then $v_2v_8, v_3v_9, v_2v_9 \in E(G)$, and if $v_6v_k \in E(G)$ or $v_7v_k \in E(G)$, then $v_kv_6v_5v_4v_1v_2v_3v_8 \overrightarrow{C} v_k$ or $v_kv_7v_6v_5v_4v_3v_2v_8 \overrightarrow{C} v_k$ is a C_{k-1}, respectively. Hence we may assume that $v_3v_8 \notin E(G)$. If $v_2v_8 \in E(G)$, then $v_1v_8 \in E(G)$ by (C), and if $v_6v_k \in E(G)$ or $v_7v_k \in E(G)$, then $v_kv_6 \overleftarrow{C} v_1v_8 \overrightarrow{C} v_k$ or $v_kv_7 \overleftarrow{C} v_2v_8 \overrightarrow{C} v_k$ is a C_{k-1}, respectively. Hence we may assume that $v_2v_8 \notin E(G)$. But then $G[\{v_2, v_3, \ldots, v_8\}]$ is an induced P_7, our final contradiction.

Case 3. i=4

By (A) we have $v_1v_6, v_kv_5 \in E(G)$. If $v_2v_6 \in E(G)$, then we obtain a C_{k-1} by (D). Hence we may assume that $v_2v_6 \notin E(G)$ and thus $v_kv_6 \in E(G)$ by (B). If $v_2v_7 \in E(G)$ or $v_2v_8 \in E(G)$ or $v_3v_7 \in E(G)$, then we obtain a C_{k-1} by (D). Hence we may assume that $v_2v_7, v_2v_8, v_3v_7 \notin E(G)$. If $v_4v_k \in E(G)$, then $v_kv_6v_5v_1v_2v_3v_4v_8 \overrightarrow{C} v_k$ is a C_{k-1}. Hence we may assume that $v_4v_8 \notin E(G)$ and thus $v_3v_8 \notin E(G)$ by (C), since $v_2v_8 \notin E(G)$. But then $G[\{v_2, v_3, \ldots, v_8\}]$ is an induced P_7, a contradiction.

Case 4. i=5

By (A) we have $v_1v_7, v_kv_6 \in E(G)$. If $v_2v_7 \in E(G)$, then we obtain a C_{k-1} by (D). Hence we may assume that $v_2v_7 \notin E(G)$ and thus $v_kv_7 \in E(G)$ by (B). If $v_2v_8 \in E(G)$ or $v_3v_8 \in E(G)$, then we obtain a C_{k-1} by (D). Hence we may assume that $v_2v_8, v_3v_8 \notin E(G)$. But then $G[\{v_2, v_3, \ldots, v_8\}]$ is an induced P_7, a contradiction.

Case 5. i=6

Since G is P_7-free, C_k contains all possible 6-chords. Thus $v_2v_8, v_6v_k \in E(G)$. By (A) we have $v_k v_7 \in E(G)$ and thus $v_k v_7 \overleftarrow{C} v_2 v_8 \overrightarrow{C} v_k$ is a C_{k-1}. ∎

Remark. The graph $G_{8.8}$ (Figure 4) shows that $k \geq 9$ in the hypothesis of Proposition 1 is sharp.

The proof of Proposition 1 gives the following two corollaries.

Corollary 2. *Let G be a CP_6-free graph on $n \geq k \geq 7$ vertices. If G contains a C_k, then G also contains a C_{k-1}.*

Corollary 3. *Let G be a CP_5-free graph on $n \geq k \geq 6$ vertices. If G contains a C_k, then G also contains a C_{k-1}.*

Remark. The cycles C_5 and C_6 and the graph $G_{8.8}$ (Figure 4) show that the assumptions "$k \geq 6$" or "$k \geq 7$" or "$k \geq 9$" in the hypothesis of Corollary 3 or Corollary 2 or Proposition 1 cannot be improved, respectively. Furthermore, the graph H of order $n = 4r$ in Figure 3 shows that the assumption that G is D-free is an essential hypothesis of Proposition 1, since it is hamiltonian, but it has no cycle of length $n - 1$.

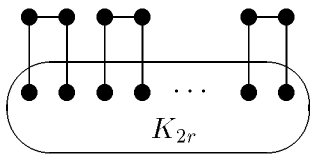


Figure 3

The next two propositions will be used to prove our main result.

Proposition 4. *Let G be a CDP_7-free graph on $n \geq 9$ vertices. If G has a hamiltonian cycle without 2-chords, then G is pancyclic.*

Proof. Let G be a CDP_7-free graph on $n \geq 9$ vertices, which has a hamiltonian cycle without 2-chords. Then G has an i-chord for some i, $3 \leq i \leq 6$, since G is P_7-free and $n \geq 9$. Among all i-chords choose one such that i is minimal. Then $n \geq 2i+1$, since G is C-free and $i \geq 3$. By Proposition 1, we know that G has a C_k for $8 \leq k \leq n$. Therefore it suffices to show that G has a C_k for $3 \leq k \leq 7$. Choose a labeling $v_1, v_2, \ldots, v_n$ of the vertices of G such that $(\{v_j v_{j+1} \mid 1 \leq j \leq n-1\} \cup \{v_n v_1, v_1 v_{i+1}\}) \subset E(G)$. We then distinguish the following four cases.

Case 1. i=6

Then $n \geq 13$ and G has all possible 6-chords, since G is P_7-free and $i = 6$. By (A) we have $v_1v_8, v_2v_9, v_3v_{10} \in E(G)$. Thus G has a C_3, C_4 and a C_7. A C_5 and a C_6 are given by $v_1v_2v_3v_9v_8v_1$ and by $v_1v_2v_3v_{10}v_9v_8v_1$, respectively.

Case 2. i=5

By (A) we have $v_1v_7, v_6v_n \in E(G)$ and thus G has a C_3, C_4, C_6 and C_7. Hence, we may assume that G has no C_5. If $v_5v_n \in E(G)$, then $v_nv_5v_6v_7v_1v_n$ is a C_5. Hence we may assume that $v_5v_n \notin E(G)$ and thus $v_7v_n \in E(G)$ by (C). Now if $v_5v_{n-1} \in E(G)$ or $v_4v_{n-1} \in E(G)$, then $v_{n-1}v_5v_6v_7v_nv_{n-1}$ or $v_{n-1}v_4v_5v_6v_nv_{n-1}$ is a C_5, respectively. Hence $v_5v_{n-1}, v_4v_{n-1} \notin E(G)$, but then $G[\{v_{n-1}, v_n, v_1, v_2, v_3, v_4, v_5\}]$ is an induced P_7, a contradiction.

Case 3. i=4

By (A) we have $v_1v_6, v_5v_n \in E(G)$ and thus G has a C_3, C_4, C_5 and C_6. Hence, we may assume that G has no C_7. Thus $v_4v_{n-2} \notin E(G)$. If $v_3v_{n-2} \in E(G)$, $v_2v_{n-2} \in E(G)$, $v_4v_{n-1} \in E(G)$, or $v_3v_{n-1} \in E(G)$, then $v_1v_2v_3v_{n-2}v_{n-1}v_nv_5v_1$, $v_1v_2v_{n-2}v_{n-1}v_nv_5v_6v_1$, $v_1v_2v_3v_4v_{n-1}v_nv_5v_1$, or $v_1v_2v_3v_{n-1}v_nv_5v_6v_1$ is a C_7, respectively. Now $v_4v_n \notin E(G)$, since $v_4v_{n-1}, v_1v_4, v_1v_{n-1} \notin E(G)$ and G is C-free. But then $G[\{v_{n-2}, v_{n-1}, v_n, v_1, v_2, v_3, v_4\}]$ is an induced P_7, a contradiction.

Case 4. i=3

By (A) we have $v_1v_5, v_4v_n \in E(G)$ and thus G has a C_3, C_4 and C_5. Hence we may assume that G has no C_6 or no C_7. If one of the edges v_3v_{n-1}, v_3v_{n-2} or v_3v_{n-3} is present, then G has a C_6 and a C_7. Now $v_3v_n \notin E(G)$, since G is C-free, $v_3v_{n-1} \notin E(G)$, and $i = 3$. If one of the edges v_2v_{n-1}, v_2v_{n-2}, v_2v_{n-3}, or v_1v_{n-3} is present, then G has a C_6 and a C_7. Now $v_1v_{n-2} \notin E(G)$, since G is C-free, $v_1v_{n-3} \notin E(G)$, and $i = 3$. By the same argument we have $v_{n-3}v_n \notin E(G)$. But then $G[\{v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_1, v_2, v_3\}]$ is an induced P_7, a contradiction. ∎

Proposition 5. *Let G be a 2-connected, CDP_7-free graph on $n \leq 13$ vertices. Then G is either pancyclic or isomorphic to one of the following graphs: $C_4, C_5, C_6, G_{6.1}, C_7, G_{7.1} - G_{7.4}, G_{8.1} - G_{8.10}, G_{9.1} - G_{9.11}, G_{10.1} - G_{10.8}, G_{11.1} - G_{11.7}, G_{12.1} - G_{12.4}, G_{13}$ (see Figure 4).*

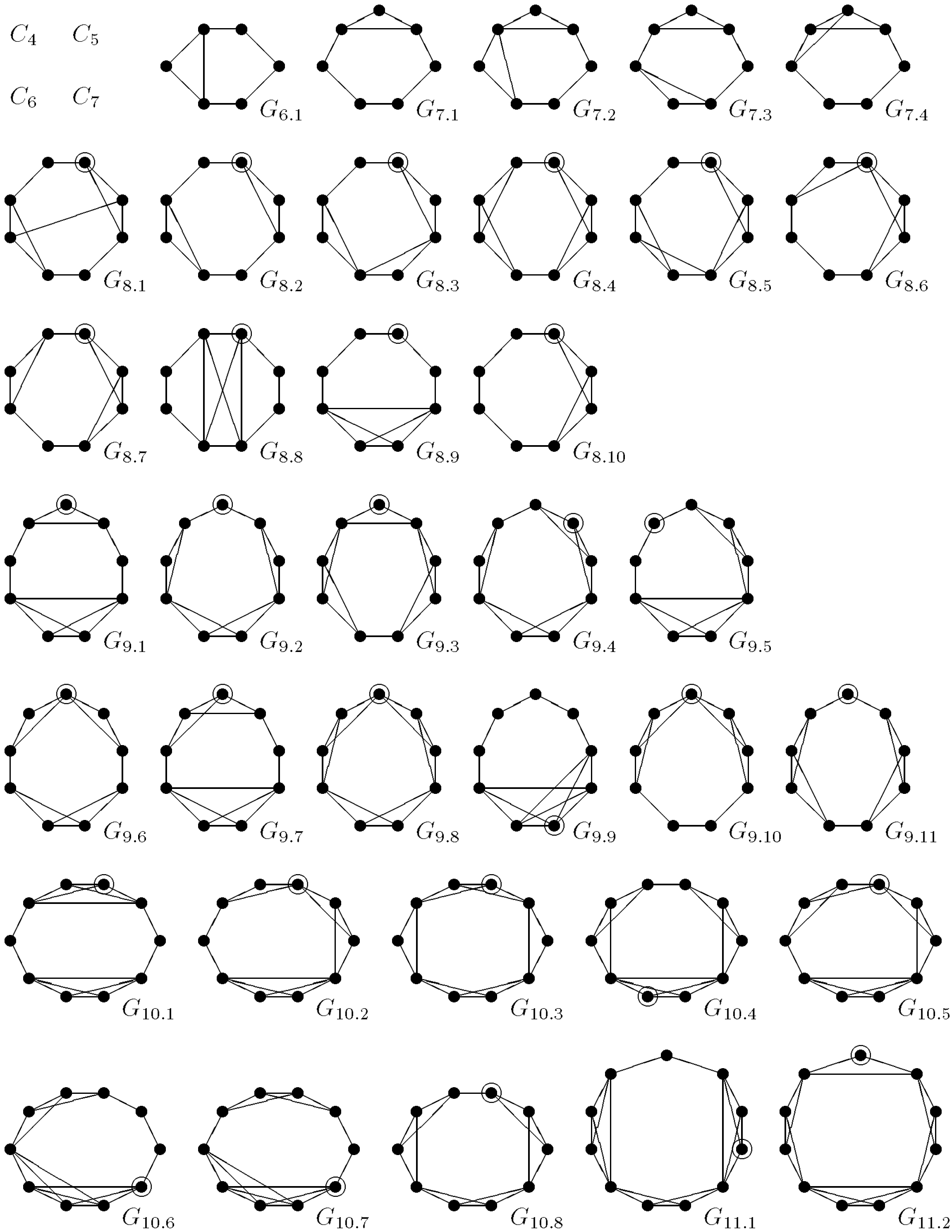
The proof of Proposition 5 is lengthy and involves a detailed case analysis, and is therefore postponed to the appendix. We are now ready to state our main result.

Theorem 6. *Let G be a 2-connected CDP_7-free graph. Then G is either pancyclic or isomorphic to one of the following graphs: $C_4, C_5, \ldots, G_{13}$ in Figure 4.*

Proof. Let G be a 2-connected CDP_7-free graph on $n \geq 3$ vertices. By Theorem E we know that G is hamiltonian. If $n \leq 7$ then G is either pancyclic or isomorphic to $C_4, C_5, C_6, G_{6.1}, C_7, G_{7.1}, G_{7.2}, G_{7.3}$ or $G_{7.4}$, which can be easily verified by a straightforward case analysis. If $n \geq 8$, then G contains all cycles from C_8 up to C_n by Proposition 1. If, moreover, G has a cycle C_k for some k with $k \geq 9$ without 2-chords, then G is pancyclic by Proposition 4. Hence, any counterexample must have $n = 8$ vertices or must have a cycle C_k with a 2-chord for some k with $k \geq 9$. All these counterexamples are given by Proposition 5. Furthermore, the proof of Proposition 5 shows that there are no counterexamples on $n \geq 14$ vertices. This completes the proof. ∎

Theorem 6 has a number of consequences. First observe that all exceptional graphs have connectivity $\kappa = 2$.

Corollary 7. *Let G be a 3-connected C-free graph. If G is DP_7-free or P_6-free, then G is pancyclic.*


C_4 C_5
C_6 C_7
$G_{6.1}$ $G_{7.1}$ $G_{7.2}$ $G_{7.3}$ $G_{7.4}$
$G_{8.1}$ $G_{8.2}$ $G_{8.3}$ $G_{8.4}$ $G_{8.5}$ $G_{8.6}$
$G_{8.7}$ $G_{8.8}$ $G_{8.9}$ $G_{8.10}$
$G_{9.1}$ $G_{9.2}$ $G_{9.3}$ $G_{9.4}$ $G_{9.5}$
$G_{9.6}$ $G_{9.7}$ $G_{9.8}$ $G_{9.9}$ $G_{9.10}$ $G_{9.11}$
$G_{10.1}$ $G_{10.2}$ $G_{10.3}$ $G_{10.4}$ $G_{10.5}$
$G_{10.6}$ $G_{10.7}$ $G_{10.8}$ $G_{11.1}$ $G_{11.2}$

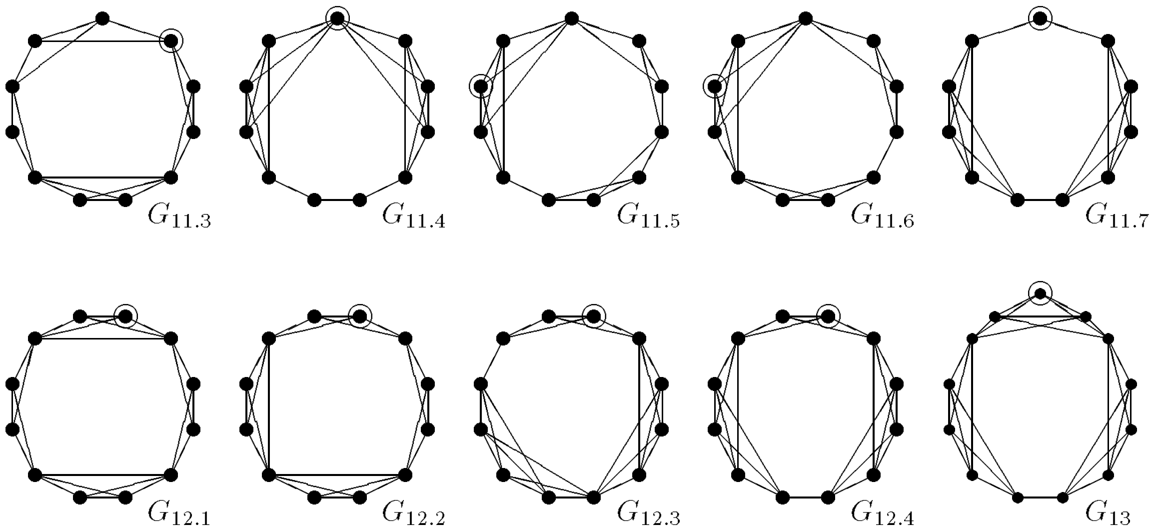


Figure 4

Corollary 8. *Let G be a 2-connected, C-free graph. If, moreover, G is DP_7-free and $n \geq 14$, G is P_6-free and $n \geq 10$, or G is P_5-free and $n \geq 6$, then G is pancyclic.*

With the use of Proposition 1, Corollary 2, or Corollary 3 we obtain the following results for pancyclic orderings, respectively.

Corollary 9. *Let G be a 2-connected CDP_7-free graph on $n \geq 8$ vertices. Then G has an 8-pancyclic ordering.*

Corollary 10. *Let G be a 2-connected CP_6-free graph on $n \geq 6$ vertices. Then G has a 6-pancyclic ordering.*

Corollary 11. *Let G be a 2-connected, CP_5-free graph on $n \geq 5$ vertices. Then G has a 5-pancyclic ordering.*

Remarks. The graph $G_{8.8}$ (Figure 4) and the graph G_2 in Figure 2 show that "8-pancyclic" and "6-pancyclic" in the conclusions of Corollary 9 and Corollary 10 are best possible, respectively. The following two classes of graphs show that "6-pancyclic" and "5-

pancyclic" in the conclusions of Corollary 10 and Corollary 11 are best possible, respectively.

Figure 5

For the class of CZ_2-free graphs we prove the following extension of Theorem C.

Theorem 12. *Let G be a 2-connected CZ_2-free graph. Then G is either 2-chord extendable or isomorphic to one of the eight graphs in Figure 6.*

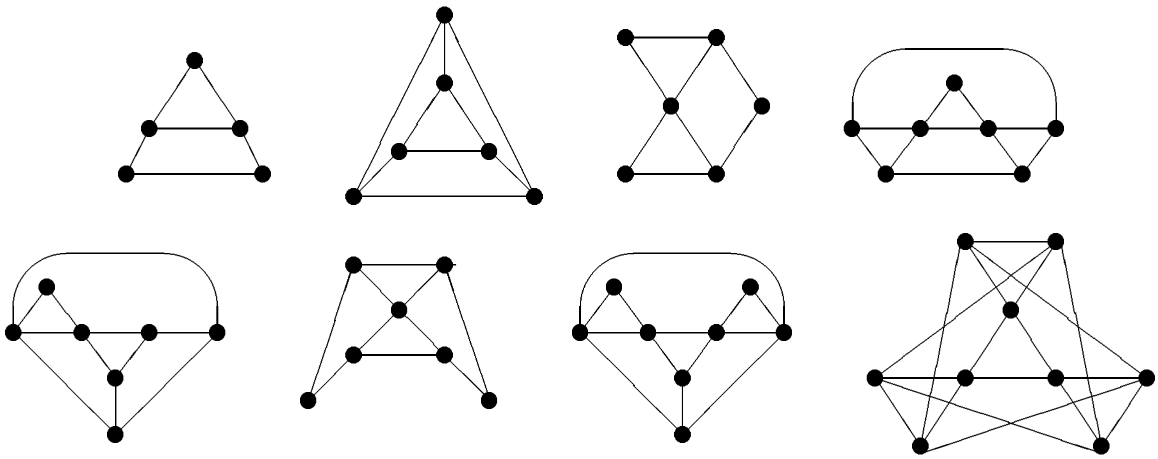


Figure 6

Proof. Let C be a cycle of length $k \geq 3$ in a 2-connected CZ_2-free graph, which cannot be extended using at most two chords. We distinguish two cases:

Case 1. $k \geq 4$

Since G is 2-connected, there is a path $v_1 x_1 x_2 \dots x_l v_2$ such that $v_1, v_2 \in V(C)$, $x_i \notin V(C)$, and $N(x_i) \cap V([v_1^+, v_2^-]) = \emptyset$ for $1 \leq i \leq l$. Now let l be minimal, and among all those paths of length l choose one such that $|V([v_1^+, v_2^-])|$ is as small as possible.

Case 1.1. l=1

Then $v_2 \neq v_1^+$, and $v_1 \neq v_2^+$, since otherwise, $v_1 x_1 v_2 \overrightarrow{C} v_1$ or $v_2 x_1 v_1 \overrightarrow{C} v_2$ would be a

0-chord extension of C. Thus $v_i^- v_i^+ \in E(G)$, for $i = 1, 2$, since G is C-free. Now $v_1^- v_2, v_1^+ v_2 \notin E(G)$, since otherwise, $v_1^- v_2 x_1 v_1 \overrightarrow{C} v_2^- v_2^+ \overrightarrow{C} v_1^-$ or $v_1 x_1 v_2 v_1^+ \overrightarrow{C} v_2^- v_2^+ \overrightarrow{C} v_1$ would be a 2-chord extension of C. Therefore, $v_1 v_2 \in E(G)$, since otherwise, $G[\{v_1^-, v_1, v_1^+, x_1, v_2\}]$ is an induced Z_2.

Let $y \in V([v_1, v_2])$ be the first vertex not adjacent to v_1 and consider $G[\{v_1, x_1, v_2, y^-, y\}]$. By the choice, $x_1 y, x_1 y^-, v_1 y \notin E(G)$, and therefore $v_2 y \in E(G)$ or $v_2 y^- \in E(G)$. Thus, there exists a vertex $z \in V([v_1^+, v_2^-])$ such that $v_1 z^-, z v_2 \in E(G)$. Now $v_1^- z^-, z v_2^+ \in E(G)$, since otherwise, $G[\{v_1^-, v_1, x_1, z\}]$ or $G[\{z, v_2, x_1, v_2^+\}]$ would be an induced claw. But then $v_1^- z^- \overleftarrow{C} v_1 x_1 v_2 \overleftarrow{C} z v_2^+ \overrightarrow{C} v_1^-$ is a 2-chord extension of C, a contradiction.

Case 1.2. $l \geq 2$

Again $v_1^- x_1, v_1^+ x_1 \notin E(G)$. Next $v_1 x_2, v_1^+ x_2 \notin E(G)$, since l is minimal. Now $v_1^- x_2 \in E(G)$, since otherwise, $G[\{v_1^-, v_1, v_1^+, x_1, x_2\}]$ would be an induced Z_2. Hence, we may assume that $l = 2$ and $v_1^+ x_2 \in E(G)$ (exchange v_1^- and v_1^+). Thus, $v_2 = v_1^+$. Since G is C-free and there is no 0-chord extension of C, we have $v_1 v_2^+, v_1^- v_2 \in E(G)$. Next $x_1 v_2^+ \notin E(G)$, since otherwise $v_1^- v_2 v_1 x_1 v_2^+ \overrightarrow{C} v_1^-$ would be a 1-chord extension of C. Then $v_1^- v_2^+ \in E(G)$, since otherwise $G[\{v_1^-, v_1, x_1, v_2^+\}]$ would be an induced claw. Now consider $G[\{v_1^-, v_2, v_2^+, x_2, x_1\}]$. Since G is Z_2-free and $x_1 v_2^+, x_1 v_1^-, x_1 v_2, x_2 v_2^+ \notin E(G)$, we have $v_1^- x_2 \in E(G)$. But then $x_2 v_2 v_1 v_2^+ \overrightarrow{C} v_1^- x_2$ is a 1-chord extension of C, a contradiction.

Case 2. k=3

Let the vertices of C be labeled v_1, v_2, v_3, and let $U_i := \{x \in V(G) - V(C) | x \in N(v_i)\}$ for $1 \leq i \leq 3$. Then

$$U_i \cap U_j = \emptyset \quad \text{for} \quad 1 \leq i < j \leq 3, \tag{1}$$

since otherwise, there would be a 0-chord extension of C. If $R = V(G) - [U_1 \cup U_2 \cup U_3 \cup \{v_1, v_2, v_3\}] \neq \emptyset$, then there is a vertex $w \in R$ contained in an induced Z_2 (together with v_1, v_2 and v_3), since G is connected, a contradiction. Thus $R = \emptyset$. Next observe that $G[u_i]$ is complete for $1 \leq i \leq 3$, since G is C-free and by equation (1).

Suppose now that $|U_1| \leq |U_2| \leq |U_3|$. Since G is 2-connected and by the previous assumption we have

$$U_2, U_3 \neq \emptyset. \tag{2}$$

If a vertex $w_i \in U_i$ is adjacent to two vertices $w_{j1}, w_{j2} \in U_j$, then $G[\{v_1, v_2, v_3, w_{j1}, w_{j2}, w_i\}]$ contains an induced Z_2, a contradiction. If a vertex $w \in U_i$ is not adjacent to two vertices $w_{j1}, w_{j2} \in U_j$, then $G[\{w_i, v_i, v_j, w_{j1}, w_{j2}\}]$ is an induced Z_2, a contradiction. Thus, by equation (2) we conclude that

$$1 \leq |U_2| \leq |U_3| \leq 2, \tag{3}$$

which implies $5 \leq n \leq 9$.

Now using all of this information we obtain the eight exceptional graphs $Z_{5.1}, Z_{6.1}, Z_{6.2}, Z_{7.1}, Z_{7.2}, Z_{7.3}, Z_{8.1}$ and $Z_{9.1}$ depicted in figure F6. ∎

Corollary 13. *If G is a 2-connected CZ_2-free graph on $n \geq 10$ vertices, then G is cycle 2-extendable.*

3. Concluding Remarks

The proof of Theorem D admits another interesting corollary which seems not to be mentioned elsewhere.

Corollary 14. *Let G be a 2-connected C-free graph. If G is HP_7-free, then G is hamiltonian.*

Unfortunately, the graph in Figure 3 shows that it is not possible to obtain an analogue of Theorem 6 replacing "D-free" by "H-free." However, the whole proof concept of Theorem 6 can be used in the same way to prove the following result.

Theorem 15. *Let G be a 2-connected, C-free graph. If, moreover, G is HP_7-free and $n \geq 9$, then G is either pancyclic or missing only one cycle.*

Sketch of Proof. We follow the proof of Theorem 6 and state only the main differences. If $n \leq 7$, then G is either pancyclic or isomorphic to $C_4, C_5, C_6, G_{6.1}, C_7, G_{7.1}, G_{7.3}$, or $G_{7,4}$. Next observe that in the proof of Theorem 1 only once a contradiction is obtained by the existence of an induced deer, namely in Case 2 ($v_1v_4 \in E(G)$). In order to avoid an induced P_7 we successively conclude that $v_5v_8, v_9v_{12}, \cdots, v_{4k-3}v_{4k}, v_{4k+1}v_{4k+4} \in E(G)$, where $n = 4k+1, n = 4k+2, n = 4k+3$, or $n = 4k+4$ (indices modulo n), respectively. As in Case 2 we know that $v_{4i-4}v_{4i}, v_{4i-4}v_{4i+1}, v_{4i-3}v_{4i+1} \in E(G)$ for each edge $v_{4i-3}v_{4i}$. Then, however, $v_nv_3v_2v_1v_5v_6 \overrightarrow{C} v_n$, $v_{n-1}v_2v_1v_nv_4v_5 \overrightarrow{C} v_{n-1}$, or $v_{n-3}v_2v_1v_{n-2}v_{n-1}v_nv_4 \overrightarrow{C} v_{n-3}$ is a C_{n-1}, if $n = 4k+1$, $n = 4k+2$, or $n = 4k+3$, respectively. Hence, we may assume that $n = 4k$, $k \geq 2$.

Now $v_1v_4v_5 \overrightarrow{C} v_1$ and $v_1v_5v_6 \overrightarrow{C} v_1$ are a C_{n-2} and a C_{n-3}, respectively. By successively replacing paths $v_{4i-3}v_{4i-2}v_{4i-1}v_{4i}$ by the edge $v_{4i-3}v_{4i}$ for $2 \leq i \leq n/4$ we exhibit all cycles C_m for $n/2 - 1 \leq m \leq n-4$. Next, for $1 \leq p \leq n/4 - 1$, we obtain a cycle C_{2p+1} by $v_1v_5 \cdots v_{4p+1}v_{4p}v_{4p-4} \cdots v_4v_1$ or a cycle C_{2p} by $v_1v_5 \cdots v_{4p-3}v_{4p}v_{4p-4} \cdots v_4v_1$, respectively.

Therefore, if $n \geq 9$, then G contains all cycles from C_8 up to C_n or is missing only one cycle. For $n = 8$ we refer to Figure 7.

A family of graphs that satisfy the assumptions of Theorem 15 and are missing exactly one cycle (namely, the cycle of length $n-1$) is given by the graphs in Figure 3. The graph $G_{8.10}$ on $n = 8$ vertices in Figure 4 shows that '$n \geq 9$' in the hypothesis of Theorem 15 cannot be improved.

C_4 C_5 C_6 $G_{6.1}$ C_7 $G_{7.1}$ $G_{7.3}$ $G_{7.4}$ $G_{8.1}$

$G_{8.2}$ $G_{8.4}$ $G_{8.7}$ $G_{8.8}$ $G_{8.9}$ $G_{8.10}$ $G_{9.1}$ $G_{9.7}$ $G_{10.1}$

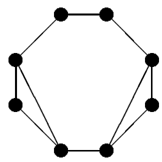


Figure 7

All 2-connected CHP_7-free graphs on $4 \leq n \leq 8$ vertices that are not pancyclic.

4. Appendix

Proof of Proposition 5. At first we generate all claw-free, P_7-free, and deer-free graphs on $n \leq 8$ vertices, which are not pancyclic [1]. Next suppose there is a claw-free, P_7-free and deer-free graph on $n+1$ vertices, $n \geq 8$, which is not pancyclic. Then, by Proposition 4, it has a 2-chord. Using this 2-chord in the RP (Proposition 1), we thus also obtain a counterexample on n vertices. Vice versa, the set of all counterexamples on $n+1$ vertices can be generated from the set of all counterexamples on n vertices as follows. Let G be a counterexample on n vertices $v_1, v_2, \ldots, v_n$ labeled such that $(\{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_n v_1\}) \subset E(G)$ (see Figure 4, where v_1 is always double-circled and vertices are labeled in clockwise orientation). We then successively replace each edge $v_i v_{i+1}$ of this C_n by a triangle with edges $v_i v_{i+1}, v_i v_{n+1}, v_{i+1} v_{n+1}$ if $1 \leq i \leq n-1$, and a triangle with edges $v_1 v_n, v_1 v_{n+1}, v_n v_{n+1}$, otherwise. Each new graph has to be checked whether it is claw-free, P_7-free, deer-free, and not pancyclic, and whether additional edges adjacent to v_{n+1} are possible. For the sake of brevity, the figures of those of these graphs, which are not counterexamples ($G_{9.10} - G_{9.41}, G_{10.6} - G_{10.31}, G_{11.6} - G_{11.8}, G_{12.3}, G_{12.4}$) will not be depicted. If such a graph is generated more than once (which will be frequently the case in the following), then its vertices are labeled according to its first occurence in this generation process.

We now distinguish seven cases.

Case 1. n=8

By the hypothesis of the proposition, the cycle C_8 contains a chord. Since G is claw-free, it contains a 2-chord or a 3-chord. Among all chords of C_8 choose an i-chord $(2 \leq i \leq 3)$ such that i is minimal. Choose a labeling $v_1, v_2, \ldots, v_8$ of the vertices of C_8 such that $(\{v_j v_{j+1} | 1 \leq j \leq 7\} \cup \{v_8 v_1, v_1 v_{i+1}\}) \subset E(G)$.

Case 1.1. i=2

Then G contains C_3, C_7 and C_8. If there is a 3-chord and a 4-chord then G is pancyclic, since a 4-chord gives a C_5 and a 3-chord gives C_4 and C_6. If there are only 4-chords then

[1] For $4 \leq n \leq 7$ it can be easily verified that all counterexamples are given by the graphs in Figure 4.

there is a pair of a 2-chord and a 4-chord that are crossing, since G is claw-free and has no 3-chord. Thus G has C_3, C_5, C_6, C_7 and C_8. If there is also a pair of a 2-chord and a 4-chord which are not crossing, then G is pancyclic. Otherwise we obtain the only counterexample $G_{8.1}$ having only 2-chords and 4-chords. If there are only 3-chords then G has C_3, C_4, C_6, C_7 and C_8. Now each pair of a 2-chord and a 3-chord whether they are crossing or not, leads to a C_5 and thus G is pancyclic, or we obtain counterexample $G_{8.9}$.

Hence we may assume that G has only 2-chords. Suppose first that there are no crossing 2-chords. Since G is P_7-free, there are at least two vertex disjoint 2-chords. If, for example, $v_1v_3, v_4v_6 \in E(G)$, then G is not deer-free. Thus the only counterexample with two 2-chords is given by $G_{8.2}$. If there are three 2-chords, for example v_1v_3, v_3v_5 and v_6v_8, then G is not deer-free. Thus the only counterexample with three 2-chords is given by $G_{8.3}$. If G has four 2-chords, then $v_3v_5, v_5v_7, v_1v_7 \in E(G)$ and G is pancyclic.

Next suppose there are crossing 2-chords. If, for example, $v_1v_3, v_2v_4, v_3v_5 \in E(G)$, then G is pancyclic. Hence we may assume that among every five successive vertices of C_8 there occur at most two 2-chords. We may assume that $v_2v_4 \in E(G)$.

If there is a pair of 2-chords with vertices from $\{v_4, v_5, \ldots, v_8, v_1\}$, which are not crossing, then G is pancyclic. Hence, we may assume that there is either a pair of crossing 2-chords or there is at most one 2-chord with vertices from $\{v_4, v_5, \ldots, v_8, v_1\}$. This gives the counterexamples $G_{8.4}, G_{8.5}, G_{8.6}, G_{8.7}$ and $G_{8.10}$.

Case 1.2. i=3

By (A) we also have $v_1v_5, v_4v_8 \in E(G)$. Thus G has C_3, C_4, C_5, C_6 and C_8. If $v_3v_8 \in E(G)$ then we obtain a C_7 by (D). Hence we may assume that $v_3v_8 \notin E(G)$, and thus $v_5v_8 \in E(G)$ by (B). Now every additional edge gives a C_7 and G will be pancyclic. Thus the only counterexample in this case is given by $G_{8.8}$.

Case 2. n=9

If we successively replace the edges of $G_{8.1}$ we obtain the graph $G_{9.42}$ (4 times), which is pancyclic, and the graph $G_{9.43}$ (4 times). The graph $G_{9.43}$ is only missing a C_4 and contains an induced deer $G[\{v_8, v_1, v_3, v_9, v_4, v_5, v_6\}]$. Any additional (possible) edge adjacent to v_9 gives a C_4.

For $G_{8.2}$ we obtain $G_{9.12}$ (4 times) and $G_{9.13}$ (4 times). The graph $G_{9.12}$ is only missing a C_5, which can be obtained by adding one of the edges $v_4v_9, v_5v_9, v_6v_9, v_7v_9$ or v_8v_9. If none of these edges is present, then $v_3v_9 \in E(G)$, since G is claw-free ($\langle v_8, v_1, v_9, v_3\rangle$), and we obtain the counterexample $G_{9.1}$. The graph $G_{9.13}$ is missing a C_4 and a C_5 which can be obtained by adding one of the edges $v_1v_9, v_2v_9, v_6v_9, v_7v_9$ or v_8v_9. If none of these edges is present, then $v_5v_9 \in E(G)$, since $G[\{v_6, v_5, v_4, v_9, v_3, v_1, v_8\}]$ is an induced deer. Thus, we obtain counterexample $G_{9.2}$.

For $G_{8.3}$ we obtain $G_{9.14}$, $G_{9.15}$, $G_{9.16}$ and $G_{9.17}$ (each of them 2 times). The graphs $G_{9.14}$, $G_{9.15}$ and $G_{9.16}$ are pancyclic, and $G_{9.17}$ contains an induced deer $G[\{v_4, v_5, v_7, v_9, v_8, v_1, v_2\}]$, and is missing only a C_4, which is obtained by adding one of the edges $v_1v_9, v_2v_9, v_3v_9, v_4v_9, v_5v_9$ or v_6v_9.

For $G_{8.4}$ we obtain $G_{9.3}$ (2 times), $G_{9.18}$ (4 times), and $G_{9.19}$ (2 times). The graph $G_{9.3}$ is only missing a C_5, which can be obtained by any additional edge adjacent to v_9. The graph $G_{9.18}$ is pancyclic and $G_{9.19}$ is missing only a C_5. However, $G_{9.19}$ is not claw-free ($\langle v_2, v_3, v_9, v_5\rangle$ and $\langle v_2, v_9, v_4, v_5\rangle$). Thus, $v_2v_9 \in E(G)$ or $v_5v_9 \in E(G)$ which gives a C_5.

For $G_{8.5}$ we obtain $G_{9.20}, G_{9.21}, G_{9.22}$ and G_{94} (each of these 2 times). In this case $G_{9.20}, G_{9.21}$, and $G_{9.22}$ are pancyclic, and counterexample $G_{9.4}$ is only missing a C_5, which can be obtained by adding one of the edges $v_1v_7, v_2v_7, v_3v_7, v_4v_7$, or v_5v_7. Thus v_7v_9 can be added and we obtain counterexample $G_{9.8}$.

For $G_{8.6}$ we obtain $G_{9.23}, G_{9.24}, G_{9.25}, G_{9.2}, G_{9.26}, G_{9.27}, G_{9.28}$, and $G_{9.29}$. The graph $G_{9.23}$ is pancyclic, and $G_{9.24}$ and $G_{9.25}$ contain an induced deer ($G[\{v_5, v_6, v_7, v_8, v_1, v_2, v_9\}]$ and $G[\{v_5, v_6, v_7, v_8, v_1, v_3, v_9\}]$), and are missing only a C_5, which is obtained by adding any edge adjacent to v_9. Counterexample $G_{9.2}$ is only missing a C_5, which can be obtained by adding one of the edges $v_1v_9, v_2v_9, v_3v_9, v_7v_9$, or v_8v_9. Thus v_6v_9 can be added and we obtain (once more) counterexample $G_{9.4}$. Also, $G_{9.26}$ contains an induced deer $G[\{v_3, v_4, v_5, v_9, v_6, v_7, v_8\}]$ and is missing only a C_5, which can be obtained by adding one of the edges v_1v_9, v_2v_9, v_3v_9, or v_8v_9, or by adding v_4v_9 and v_7v_9. Thus either $v_4v_9 \in E(G)$ or $v_7v_9 \in E(G)$, and we obtain (once more) counterexamples $G_{9.4}$ and $G_{9.3}$. The graph $G_{9.27}$ contains an induced P_7 ($G[\{v_8, v_1, v_2, v_4, v_5, v_6, v_9\}]$) and is missing only a C_5, which can be obtained by adding one of the edges $v_1v_9, v_2v_9, v_3v_9, v_4v_9$, or v_8v_9. Thus $v_5v_9 \in E(G)$, and we obtain (once more) counterexample $G_{9.3}$. By the same arguments $G_{9.28}$ and $G_{9.29}$ can be handled to obtain in both cases counterexample $G_{9.5}$.

For $G_{8.7}$ we obtain $G_{9.30}, G_{9.31}, G_{9.26}, G_{9.6}, G_{9.32}, G_{9.33}, G_{9.27}$, and $G_{9.34}$. The graph $G_{9.30}$ is not claw-free ($\langle v_1, v_2, v_9, v_4\rangle$ and $\langle v_1, v_9, v_3, v_4\rangle$) and is missing only a C_5, which can be obtained by adding any edge adjacent to v_9. Both $G_{9.31}$ and $G_{9.34}$ are pancyclic. Counterexample $G_{9.6}$ is missing only a C_5 and is a subgraph of counterexample $G_{9.4}$. The graph $G_{9.32}$ contains two induced P_7 ($G[\{v_9, v_7, v_8, v_1, v_2, v_4, v_5\}]$ and $G[\{v_9, v_7, v_8, v_1, v_3, v_4, v_5\}]$) and is missing only a C_5, which can be obtained by adding one of the edges $v_1v_9, v_2v_9, v_3v_9, v_4v_9$, or v_5v_9. Thus v_8v_9 can be added to obtain counterexample $G_{9.7}$. A repeat of the previous argument handles $G_{9.33}$.

For $G_{8.8}$ we obtain $G_{9.35}$ (4 times), $G_{9.36}$ (2 times), and $G_{9.37}$ (2 times), which are all pancyclic.

For $G_{8.9}$ we obtain $G_{9.38}$, $G_{9.39}$ (2 times), $G_{9.40}$ (2 times), $G_{9.41}$ (2 times), and $G_{9.1}$. Both $G_{9.40}$ and $G_{9.41}$ are pancyclic, $G_{9.39}$ contains an induced P_7 ($G[\{v_9, v_2, v_1, v_8, v_7, v_6, v_5\}]$), and $G_{9.38}$ contains an induced deer ($G[\{v_4, v_3, v_2, v_9, v_1, v_8, v_7\}]$). Both are missing only a C_5, which can be obtained by adding any edge adjacent to v_9, except for v_1v_9 in $G_{9.39}$ and v_3v_9 or v_8v_9 in $G_{9.38}$, which leads to counterexamples $G_{9.5}$ and $G_{9.5}$ or $G_{9.7}$, respectively. Counterexample $G_{9.1}$ can again be extended to counterexample $G_{9.7}$.

For $G_{8.10}$ we obtain $G_{9.44}$ (2 times), $G_{9.45}$, $G_{9.46}$ (2 times), $G_{9.47}$ (2 times) and $G_{9.48}$, which are all missing a C_5 and a C_6. Moreover, all of them have an induced claw, an induced deer or an induced P_7. Adding edges adjacent to v_9, these induced subgraphs disappear and we obtain counterexample $G_{9.9}$ in the case of $G_{9.44}$ and $G_{9.45}$, counterexample $G_{9.10}$ in the case of $G_{9.46}$ and $G_{9.47}$ and counterexample $G_{9.11}$ in the case of $G_{9.47}$ and $G_{9.48}$.

Case 3. n=10

The graph $G_{9.1}$ is only missing a C_5, which is obtained if we replace one of the edges v_4v_5, v_5v_6 or v_6v_7 by a triangle. Otherwise, $G_{10.32}, G_{10.9}$ and $G_{10.10}$ are obtained (each of them 2 times). Also, $G_{10.32}$ contains an induced P_7 $G[\{v_3, v_4, v_7, v_8, v_9, v_1, v_{10}\}]$ and is missing only a C_5, which can be obtained by adding one of the edges $v_3v_{10}, v_4v_{10}, v_5v_{10}, v_6v_{10}, v_7v_{10}$, or

v_8v_{10}. Thus v_9v_{10} can be added to obtain counterexample $G_{10.1}$. The graph $G_{10.9}$ contains an induced deer $G[\{v_8, v_9, v_2, v_{10}, v_3, v_4, v_5\}]$ and is missing only a C_5, which can be obtained by adding one of the edges $v_1v_{10}, v_5v_{10}, v_6v_{10}, v_7v_{10}, v_8v_{10}$, or v_9v_{10}. Thus v_4v_{10} can be added to obtain counterexample $G_{10.2}$. Using the same argumentation v_2v_{10} can be added in $G_{10.10}$ to obtain $G_{10.2}$.

The graph $G_{9.2}$ is only missing a C_5, which is obtained if we replace one of the edges v_4v_5 or v_6v_7 by a triangle. Otherwise, $G_{10.11}$ (2 times), $G_{10.12}$ (2 times), $G_{10.13}$ (2 times), and $G_{10.14}$ are generated, each of them missing only a C_5. The graph $G_{10.11}$ has an induced deer $G[\{v_4, v_5, v_7, v_8, v_9, v_1, v_{10}\}]$, and any additional edge adjacent to v_{10} gives a C_5, Also, $G_{10.12}$ has an induced P_7 $G[\{v_{10}, v_3, v_4, v_6, v_7, v_9, v_1\}]$, and a C_5 can be obtained by adding one of the edges $v_1v_{10}, v_5v_{10}, v_6v_{10}, v_7v_{10}, v_8v_{10}$, or v_9v_{10}. Thus v_4v_{10} can be added to obtain $G_{10.2}$ (once more). By the same argument v_2v_{10} can be added in $G_{10.13}$ to obtain $G_{10.2}$. Also, $G_{10.14}$ contains an induced deer $G[\{v_{10}, v_6, v_7, v_8, v_9, v_1v_2\}]$ and any edge adjacent to v_{10} gives a C_5.

The graph $G_{9.3}$ is only missing a C_5, which is obtained if we replace one of the edges v_2v_3, v_4v_5, v_6v_7 or v_8v_9 be a triangle. Otherwise, $G_{10.15}$ (2 times), $G_{10.16}$ (2 times) and $G_{10.3}$ are generated. Now, $G_{10.15}$ has an induced P_7 $G[\{v_4, v_5, v_6, v_7, v_9, v_1, v_{10}\}]$, and every additional edge v_iv_{10} gives a C_5, except for v_9v_{10}, which leads to counterexample $G_{10.4}$. Also, $G_{10.16}$ has an induced deer $G[\{v_6, v_7, v_9, v_1, v_2, v_3, v_{10}\}]$ and every additional edge v_iv_{10} gives a C_5. The same holds for counterexample $G_{10.3}$.

The graph $G_{9.4}$ is only missing a C_5, which is obtained if we replace one of the edges v_1v_9, v_2v_3, v_3v_4 or v_5v_6 by a triangle. Otherwise, $G_{10.17}, G_{10.18}, G_{10.19}, G_{10.20}$ and $G_{10.31}$ are generated. Both $G_{10.17}$ and $G_{10.18}$ are not claw-free ($\langle v_9, v_1, v_{10}, v_3\rangle$ and $\langle v_3, v_4, v_{10}, v_6\rangle$), and every additional edge v_iv_{10} gives a C_5. In $G_{10.19}$ and $G_{10.20}$ every additional edge v_iv_{10} gives a C_5 except for v_8v_{10} in $G_{10.19}$ and v_6v_{10} in $G_{10.20}$, which both lead to counterexample $G_{10.5}$. In $G_{10.31}$ $G[\{v_{10}, v_9, v_2, v_3, v_5, v_6, v_7\}]$ is an induced P_7.

The graph $G_{9.5}$ is only missing a C_5, which is obtained if we replace on of the edges $v_2v_3, v_4v_5, v_5v_6, v_6v_7$, or v_7v_8 by a triangle. Otherwise, $G_{10.2}, G_{10.21}, G_{10,22}$, and $G_{10.23}$ are generated. In counterexample $G_{10.2}$ every edge v_iv_{10} gives a C_5, except for v_9v_{10}, which leads to counterexample $G_{10.5}$. The graph $G_{10.21}$ is not C-free ($\langle v_2, v_3, v_{10}, v_5\rangle$), and adding v_2v_{10} or v_5v_{10} gives a C_5. Also, $G_{10.22}$ has an induced P_7 $G[\{v_6, v_5, v_4, v_2, v_1, v_9, v_{10}\}]$, and $G_{10.23}$ an induced deer $G[\{v_6, v_8, v_9, v_{10}, v_1, v_2, v_3\}]$. Every additional edge v_iv_{10} gives a C_5 except for v_1v_{10} in $G_{10.22}$, which leads to counterexample $G_{10.4}$, and except for v_2v_{10} and v_8v_{10} in $G_{10.23}$. Adding both edges gives a C_5, whereas adding only one edge leads to counterexamples $G_{10.5}$ and $G_{10.4}$, respectively.

The graph $G_{9.6}$ is missing only a C_5, which is obtained if one of the edges v_4v_5 or v_6v_7 is replaced by a triangle. Otherwise, $G_{10.24}$ (2 times), $G_{10.25}$ (2 times), $G_{10.26}$ (2 times), and $G_{10.27}$ are generated. Both $G_{10.24}$ and $G_{10.25}$ are not C-free ($\langle v_9, v_1, v_{10}, v_3\rangle$ or $\langle v_1, v_{10}, v_3, v_4\rangle$), and adding only v_3v_{10} or v_1v_{10} gives no C_5, and leads to $G_{10.6}$, which has an induced P_7. The graph $G_{10.26}$ has an induced deer $G[\{v_9, v_1, v_3, v_{10}, v_4, v_5, v_7\}]$ and $G_{10.27}$ is not C-free ($\langle v_4, v_5, v_{10}, v_7\rangle$). In both graphs each additional edge v_iv_{10} gives a C_5.

Also, $G_{9.7}$ is only missing a C_5 which is obtained if we replace one of the edges v_1v_2, v_4v_5, v_5v_6, v_6v_7 or v_8v_9 be a triangle. Otherwise, $G_{10.28}$, $G_{10.5}$, $G_{10.7}$, and $G_{10.29}$ are generated. The graph $G_{10.28}$ has an induced deer $G[\{v_8, v_1, v_2, v_{10}, v_3, v_4, v_5\}]$, and every additional edge v_iv_{10} gives a C_5 except for v_4v_{10}, which leads to counterexample $G_{10.5}$. Every additional edge v_iv_{10} also gives a C_5 in the other three graphs except for v_2v_{10} in $G_{10.6}$, which leads (again) to counterexample $G_{10.5}$. The graph $G_{10.7}$ has an induced P_7.

The graph $G_{9.8}$ is also only missing a C_5, which is obtained if we replace one of the edges $v_1v_2, v_3v_4, v_4v_5, v_6v_7, v_7v_8$ or v_9v_1 by a triangle. Otherwise, $G_{10.30}$ is generated (3 times), in which every additional edge v_iv_{10} gives a C_5.

The graph $G_{9.9}$ is only missing a C_6, which is obtained if we replace one of the edges v_1v_2, v_2v_3, v_8v_9, or v_9v_1 by a triangle. Otherwise, all generated graphs are not deer-free and we obtain counterexamples $G_{10.6}$ and $G_{10.7}$.

The graph $G_{9.10}$ is missing a C_5 and a C_6. Moreover, all of them have an induced claw, an induced deer or an induced P_7. Adding edges adjacent to v_{10}, these induced subgraphs disappear and we obtain counterexamples $G_{10.6}$ and $G_{10.8}$.

Finally, the graph $G_{9.11}$ is only missing a C_5 and a C_6. A repeat of the previous arguments this time leads to counterexamples $G_{10.7}$ and $G_{10.8}$.

Case 4. n=11

The graph $G_{10.1}$ is only missing a C_5, which is obtained if we replace one of the edges $v_1v_2, v_4v_5, v_5v_6, v_6v_7, v_9v_{10}$, or $v_{10}v_1$ by a triangle. Otherwise, $G_{11.8}$ is generated (4 times). Also, $G_{11.6}$ has an induced deer $G[\{v_8, v_9, v_2, v_{11}, v_3, v_4, v_5\}]$, and every additional edge v_iv_{11} gives a C_5, except for v_4v_{11}, which leads to counterexample $G_{11.1}$.

The graph $G_{10.2}$ is only missing a C_5, which is obtained if we replace one of the edges $v_1v_2, v_3v_4, v_4v_5, v_5v_6$ or v_6v_7 by a triangle. If we replace v_2v_3 (by a triangle), we obtain a graph that is not claw-free, and every additional edge v_iv_{11} gives a C_5. Replacing v_7v_8 or v_8v_9 we obtain two graphs that are not D-free. Every additional edge v_iv_{11} gives a C_5 except for v_9v_{11} or v_7v_{11}, respectively, which leads (in both cases) to counterexample $G_{11.2}$. Replacing v_9v_{10} or v_1v_{10} we obtain two graphs that are not C-free. Every additional edge v_iv_{11} gives a C_5 except for v_1v_{11} or v_9v_{11}, respectively, which leads (in both cases) to counterexample $G_{11.1}$.

The graph $G_{10.3}$ is only missing a C_5, which is obtained if we replace one of the edges v_1v_2, v_4v_5, v_6v_7, or v_9v_{10} by a triangle. Otherwise, $G_{11.9}$ (2 times) and $G_{11.10}$ (4 times) are generated. Both of them are not C-free and every additional edge gives a C_5.

The graph $G_{10.4}$ is only missing a C_5, which is obtained if we replace one of the edges $v_1v_2, v_2v_3, v_4v_5, v_6v_7, v_8v_9, v_9v_{10}$, or v_1v_{10} by a triangle. Replacing v_3v_4 or v_7v_8 we obtain a graph, which is not C-free, and every additional edge gives a C_5. If we replace v_5v_6, we obtain counterexample $G_{11.2}$, and every additional edge v_iv_{11} gives a C_5.

The graph $G_{10.5}$ is only missing a C_5 which is obtained if we replace one of the edges $v_1v_2, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_8v_9$, or v_1v_{10} by a triangle. Replacing v_2v_3 or v_9v_{10} we obtain two graphs, which are not C-free, and every additional edge v_iv_{11} gives a C_5. Replacing v_7v_8 we obtain counterexample $G_{11.3}$, and every additional edge v_iv_{11} gives a C_5.

The graph $G_{10.6}$ is only missing a C_6 which is obtained if we replace one of the edges v_1v_2, v_2v_3, v_3v_4, or v_4v_5 by a triangle. Otherwise, all generated graphs have an induced claw, an induced deer or an induced P_7. Adding edges adjacent to v_{11}, these induced subgraphs disappear and we obtain counterexamples $G_{11.4}$, $G_{11.5}$ and $G_{11.6}$.

Also, the graph $G_{10.7}$ is only missing a C_6 and can be treated like $G_{10.6}$. We this time obtain counterexamples $G_{11.5}$ and $G_{11.7}$.

Finally, the graph $G_{10.8}$ is missing a C_5 and a C_6. Replacing successively edges by a triangle, we obtain graphs which have an induced claw, an induced deer or an induced P_7. Adding edges adjacent to v_{11}, these induced subgraphs disappear and we obtain counterexamples $G_{11.5}$ and $G_{11.6}$.

Case 5. n=12

The graph $G_{11.1}$ is only missing a C_5, which is obtained if we replace one of the edges $v_1v_2, v_2v_3, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_{10}v_{11}$, or v_1v_{11} by a triangle. Otherwise, $G_{12.3}$ and $G_{12.4}$ (2 times) are generated, and $G_{12.3}$ is not C-free and $G_{12.4}$ is not deer-free. Every additional edge v_iv_{12} gives a C_5 except for $v_{10}v_{12}$ in $G_{12.4}$, which leads to counterexample $G_{12.1}$.

The graph $G_{11.2}$ is only missing a C_5 which is obtained if we replace one of the edges $v_2v_3, v_4v_5, v_5v_6, v_6v_7, v_7v_8, v_8v_9$, or $v_{10}v_{11}$ by a triangle. Replacing $v_1v_2, v_3v_4, v_9v_{10}$ or v_1v_{11} we obtain graphs that are not C-free and every additional edge v_iv_{12} gives a C_5, except for $v_{11}v_{12}$ or v_2v_{12} in the cases of v_1v_2 or v_1v_{11}, respectively. This leads (in both cases) to counterexample $G_{12.1}$.

The graph $G_{11.3}$ is only missing a C_5 which is obtained if we replace one of the edges $v_1v_2, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_9v_{10}$, or v_1v_{11} by a triangle. Replacing v_2v_3, v_7v_8, v_8v_9 or $v_{10}v_{11}$ we obtain graphs that are not C-free and every additional edge v_iv_{12} gives a C_5, except for v_9v_{12} or v_7v_{12} in the cases of v_7v_8 or v_8v_9, respectively. This leads (in both cases) to counterexample $G_{12.2}$.

The graph $G_{11.4}$ is only missing a C_6 which is obtained if we replace one of the edges $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_8v_9, v_9v_{10}, v_{10}v_{11}$, or $v_{11}v_1$ by a triangle. Otherwise, three graphs are generated which are not deer-free. Then every additional edge $v_{12}v_i$ gives a C_6 or we obtain counterexample $G_{12.3}$.

Also, the graph $G_{11.5}$ is only missing a C_6 which is obtained if we replace one of the edges $v_1v_2, v_2v_3, v_{10}v_{11}$, or $v_{11}v_1$ by a triangle. Otherwise, all generated graphs have an induced claw, an induced deer or an induced P_7. Adding edges adjacent to v_{12}, these induced subgraphs disappear and we obtain counterexamples $G_{12.3}$ and $G_{12.4}$.

Also, $G_{11.6}$ is only missing a C_6 and can be treated like $G_{11.5}$. We this time obtain counterexample $G_{12.3}$.

Finally, $G_{11.7}$ is only missing a C_6 and can be treated like $G_{11.4}$. We this time obtain counterexample $G_{12.4}$.

Case 6. n=13

Both $G_{12.1}$ and $G_{12.2}$ are only missing a C_5. Replacing an edge by a triangle gives a C_5 except for v_3v_4 and v_9v_{10} in $G_{12.1}$ and v_3v_4 and v_1v_{12} in $G_{12.2}$. In all four cases we obtain graphs that are not C-free and every additional edge v_iv_{13} gives a C_5.

The graph $G_{12.3}$ is only missing a C_6 which is obtained if we replace one of the edges $v_2v_3, \ldots, v_5v_6, v_7v_8, \ldots, v_{10}v_{11}$ by a triangle. Otherwise, all generated graphs have an induced claw, an induced deer or an induced P_7. Adding edges adjacent to v_{13}, these induced subgraphs disappear and we obtain the counterexample G_{13}.

Finally, $G_{12.4}$ is only missing a C_6 and can be treated like $G_{12.3}$.

Case 7. n=14

The graph G_{13} is only missing a C_6. Replacing an edge by a triangle gives a C_6 except for v_6v_7. In this case the generated graph is not deer-free and every additional edge v_iv_{14} gives a C_6. ∎

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