Toughness and hamiltonicity in almost claw-free graphs

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Abstract

Some known results on claw-free ($K_{1,3}$-free) graphs are generalized to the larger class of almost claw-free graphs which were introduced by Ryjáček. In particular, we show that a 2-connected almost claw-free graph is 1-tough, and that a 2-connected almost claw-free graph on n vertices is hamiltonian if $\delta \geq \frac{1}{3}(n-2)$, thereby (partly) generalizing results of Matthews and Sumner. Finally, we use a result of Bauer et al. to show that a 2-connected almost claw-free graph on n vertices is hamiltonian if $d(u) + d(v) + d(w) \geq n$ for all independent sets of vertices u, v and w.

Keywords: Hamilton cycle, hamiltonian graph, tough graph, (almost) claw-free graph.

AMS Subject Classifications (1991): 05C45.

1 Introduction

We use Bondy & Murty [3] for terminology and notation not defined here and consider simple graphs only.

Throughout, let G be a graph of order n. The connectivity of G is denoted by $\kappa(G)$, the number of vertices in a maximum independent set of G by $\alpha(G)$, the set of vertices adjacent to a vertex v by $N(v)$, and the degree of v by $d(v) = |N(v)|$. We denote by $\sigma_k(G)$ the minimum value of the degree-sum of any k pairwise nonadjacent vertices if $k \leq \alpha(G)$; if $k > \alpha(G)$, we put $\sigma_k(G) = k(n-1)$. Instead of $\sigma_1(G)$ we use the more common notation $\delta(G)$. If G has a Hamilton cycle (a cycle containing every vertex of G), then G is called *hamiltonian*; G is called *hamiltonian-connected* if every two vertices of G are connected by a Hamilton path (a path containing every vertex of G). The graph G is *t-tough* ($t \in \mathbf{R}$, $t \geq 0$) if $|S| \geq t \cdot \omega(G-S)$ for every subset S of $V(G)$ with $\omega(G-S) > 1$, where $\omega(G-S)$ denotes the number of components of $G-S$. The *toughness* of G, denoted $\tau(G)$, is the maximum value of t for which G is t-tough ($\tau(K_n) = \infty$ for all $n \geq 1$). A *dominating set* of G is a subset S of $V(G)$ such that every vertex of G belongs to S or is adjacent to a vertex of S. The *domination number*, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. The graph G is *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$. A vertex $v \in V(G)$ is a *center of a claw* if v has three pairwise nonadjacent neighbors. If H is a subgraph of G and S is a subset of $V(G)$ or a subgraph of G, then $N_H(S)$ denotes the set of all vertices of H having a neighbor in S; if $S = \{v\}$, we write $N(v,G)$ for the subgraph induced by $N_G(\{v\})$. We define the *local independence number* $\alpha_L(G)$ and the *local domination number* $\gamma_L(G)$ as follows:

$$\alpha_L(G) = \max\{\alpha(N(v,G)) \mid v \in V(G)\},$$

$$\gamma_L(G) = \max\{\gamma(N(v,G)) \mid v \in V(G)\}.$$

Obviously, $\gamma(G) \leq \alpha(G)$ and $\gamma_L(G) \leq \alpha_L(G)$ for every graph G. Moreover, it is easy to see that G is claw-free if and only if $\alpha_L(G) \leq 2$.

Following Ryjáček [11], we say a graph G is *almost claw-free* if there exists an independent set $A \subset V(G)$ such that $\alpha(N(v,G)) \leq 2$ for every $v \notin A$ and $\gamma(N(v,G)) \leq 2 < \alpha(N(v,G))$ for every $v \in A$. Equivalently, G is almost claw-free if $\gamma_L(G) \leq 2$ and the set A consisting of the centers of all claws is an independent set. Clearly, every claw-free graph is almost claw-free, and there exist almost claw-free graphs which are not claw-free. In [11] it was shown that every almost claw-free graph has no induced subgraph isomorphic to $K_{1,5}$ or $K_{1,1,3}$.

Our objective is to generalize results on claw-free graphs to almost claw-free graphs. In Section 2 we prove that in a noncomplete almost claw-free graph G, $\tau(G) \geq \min\{1, \frac{1}{2}\kappa(G)\}$, thereby (partly) generalizing a result of Matthews and Sumner [8]. In Section 4 we prove that a 2-connected almost claw-free graph G is hamiltonian if $\delta(G) \geq \frac{1}{3}(n-2)$. This result generalizes another result of Matthews and Sumner [9]. Finally, we use a result of Bauer et al. [1] to show that a 2-connected almost claw-free graph G is hamiltonian if $\sigma_3(G) \geq n$.

2 Toughness

Let G be a noncomplete graph. Then it is obvious that $\tau(G) \leq \frac{1}{2}\kappa(G)$. If G is claw-free, then equality holds, as was shown by Matthews and Sumner.

Theorem 1 *[8] If G is a noncomplete claw-free graph, then $\tau(G) = \frac{1}{2}\kappa(G)$.*

In the same paper they conjecture that every 4-connected claw-free graph is hamiltonian. This conjecture is a special case of the following well-known conjecture due to Chvátal.

Conjecture 2 *[5] Every 2-tough graph on $n \geq 3$ vertices is hamiltonian.*

Even in the case of claw-free graphs, a possible proof of the conjecture seems to be very difficult. Before we discuss some results on hamiltonicity involving degree conditions in Section 3, we first prove the following result which generalizes Theorem 1 in case $\kappa(G) \leq 2$.

Theorem 3 *If G is a noncomplete almost claw-free graph, then* $\tau(G) \geq \min\{1, \frac{1}{2}\kappa(G)\}$.

Proof. In any noncomplete graph G, clearly $\tau(G) \leq \frac{1}{2}\kappa(G)$. If G is not connected, then $\tau(G) = \frac{1}{2}\kappa(G) = 0$. Suppose $G \neq K_n$ is a connected almost claw-free graph and S is a cutset of G such that $\tau(G) = \frac{|S|}{\omega(G-S)} < \min\{1, \frac{1}{2}\kappa(G)\}$. Let $H_1, \ldots, H_p$ be the components of $G-S$. There exist at least $\kappa(G)$ disjoint paths from $u \in V(H_i)$ to $v \in V(H_j)$ for any $i, j \in \{1, \ldots, p\}$ with $i \neq j$. Each of these paths contains a vertex of S. Hence for each $i \in \{1, \ldots, p\}$ there are at least $\kappa(G)$ edges joining vertices of H_i to distinct vertices of S. Thus there are at least $p\kappa(G)$ edges from $G-S$ to S, counting at most one from any component of $G-S$ to a particular vertex of S. Suppose every vertex $v \in S$ has neighbors in at most two components of $G-S$. Then there are at most $2|S|$ edges from $G-S$ to S, counting at most one from any component of $G-S$ to a particular vertex of S. Then $p\kappa(G) \leq 2|S|$ or $\frac{1}{2}\kappa(G) \leq \frac{|S|}{p} = \tau(G)$, a contradiction.
Hence S contains a center x of a claw with neighbors in at least three components of $G-S$. Since G is almost claw-free, $\gamma(N(x, G)) \leq 2$. This implies that there exists a neighbor y of x in S, and, moreover, that x has neighbors in at least three components of $G-S$, and y is adjacent to vertices in precisely two of these components. But then $T = S - \{y\}$ is a cutset of G with $\omega(G-T) = \omega(G-S) - 1$, so that $\tau(G) \leq \frac{|T|}{\omega(G-T)} = \frac{|S|-1}{\omega(G-S)-1} < \frac{|S|}{\omega(G-S)} = \tau(G)$, a contradiction. Hence $\tau(G) \geq \min\{1, \frac{1}{2}\kappa(G)\}$. □

The graph G_0 of Figure 1 shows that we cannot prove an analogue of Theorem 1 for almost claw-free graphs with connectivity exceeding two. G_0 is almost claw-free (the set of centers of claws is $A = \{s_2, s_3\}$) and 3-connected, but if we let $S = \{s_1, s_2, s_3, s_4\}$, then $|S| = 4$, $\omega(G-S) = 3$ and hence

$$\tau(G) \leq \frac{|S|}{\omega(G-S)} = \frac{4}{3} < \frac{3}{2} = \frac{\kappa(G)}{2}.$$

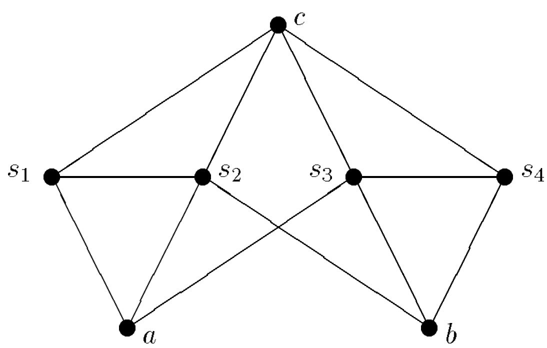

Figure 1. A 3-connected almost claw-free graph G_0 which is not $\frac{3}{2}$-tough.

3 Hamilton cycles

There are many results showing that claw-free graphs have interesting hamiltonian properties under certain additional assumptions. Here we focus on degree conditions ensuring hamiltonicity. The following result is due to Matthews and Sumner.

Theorem 4 *[9] If G is a 2-connected claw-free graph with $\delta(G) \geq \frac{1}{3}(n-2)$, then G is hamiltonian.*

The following generalization of Theorem 4 was independently obtained by Broersma and Zhang.

Theorem 5 *[4, 13] If G is a 2-connected claw-free graph with $\sigma_3(G) \geq n-2$, then G is hamiltonian.*

More generally, Zhang [13] proved that G is hamiltonian if G is a k-connected claw-free graph with $\sigma_{k+1}(G) \geq n-k$ $(k \geq 2)$.

Theorem 4 was extended to classes of graphs containing a restricted number of claws by Flandrin and Li [7].

An analogue of Theorem 5 for $K_{1,1,3}$-free and $K_{2,3}$-free graphs was obtained by Flandrin, Jung and Li [6].

In Section 4 we prove the following two results; the first generalizes Theorem 4, the second is an analogue of Theorem 5. These results are independent of the aforementioned results of Flandrin and Li, and Flandrin, Jung and Li, respectively.

Theorem 6 *If G is a 2-connected almost claw-free graph with $\delta(G) \geq \frac{1}{3}(n-2)$, then G is hamiltonian.*

Theorem 7 *If G is a 2-connected almost claw-free graph with $\sigma_3(G) \geq n$, then G is hamiltonian.*

Theorem 6 is best possible, but we do not know whether Theorem 7 is best possible. Perhaps Theorem 5 can be generalized to almost claw-free graphs.

The examples we know showing that Theorem 6 is best possible are the same examples that show Theorem 4 to be best possible, and they all have connectivity 2. It is likely that the degree bound in Theorem 6 can be improved for 3-connected graphs, as it is the case with Theorem 4 (as shown by Zhang's result). To show that Theorem 6 is more general than Theorem 4, consider the following graphs, one of which is drawn in Figure 2.

Let H_1, H_2, H_3 be three vertex disjoint copies of K_δ $(\delta \geq 2)$ and join two new vertices x and y to all vertices of H_2 and H_3. Join x also to y and to all vertices of H_1. Let G be a graph obtained from this graph by adding $k \geq 1$ edges such that $N_{H_3}(H_2) = \emptyset$, $N_{H_1}(H_2) \cap N_{H_1}(H_3) = \emptyset$ and $N_{H_1}(H_2) \cup N_{H_1}(H_3) \neq V(H_1)$. (For an example with $\delta = 3$, $k=3$, $N_{H_1}(H_2) = \{a\}$ and $N_{H_1}(H_3) = \{b\}$, see Figure 2.) Then G is a 2-connected almost claw-free graph with $n = 3\delta + 2$ vertices and hence G satisfies the assumptions of

Theorem 6, but it does not satisfy the assumptions of Theorem 4 since it is not claw-free. (Note that G also does not satisfy the assumptions of the main result of [7].)

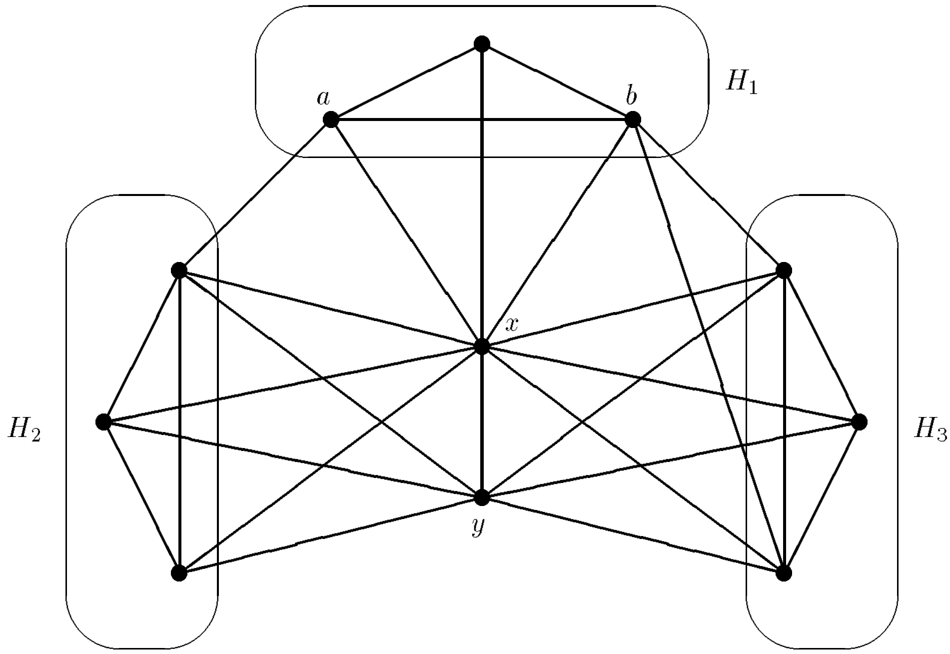

Figure 2. A graph satisfying the assumptions of Theorem 6 but not of Theorem 4.

4 Proofs of Theorems 6 and 7

We first introduce some additional notation and prove two auxiliary results.

Let C be a cycle of G. By $\overrightarrow{C}$ we denote the cycle C with a given orientation, and by $\overleftarrow{C}$ the same cycle with the reversed orientation. If $u, v \in V(C)$, then $u\overrightarrow{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\overrightarrow{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor.

Lemma 8 *Let $\overrightarrow{C}$ be a longest cycle in an almost claw-free graph G. Let $y \in V(G) - V(C)$ and let x be a neighbor of y on C such that $x^-x^+ \notin E(G)$. Then there exists a vertex $d \in V(C) \cap N(x^-) \cap N(x) \cap N(x^+)$ with the following properties: Either $d^+ = x^-$ or $d^- = x^+$, or there is a path Q_1 between d^- and d^+ and a path Q_2 between $x^-(x^+)$ and x such that $V(Q_1) \cap V(Q_2) = \emptyset$ and $V(Q_1) \cup V(Q_2) = \{x^-, x, x^+, d^-, d, d^+\}$.*

Proof. Suppose first that y and x^- have a common neighbor v in $N(x)$. It is clear that the choice of C implies $v \in V(C)$, and $yv^-, yv^+ \notin E(G)$. Since G is almost claw-free and x is a center of a claw, v is not a center of a claw, implying that $v^-v^+ \in E(G)$. We can extend C by replacing v^-vv^+ by v^-v^+, and x^-x by x^-vyx, a contradiction. Hence y and x^- have no common neighbor in $N(x)$. By symmetry, y and x^+ have no common neighbor in $N(x)$. Since $\gamma_L(G) \leq 2$, there is a vertex $d \in N(x)$ dominating both x^- and x^+. It is obvious that $d \in V(C)$ and that d is not a center of a claw. If $d^+ = x^-$ or $d^- = x^+$, then we are done. Suppose $d^+ \neq x^-$ and $d^- \neq x^+$. Consider the subgraph of G induced by $\{d^-, d, d^+, x^+\}$. At least one of the edges d^-d^+, d^-x^+ and d^+x^+ belongs to G. If $d^-d^+ \in E(G)$, then put $Q_1 = d^-d^+$ and $Q_2 = x^-dx^+x$. If $d^-x^+ \in E(G)$, then put $Q_1 = d^-x^+dd^+$ and $Q_2 = x^-x$. If $d^+x^+ \in E(G)$, then put $Q_1 = d^-dx^+d^+$ and $Q_2 = x^-x$.

The similar statement for x^+ follows by symmetry. □

In the sequel, let G be a nonhamiltonian 2-connected almost claw-free graph, let $\overrightarrow{C}$ be a longest cycle in G, and let H be a component of $G - V(C)$. Denote by $x_1, \ldots, x_k$ the vertices of $N_C(H)$ occurring on $\overrightarrow{C}$ in the order of their indices, and let $S_i = x_i^+ \overrightarrow{C} x_{i+1}^-$ and $s_i = |S_i|$. Clearly, $k \geq 2$. Let l_i denote the length of a longest path between x_i and x_{i+1} with all internal vertices in H ($i = 1, \ldots, k$; indices mod k). Note that in the proof of Lemma 9 we sometimes apply Lemma 8 to $\overleftarrow{C}$.

Lemma 9 $\sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} l_i + k$.

Proof. Let $i \in \{1, \ldots, k\}$ and let L_i denote a path of length l_i between x_i and x_{i+1} with all internal vertices in H (indices mod k). If we compare the length of C with the length of the cycle obtained from C by replacing $x_iS_ix_{i+1}$ by $x_iL_ix_{i+1}$, we obtain that $s_i \geq l_i - 1$ by the choice of C ($i = 1, \ldots, k$). We can however sometimes increase this lower bound on s_i by looking more carefully at the "configuration" concerning the vertices of S_i and its neighbors x_i and x_{i+1}. If, e.g., $x_i^-x_i^+ \in E(G)$, then we can increase the lower bound on s_i by 1 by observing that instead of replacing $x_iS_ix_{i+1}$ in C by $x_iL_ix_{i+1}$, we can replace $x_i^-x_iS_ix_{i+1}$ in C by $x_i^-x_i^+x_iL_ix_{i+1}$. In this case we say that the "left gain" on s_i is 1. Similarly, if $x_{i+1}^-x_{i+1}^+ \in E(G)$, we can increase the lower bound on s_i by 1. In this case we say that the "right gain" on s_i is 1. Note that in this case the left and right gain are additive, i.e. we can increase the lower bound on s_i by 2 if both $x_i^-x_i^+ \in E(G)$ and $x_{i+1}^-x_{i+1}^+ \in E(G)$.

More generally, we define the *left gain* $g_L(s_i)$ on s_i as the amount we can add to the lower bound $l_i - 1$ on s_i by only looking at the configuration concerning x_i (to be specified later), and the *right gain* $g_R(s_i)$ on s_i as the amount we can add to this lower bound by only looking at the configuration concerning x_{i+1} ($i = 1, \ldots, k$; indices mod k).

We will show that the left and right gains on s_i we obtain in the sequel are additive with one exception which needs a more careful analysis.

We first obtain values for $g_R(s_i)$ and $g_L(s_{i+1})$ by looking at the configuration concerning x_{i+1} ($i = 1, \ldots, k$; indices mod k). For this purpose we distinguish the following possible configurations concerning x_{i+1}.

A. $x_{i+1}^-x_{i+1}^+ \in E(G)$. We already showed that in this case $g_R(s_i) = 1$ and $g_L(s_{i+1}) = 1$.

B. $x_{i+1}^- x_{i+1}^+ \notin E(G)$. Then x_{i+1} is a center of a claw, and there exists a vertex $d \in V(C)$ associated to x_{i+1} with the properties given by Lemma 8. The following cycles, respectively, show that $d \notin \{x_{i+2}^-, x_{i+2}, x_{i+2}^+\}$: $x_{i+1}^- d \overleftarrow{C} x_{i+1} L_{i+1} x_{i+2} \overrightarrow{C} x_{i+1}^-$, $x_{i+1}^- d L_{i+1} x_{i+1} \overrightarrow{C} x_{i+2}^- x_{i+2}^+ \overrightarrow{C} x_{i+1}^-$, $x_{i+1} L_{i+1} x_{i+2} \overleftarrow{C} x_{i+1}^+ d \overrightarrow{C} x_{i+1}$.

B1. $d \in S_{i+1}$. Then $g_R(s_i) = 0$ and $g_L(s_{i+1}) = d_C(d, x_{i+1}) \geq 2$, since otherwise replacing $x_{i+1}^- x_{i+1} S_{i+1} x_{i+2}$ in C by $x_{i+1}^- d \overleftarrow{C} x_{i+1} L_{i+1} x_{i+2}$ we obtain a longer cycle. Here $d_C(u, v)$ denotes the distance along C between two vertices of C.

B2. $d \in S_i$. By similar arguments as in B1, we obtain $g_L(s_{i+1}) = 0$ and $g_R(s_i) = d_C(d, x_{i+1}) \geq 2$.

B3. In the other cases, $g_L(s_{i+1}) = 1$; otherwise, (using the terminology of Lemma 8) replacing $d^- d d^+$ by Q_1 and $x_{i+1}^- x_{i+1} S_{i+1} x_{i+2}$ by $x_{i+1}^- Q_2 x_{i+1} L_{i+1} x_{i+2}$ we obtain a longer cycle. Similarly, $g_R(s_i) = 1$.

As we argued before, $g_L(s_i)$ and $g_R(s_i)$ are additive if $x_i^- x_i^+ \in E(G)$ and $x_{i+1}^- x_{i+1}^+ \in E(G)$. It is not difficult to check that the same is true if only one of those edges is present, or in case $g_L(s_i) = 0$ or $g_R(s_i) = 0$. We can however not always guarantee the additivity in case $x_i^- x_i^+, x_{i+1}^- x_{i+1}^+ \notin E(G)$ and $g_L(s_i), g_R(s_i) > 0$. Then x_i and x_{i+1} are (nonadjacent) centers of a claw, and there are vertices $d_1, d_2 \in V(C)$ associated to x_i and x_{i+1}, respectively, with the properties given by Lemma 8. As before, it is clear that $d_1, d_2 \notin \{x_i^-, x_i, x_i^+, x_{i+1}^-, x_{i+1}, x_{i+1}^+\}$. We give a more detailed analysis of the possible cases.

Case 1. $d_1 \in x_{i+1} \overrightarrow{C} x_i$.

Suppose first $d_1 = d_2$. Since d_1 is not a center of a claw, at least one of $d_1^- x_i^-$ and $d_1^- x_i^+$ is an edge of G. Then, however, the cycles $d_1 \overrightarrow{C} x_i^- d_1^- \overleftarrow{C} x_{i+1} L_i x_i \overrightarrow{C} x_{i+1}^- d_1$ and $d_1 \overrightarrow{C} x_i L_i x_{i+1} \overleftarrow{C} x_i^+ d_1^- \overleftarrow{C} x_{i+1}^+ d_1$, respectively, contradict the choice of C. Hence $d_1 \neq d_2$. Suppose next $d_1 d_2 \in E(C)$. If $d_1 = d_2^+$, then the cycle $d_1 \overrightarrow{C} x_i L_i x_{i+1} \overrightarrow{C} d_2 x_{i+1}^- \overleftarrow{C} x_i^+ d_1$ contradicts the choice of C; if $d_1 = d_2^-$, then the cycle $d_1 \overleftarrow{C} x_{i+1} L_i x_i \overleftarrow{C} d_2 x_{i+1}^- \overleftarrow{C} x_i^+ d_1$ contradicts the choice of C. Hence $d_1 d_2 \notin E(C)$. Using the properties given in Lemma 8, the above observations yield that $g_L(s_i)$ and $g_R(s_i)$ are additive in this case.

Case 2. $d_1 \in x_i \overrightarrow{C} x_{i+1}$.

If $d_2 \notin x_i \overrightarrow{C} d_1$, then $g_L(s_i)$ and $g_R(s_i)$ are again additive. Suppose now $d_2 \in x_i \overrightarrow{C} d_1$. We first show that $d_1 \notin \{d_2, d_2^+\}$. If $d_1 = d_2$, then the following cycles, respectively, show that $d_1^- d_1^+ \notin E(G)$ and $x_i^- d_1^+ \notin E(G)$: $x_i L_i x_{i+1} \overleftarrow{C} d_1^+ d_1^- \overleftarrow{C} x_i^+ d_1 x_{i+1}^+ \overrightarrow{C} x_i$ and $x_i^- d_1^+ \overrightarrow{C} x_{i+1}^- d_1 \overleftarrow{C} x_i L_i x_{i+1} \overrightarrow{C} x_i^-$. Then, since d_1 is not a center of a claw, we obtain $x_i^- d_1^- \in E(G)$, and, by symmetry, $x_{i+1}^+ d_1^+ \in E(G)$. Now the cycle $x_i d_1^- \overleftarrow{C} x_i L_i x_{i+1} d_1 x_{i+1}^- \overleftarrow{C} d_1^+ x_{i+1}^+ \overrightarrow{C} x_i$ contradicts the choice of C. If $d_1 = d_2^+$, then the cycle $x_i^- d_1 \overrightarrow{C} x_{i+1}^- d_2 \overleftarrow{C} x_i L_i x_{i+1} \overrightarrow{C} x_i^-$ contradicts the choice of C. From these observations we conclude that $g_L(s_i) = d_C(x_i, d_1) \geq 4$ and $g_R(s_i) = d_C(x_{i+1}, d_2) \geq 4$. Clearly, $g_L(s_i)$ and $g_R(s_i)$ are not additive in this case, but we could regard the total gain on s_i as two additive gains of at least 2 at each side of S_i. In such cases we redefine $g_L(s_i) = 2$ and $g_R(s_i) = 2$.

If we adapt the definition of $g_L(s_i)$ and $g_R(s_i)$ in the way we discussed above (in Case 2), we obtain the following conclusion.

$$\sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k}(l_i - 1) + \sum_{i=1}^{k}(g_L(s_i) + g_R(s_i)) \geq \sum_{i=1}^{k} l_i - k + \sum_{i=1}^{k}(g_L(s_i) + g_R(s_{i+1})) \geq \sum_{i=1}^{k} l_i + k.$$

$\square$

Proof of Theorem 6. Assume $\delta(G) \geq \frac{1}{3}(n-2)$. Using Lemma 9, we obtain $n \geq \sum_{i=1}^{k} s_i + k + 1 \geq \sum_{i=1}^{k} l_i + 2k + 1 \geq 4k + 1 \geq 9$. Suppose $V(H) = \{v\}$. Then $\frac{1}{3}(n-2) \leq \delta(G) \leq d(v) \leq k \leq \frac{1}{4}(n-1)$, a contradiction. Hence no component of $G - V(C)$ is an isolated vertex. We may assume $|V(H)| \geq 2$. Among the pairs $v_1, v_2 \in V(H)$ for which

(1) $|N_C(v_1)| + |N_C(v_2)|$ is as large as possible

choose a pair u, v such that

(2) $|N_C(u) \cup N_C(v)|$ is as large as possible.

If $|N_C(u) \cup N_C(v)| \leq 1$, then (1) and (2) imply $|N_C(H)| \leq 1$, a contradiction. Hence $|N_C(u) \cup N_C(v)| \geq 2$. Moreover, by the 2-connectedness of G, we may assume u and v are chosen in such a way that $uy_1, vy_2 \in E(G)$ for two distinct vertices $y_1, y_2 \in V(C)$. Let $p = |N_C(u)|$, $q = |N_C(v)|$, $r = |N_C(u) \cap N_C(v)|$. Assume $p \geq q$ without loss of generality, and let $l(u,v)$ denote the length of a longest path between u and v in H. Denote $N_C(u) \cup N_C(v)$ by $\{x_1, \ldots, x_t\}$, where the vertices occur on $\overrightarrow{C}$ in the order of their indices. Then, using Lemma 9 for this subset $\{x_1, \ldots, x_t\}$ of $N_C(H)$, we obtain

$$\begin{aligned} n &\geq |V(H)| + |V(C)| \geq |V(H)| + \sum_{i=1}^{t} s_i + t \\ &\geq |V(H)| + \sum_{i=1}^{t} l_i + 2t \geq |V(H)| + 4t + \max\{2, r\} \cdot l(u,v). \end{aligned} \tag{3}$$

We distinguish two cases.

Case 1. $p + q \leq \delta(G) - 1$.
By the choice of u and v, $d_H(v_1) + d_H(v_2) \geq 2\delta(G) - (p+q) \geq \delta(G) + 1 \geq \frac{1}{3}(n+1)$ for all $v_1, v_2 \in V(H)$. By Theorem 3, G is 1-tough. Using a result of Bauer and Schmeichel [2], and Tian and Zhao [12], $|V(C)| \geq 2\delta(G) + 2$, hence $|V(H)| \leq n - (2\delta(G) + 2) \leq \frac{1}{3}(n-2)$. Thus $d_H(v_1) + d_H(v_2) \geq |V(H)| + 1$ for all $v_1, v_2 \in V(H)$, implying H is hamiltonian-connected by a result of Ore [10]. In particular, $l(u,v) = |V(H)| - 1$. Using (3), we have

$$n \geq |V(H)| + 4t + 2l(u,v) \geq 3|V(H)| + 4t - 2.$$

Clearly,

(4) $\delta(G) + 1 - q \leq |V(H)|$.

Hence

$$\begin{aligned} n &\geq 3\delta(G) + 4t - 3q + 1 = 3\delta(G) + t + 3(t-q) + 1 \\ &\geq 3\delta(G) + 3 \geq n + 1, \end{aligned}$$

a contradiction.

Case 2. $p + q \geq \delta(G)$.
Using (3) and (4), we have

$$\begin{aligned} n &\geq |V(H)| + 4t + \max\{2, r\} \cdot l(u,v) \\ &\geq \delta(G) + 1 - q + 4(p + q - r) + \max\{2, r\} \cdot l(u,v) \\ &= \delta(G) + 1 + 2(p+q) + (p + q - r) + p - 3r + \max\{2, r\} \cdot l(u,v) \\ &\geq 3\delta(G) + 3 + p - 3r + \max\{2, r\} \cdot l(u,v) \\ &\geq n + 1 + \max\{2, r\} \cdot l(u,v) - 2r. \end{aligned}$$

This clearly yields a contradiction in case $l(u,v) \geq \min\{2, r\}$. For the remaining cases assume $l(u,v) = 1$ and $r \geq l(u,v) + 1$. Then $N_H(u) \cap N_H(v) = \emptyset$, hence $|V(H)| \geq 2\delta(G) - (p+q)$. By (3),

$$\begin{aligned} n &\geq |V(H)| + 4t + r \\ &\geq 2\delta(G) - (p+q) + 4(p + q - r) + r \\ &= 2\delta(G) + (p+q) + (p + q - r) + (p + q - 2r) \\ &\geq 3\delta(G) + 2 \geq n. \end{aligned}$$

This implies $p = q = r = 2$, $\delta(G) = 4$, $n = 14$ and $|V(H)| = 4$. Now u and v have neighbors w_1 and w_2 in H, respectively, such that $w_1w_2, vw_1, uw_2 \notin E(G)$ (since $l(u,v) = 1$). Furthermore, $d_H(w_1) + d_H(w_2) = 2$ since $|V(H)| = 4$, while on the other hand the choice of u and v implies $d_H(w_1) + d_H(w_2) \geq 2\delta(G) - (p+q) = 8 - 4 = 4$, a contradiction. $\square$

Proof of Theorem 7. Assume $\sigma_3(G) \geq n$. By Theorem 3, G is 1-tough. We use the following lemma. The first part of this lemma is [1, Theorem 5] and the second part is implicit in the proof of [1, Theorem 9].

Lemma. Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Then every longest cycle of G is a dominating cycle. Moreover, if G is nonhamiltonian, G contains a longest cycle C such that $\max\{d(v) \mid v \in V(G) - V(C)\} \geq \frac{1}{3}\sigma_3(G)$.

Let C be a dominating cycle such that there is a vertex $v \in V(G) - V(C)$ with $d(v) \geq \frac{1}{3}\sigma_3(G) \geq \frac{1}{3}n$. By Lemma 9 (with $d(v) = k$), $n \geq \sum_{i=1}^{k} s_i + k + 1 \geq \sum_{i=1}^{k} l_i + 2k + 1 \geq 4k + 1 \geq \frac{4}{3}n + 1$, a contradiction. $\square$

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