The flower conjecture in special classes of graphs

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May 17, 1994

Abstract

We say that a spanning eulerian subgraph $F \subset G$ is a *flower* in a graph G if there is a vertex $u \in V(G)$ (called the center of F) such that all vertices of G except u are of degree exactly 2 in F. A graph G has the *flower property* if every vertex of G is a center of a flower.

Kaneko conjectured that G has the flower property if and only if G is hamiltonian. In the present paper we prove this conjecture in several special classes of graphs, among others in squares and in a certain subclass of claw-free graphs.

1. INTRODUCTION

We consider only finite undirected graphs without loops and multiple edges. For terminology and notation not defined here we refer to [1].

If $x \in V(G)$, then by $d_G(x)$ we denote the degree of x and by $N_G(x)$ (or simply $N(x)$) we denote the set of all vertices of G that are adjacent to x. Unlike in [1], we denote the induced subgraph on a set $M \subset V(G)$ by $\langle M \rangle$. If for every $x \in V(G)$, $\langle N(x) \rangle$ has a property P, then we say that G is *locally* P.

*This research was done partly while the second author was visiting the University of West Bohemia and partly while the first author was visiting RWTH Aachen. Research supported by EC-grant No 927.

The *square* of a connected graph H is the graph $G = H^2$ such that $V(G) = V(H)$ and two vertices x, y are adjacent in G if and only if x, y are at distance at most 2 in H. If G and G' are graphs, then we say that G is *G'-free* if G contains no induced subgraph isomorphic to G'. Specifically, in the case that $G' = K_{1,3}$ we say that G is *claw-free* and the star $K_{1,3}$ will be also referred to as *the claw.*

Let G be a graph of order $n \geq 3$ and $u \in V(G)$. If there is a spanning eulerian subgraph F of G such that $d_F(u) \geq 2$ and $d_F(v) = 2$ for all $v \in V(G)$, $v \neq u$, then F is called a *flower at u* and the vertex u is called the *center* of F. If F is a flower at u then the components of the graph $F - u$ will be called the *leaves* of F. Since $1 \leq d_{F-u}(x) \leq 2$ for every $x \neq u$, every leaf of F is a path.

We say that a graph G has the *flower property* if G has a flower at u for every $u \in V(G)$.

Obviously, every hamiltonian cycle of G is a flower and hence every hamiltonian graph has the flower property. Kaneko [4] conjectured that these properties are equivalent.

Conjecture [4] (The Flower Conjecture). A graph G has the flower property if and only if G is hamiltonian.

Kaneko and Ota [5] proved that if G has the flower property, then G is 1-tough and has a 2-factor.

In the present paper we prove the flower conjecture in several special classes of graphs.

2. OBSERVATIONS

Proposition 1. Let G be a graph with minimum degree $\delta(G) \leq 3$. Then G has the flower property if and only if G is hamiltonian.

Proof. If $x \in V(G)$ is a vertex such that $d_G(x) \leq 3$ then every flower at x is a hamiltonian cycle. ■

Proposition 2. Let G be a graph with connectivity $\kappa(G) \leq 2$. Then G has the flower property if and only if G is hamiltonian.

Proof. If $\kappa(G) = 1$ then G is neither hamiltonian nor has the flower property and thus we can assume that $\kappa(G) = 2$. Suppose that G has the flower property. Let $\{x, y\}$ be a 2-vertex cut set of G. By the result of Kaneko and Ota [5], G is 1-tough and hence $G - \{x, y\}$ has two components H_1, H_2. Choose $z_i \in H_i$ and let F_i be a flower of G at z_i, $i = 1, 2$. Then $P_1 = F_1 - H_1$ is a hamiltonian $\{x, y\}$-path in $G - H_1$ and, similarly, $P_2 = F_2 - H_2$ is a hamiltonian $\{y, x\}$-path in $G - H_2$. But then the cycle $C = xP_1yP_2x$

is a hamiltonian cycle in G. ■

Proposition 3. Let G be a bipartite graph. Then G has the flower property if and only if G is hamiltonian.

Proof. Let (X, Y) be the bipartition of G. If F is a flower at $u \in X$, then

$$\sum_{x \in X} d_F(x) = |E(F)| = \sum_{y \in Y} d_F(y),$$

from which

$$d_F(u) + 2|X - \{u\}| = 2|Y|,$$

or, equivalently,

$$d_F(u) - 2 + 2|X| = 2|Y|,$$

which implies $|X| \leq |Y|$. Taking a flower F' at $v \in Y$, we get analogously $|X| \geq |Y|$ and hence $|X| = |Y|$. This implies $d_F(u) = 2$ and hence F is a hamiltonian cycle. ■

Proposition 4. Let G be a graph and let $x \in V(G)$ be such that $\langle N(x) \rangle$ is a complete graph. Then G has the flower property if and only if G is hamiltonian.

Proof. Suppose that G has the flower property and let F be a flower at x such that $d_F(x)$ is minimum. Suppose that $d_F(x) > 2$ and let z_1, z_2 be endvertices of two different leaves of F. Then, deleting from F the edges xz_1, xz_2 and adding z_1z_2, we get a flower F' with $d_{F'}(x) < d_F(x)$, which contradicts the minimality of F. Thus, $d_F(x) = 2$ and F is a hamiltonian cycle. ■

3. SQUARES

Fleischner [2] proved the following theorem.

Theorem A. **[2]** If H is a 2-connected graph and $G = H^2$, then G is hamiltonian.

The following statement is also due to Fleischner and follows from Theorem 3 of [3].

Theorem B. **[3]** Let y be an arbitrary vertex of a 2-connected graph H. Then the graph $G = H^2$ contains a hamiltonian cycle C such that both edges of C containing y are in

$E(H)$.

Using these two theorems, we can prove the following.

Theorem 5. Let H be a graph and $G = H^2$. Then G has the flower property if and only if G is hamiltonian.

Proof. Suppose that $G = H^2$ and G has the flower property.

If H is 2-connected, then G is hamiltonian by Theorem A. Hence $\kappa(H) = 1$.

If H has a vertex x with $d_H(x) = 1$, then $\langle N_G(x) \rangle$ is a complete graph and G is hamiltonian by Proposition 4. Hence $\delta(H) \geq 2$.

If H has a cut edge (i.e. an edge which is a block) $xy \in E(H)$, then, since $\delta(H) \geq 2$, $\{x, y\}$ is a 2-vertex cut set of G and G is hamiltonian by Proposition 2.

Hence we can assume that H has connectivity $\kappa(H) = 1$, minimum degree $\delta(H) \geq 2$ and every block of H has at least three vertices.

Let H_1 be an endblock (i.e. a block containing exactly one cutvertex) of H and let x be the cutvertex of H in H_1. By Theorem B, there is a hamiltonian cycle C_1 in H_1^2 such that $xx^- \in E(H)$ and $xx^+ \in E(H)$ (here we denote by x^- and x^+ the predecessor and successor of x on C).

Put $H_2 = H - (H_1 - x)$, choose a vertex $y \in N_{H_1}(x)$ and let F be a flower in G at y. We consider the subgraph $F' = F - (H_1 - x)$. Since $1 \leq d_{F'}(v) \leq 2$ for every $v \in V(H_2)$ and $d_{F'}(v) = 1$ if and only if $v = x$ or $v \in N(x)$, F' is a collection of paths P_i, $i = 1, \ldots, \ell$, with endvertices $a_i, b_i \in N(x) \cup \{x\}$, $i = 1, \ldots, \ell$.

If all the vertices a_i, b_i, $i = 1, \ldots, \ell$, are distinct from x, then, since $\langle N(x) \cup \{x\} \rangle$ is a clique in G, $C' = xa_1P_1b_1a_2P_2b_2 \ldots a_\ell P_\ell b_\ell x^+ Cx$ is a hamiltonian cycle in G. Hence there is an i_0 such that $x = a_{i_0}$ (or, similarly, $x = b_{i_0}$). We can assume without loss of generality that $x = a_1$ and then analogously $C' = xP_1b_1a_2P_2b_2 \ldots a_\ell P_\ell b_\ell x^+ Cx$ is a hamiltonian cycle in G. ■

4. CLAW-FREE GRAPHS

Theorem 6 Let G be a graph and let $x \in V(G)$ be such that $\langle N(x) \rangle$ is connected and x is not a vertex of an induced claw in G. Then G has the flower property if and only if G is hamiltonian.

Proof. Suppose that G has the flower property but is not hamiltonian and let F be a flower at x such that $d_F(x)$ is minimum. Let $P_1, \ldots, P_\ell$ be the leaves of F and denote by x_i^1, x_i^2 the endvertices of P_i, $i = 1, \ldots, \ell$. If some endvertices $x_{i_1}^{j_1}, x_{i_2}^{j_2}$ ($i_1 \neq i_2$) of two different leaves P_{i_1}, P_{i_2} are adjacent, then, deleting from F the edges $xx_{i_1}^{j_1}, xx_{i_2}^{j_2}$ and adding

$x_{i_1}^{j_1}x_{i_2}^{j_2}$, we get a flower F' with $d_{F'}(x) < d_F(x)$. Hence no endvertices of two different leaves of F can be adjacent. This implies that $\ell = 2$ since otherwise $\langle x, x_1^1, x_2^1, x_3^1\rangle$ is an induced claw centred at x. Moreover, $x_1^1x_1^2 \in E(G)$ (since otherwise $\langle x, x_1^1, x_1^2, x_2^1\rangle$ is an induced claw centred at x) and, similarly, $x_2^1x_2^2 \in E(G)$. Denote $x_i^1x_i^2 = e_i$, $i = 1, 2$.

Since $\langle N(x)\rangle$ is connected, there is a path P in $\langle N(x)\rangle$ joining e_1 to e_2. Suppose that the flower F and the path P are chosen such that, among all flowers F at x with minimum $d_F(x)$, the e_1, e_2-path P is shortest possible. We can assume without loss of generality that P is an x_1^1, x_2^1-path. Let $x_1^1 = z_0, z_1, \ldots, z_k = x_2^1$ be the vertices of P.

Suppose first that there is an integer i, $1 \le i \le k$, such that $z_{i-1}z_i \in E(F)$. If $z_{i-1}z_i \in E(P_1)$, then, deleting from F the edges $z_{i-1}z_i$, xx_1^1 and xx_1^2 and adding the edges $x_1^1x_1^2$, xz_{i-1} and xz_i (not excluding the possible case $i = 1$), we get a contradiction with the minimality of P. Similarly we show that $z_{i-1}z_i \notin E(P_2)$ and hence $z_{i-1}z_i \notin E(F)$ for any i, $1 \le i \le k$, i.e., no two consecutive vertices of P are consecutive on F.

We now consider the subgraph $\langle z_1, x, z_1^-, z_1^+\rangle$, where z_1^-, z_1^+ are the predecessor and successor of z_1 on F. If $z_1^-z_1^+ \in E(G)$, then, deleting from F the edges $z_1z_1^-, z_1z_1^+$ and xz_0 and adding the edges z_0z_1, z_1x and $z_1^-z_1^+$, we get a flower that contradicts the minimality of P. Hence $z_1^-z_1^+ \notin E(G)$. Since $\langle z_1, x, z_1^-, z_1^+\rangle$ cannot be an induced claw centred at z_1, we have $xz_1^- \in E(G)$ or $xz_1^+ \in E(G)$. We distinguish the following four cases.

Case	*Deleted edges*	*Added edges*
$xz_1^- \in E(G), z_1 \in V(P_1)$	$z_1z_1^-, xx_1^1, xx_1^2$	$xz_1^-, xz_1, x_1^1x_1^2$
$xz_1^- \in E(G), z_1 \in V(P_2)$	$z_1z_1^-, xx_2^1, xx_2^2$	$xz_1^-, xz_1, x_2^1x_2^2$
$xz_1^+ \in E(G), z_1 \in V(P_1)$	$z_1z_1^+, xx_1^1, xx_1^2$	$xz_1^+, xz_1, x_1^1x_1^2$
$xz_1^+ \in E(G), z_1 \in V(P_2)$	$z_1z_1^+, xx_2^1, xx_2^2$	$xz_1^+, xz_1, x_2^1x_2^2$

In each of these cases we get a contradiction with the minimality of P. ■

Corollary 7. Let G be a claw-free graph which is not locally disconnected. Then G has the flower property if and only if G is hamiltonian.

Proof follows immediately from Theorem 6.

Remark 8. It is easy to observe that if G is a locally disconnected claw-free graph, then, for every $x \in V(G)$, $\langle N(x)\rangle$ consists of two vertex disjoint cliques and hence G is a line graph. Moreover, if $G = L(H)$, then G is locally disconnected if and only if H is triangle-free. Thus, according to Theorem 6, for the proof of the flower conjecture in claw-free graphs, it remains to prove it in the case that G is a line graph of a triangle-free graph. Hence we have the following corollary.

Corollary 9. Let G be a claw-free graph that is not a line graph of a triangle-free graph.

Then G has the flower property if and only if G is hamiltonian.

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