Shortest walks in almost claw-free graphs

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Abstract

There have been many results concerning claw-free graphs and hamiltonicity. Recently, Jackson and Wormald have obtained more general results on walks in claw-free graphs. In this paper, we consider the family of almost claw-free graphs that contains the previous one, and give some results on walks, especially on shortest covering walks visiting only once some given vertices.

Throughout the paper, we deal with simple undirected graphs without loops and multiple edges. A *(x_0, x_l)-walk* (or simply a *walk*) in G is a sequence of vertices $C = x_0x_1...x_l$ such that $x_ix_{i+1} \in E(G)$ for every $i = 0,\ 1, ...,\ l-1$. If $x \in V(G)$ occurs in this sequence, then we write $x \in C$ and we say that x is *visited* by C. The number l will be called the *length* of C and denoted by $l(C)$. If $x_0 = x_l$, then we say that C is a *closed walk*. For every closed walk $C = x_0x_1...x_l$ and $x \in V(G)$, we put $v(x, C) = |\{i = 1, ..., l; x_i = x\}|$. If $v(x, C) = t$ then we say that C *visits* t-times the vertex x. The length of the closed walk can be expressed as $l(C) = \sum_{x \in C} v(x, C)$. A closed walk C such that $v(x, C) \geq 1$ for every $x \in V(G)$ is said to be a *covering walk of G* (or simply a covering walk). A covering walk C is said to be a *k-walk of G* or simply a *k-walk* (k being an integer), if $v(x, C) \leq k$ for every $x \in V(G)$. Clearly, every connected graph has a k-walk for some $k \geq 1$ and every hamiltonian cycle is a 1-walk.

We consider every walk to be oriented in the natural way by increasing subscripts (taken modulo l for closed walks) and, for any $x \in C$, we denote by x^- and x^+ the predecessor and successor of x, respectively, on C in this orientation. If we consider a walk C in the opposite orientation, then we denote it by $\overleftarrow{C}$.

If C is a walk, every (a, b)-walk P such that P is a subwalk of C (i.e. a subsequence of consecutive vertices of C) will be denoted by aCb. If moreover C is a closed walk and

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for a vertex x we have $a^- = x$, $b^+ = x$ and $x \notin P$, then the walk aCb will be called a *branch of C at x* (i.e. branches of C at x are the "parts of C" between two consecutive visits of x by C). The vertices a and b are the *endvertices of the branch P*. Clearly, for every $x \in C$, there are $v(x, C)$ branches of C at x.

We say that the vertices x and y are *related* (denoted $x \sim y$), if $x = y$ or $xy \in E(G)$. When $x \sim y$ then $\widehat{xy}$ denotes the sequence xy or the single vertex x, respectively.

For any $M \subset V(G)$, $\langle M \rangle$ denotes the subgraph induced by M in G. For any $x \in V(G)$, the *neighbourhood* of x, denoted by $N(x)$, is the set of vertices which are adjacent to x. If $\langle N(x) \rangle$ is connected then we say that x is a *locally connected vertex*. The graph G is *locally connected* if all its vertices are locally connected.

A set $A \subset V(G)$ is *independent* if any $x, y \in A$ are non-adjacent. The size of a maximum independent set in G is denoted by $\alpha(G)$ and referred to as the *independence number* of G. A set $B \subset V(G)$ is *dominating* if every vertex of G belongs to B or has a neighbour in B. The size of a minimum dominating set is called the *domination number of G* and denoted by $\gamma(G)$. If $\gamma(G) \leq k$, we say that G is *k-dominated.*

If H is a graph, then G is said to be *H-free* if G does not contain a copy of H as an induced subgraph. The complete bipartite graph $K_{1,3}$ will be referred to as the *claw.* Clearly, G is claw-free if and only if $\alpha(\langle N(x) \rangle) \leq 2$ for every $x \in V(G)$.

Claw-free graphs are known to have many interesting properties. Oberly and Sumner [6] proved the following result:

Theorem A: Every connected locally connected claw-free graph on at least three vertices is hamiltonian.

Clark [2] showed that, under the same assumptions, G is vertex-pancyclic. Hendry [3] further strengthened this result showing that G is fully cycle extendable.

Jackson and Wormald [4] removed the hypothesis "G is locally connected" and obtained the following result:

Theorem B: Every connected claw-free graph has a 2-walk.

In [5], the class of claw-free graphs was extended in the following way: we say that G is *almost claw-free* if there is an independent set $A \subset V(G)$ such that $\alpha(\langle N(x) \rangle) \leq 2$ for $x \notin A$ and $\gamma(\langle N(x) \rangle) \leq 2 < \alpha(\langle N(x) \rangle)$ for $x \in A$. Equivalently, G is almost claw-free if the centers of induced claws are independent and their neighbourhoods are 2-dominated. Clearly, every claw-free graph is almost claw-free.

It can be shown (see [5]) that every almost claw-free graph is $K_{1,5}$-free and $K_{1,1,3}$-free and that, for every $x \in A$, $\gamma(\langle N(x) \rangle) = 2$. In [5] and [1], several results in claw-free graphs are extended to the class of almost claw-free graphs.

In the present paper we proceed on in the investigations which were done in [4]. Our theorem 7 is a common generalization of Theorems A and B.

First we show that in almost claw-free graphs every covering walk can be shortened until it becomes a 2-walk.

<u>**Proposition 1**</u> (reduction lemma): Let G be an almost claw-free graph, $x \in V(G)$ and C a covering walk. If $v(x, C) \geq 3$ then there exists a covering walk C' such that $l(C') \leq l(C) - 1$, $v(x, C') = v(x, C) - 1$ and $v(y, C') \leq v(y, C)\ \forall y \neq x$.

Proof: Suppose that $v(x, C) \geq 3$ and denote by $P_1, \ldots, P_s$ $(s \geq 3)$ the branches of C at x and by x_i^k the endvertices of P_i $(i = 1, \ldots, s;\ k = 1,\ 2)$ in accordance with the orientation of C.

If there are $i,\ j,\ k,\ h$ such that $i \neq j$ and $x_i^k \sim x_j^h$, then we can suppose without loss of generality (changing if necessary the orientation of some branches and the order of the branches) that $i = 1$, $j = 2$, $k = 2$ and $h = 1$. The closed walk $C' = xP_1\widehat{x_1^2 x_2^1}P_2xP_3x\ldots P_sx$, obtained from C by deleting the edges x_1^2x and xx_2^1 and adding the edge or vertex $\widehat{x_1^2 x_2^1}$, has the required property.

We thus suppose that x_i^k and x_j^h are not related for all $i \neq j,\ \ i,\ j \in \{1,\ 2, \ldots, s\}$ and for all $k,\ h \in \{1,\ 2\}$ (but x_i^1 and x_i^2 can be related). This implies $x \in A$. Hence there are vertices d_1 and d_2 in $N(x)$ such that every vertex in $N(x)$ is adjacent to d_1 or d_2. As the set A is independent, neither d_1 nor d_2 can center a claw and therefore at least one of them (say, d_1) is adjacent to both endvertices of some P_i (say, P_1) and to at least one other vertex x_j^k (say, x_2^1). Since C is a covering walk, there exists a branch P_{i_0} that visits d_1. Clearly $d_1 \notin \{x_{i_0}^1,\ x_{i_0}^2\}$ for otherwise endvertices of different branches would be related.

If $d_1^+ \sim d_1^-$, then C' is obtained from C by replacing the subwalk $d_1^- d_1 d_1^+$ by $\widehat{d_1^- d_1^+}$ and the subwalk $x_1^2 x x_2^1$ by $x_1^2 d_1 x_2^1$.

Henceforth d_1^- and d_1^+ are not related.

If $d_1 \in P_1$, then, considering $\langle d_1,\ d_1^-,\ d_1^+,\ x_2^1 \rangle$ and the fact that $d_1 \notin A$, we see that $d_1^- \sim x_2^1$ or $d_1^+ \sim x_2^1$. In the first case we take $C' = x_1^1 P_1 \widehat{d_1^- x_2^1} P_2 x P_3 x \ldots P_s x \overleftarrow{P_1}\ d_1 x_1^1$ which is obtained from C by deleting the edges xx_1^1, xx_2^1, $d_1^- d_1$ and adding $\widehat{d_1^- x_2^1}$ and $d_1 x_1^1$. In the second case similarly $C' = xP_1 d_1 x_1^2 \overleftarrow{P_1}\ \widehat{d_1^+ x_2^1} P_2 x P_3 x \ldots P_s x$.

If $d_1 \in P_i$ for some $i \geq 2$, then, considering $\langle d_1,\ d_1^-,\ d_1^+,\ x_1^1 \rangle$, we see that $d_1^- \sim x_1^1$ or $d_1^+ \sim x_1^1$. In the first case $C' = x_1^1 P_1 x_1^2 d_1 P_i x P_{i+1} x \ldots P_s x P_2 x \ldots P_{i-1} x P_i \widehat{d_1^- x_1^1}$ is obtained from C by deleting the edges xx_1^1, $x_1^2 x$, $d_1^- d_1$ and adding $x_1^2 d_1$ and $\widehat{d_1^- x_1^1}$. In the second case, $C' = xP_2 x P_3 x \ldots P_i d_1 x_1^2 \overleftarrow{P_1}\ \widehat{x_1^1 d_1^+} P_i x P_{i+1} x \ldots P_s x$ is obtained from C by deleting the edges xx_1^1, $x_1^2 x$, $d_1 d_1^+$ and adding $x_1^2 d_1$ and $\widehat{x_1^1 d_1^+}$.

In all cases, the new walk C' has the required properties. $\square$

If we apply the reduction lemma while there exists some vertex which is visited at least three times by C, we obtain the following three corollaries. The second one extends Theorem A.

<u>**Corollary 2**</u>: Every shortest covering walk of a connected almost claw-free graph is a 2-walk.

Corollary 3: Every connected almost claw-free graph has a 2-walk.

Corollary 4: In a connected almost claw-free graph, if there is a covering walk visiting exactly once the vertex x, then there is a 2-walk visiting x exactly once.

In the following, we are interested in those vertices which are visited exactly once by some 2-walk. It is easy to see that such vertices cannot be cutvertices of G. The next proposition shows that in almost claw-free graphs this trivial necessary condition is also sufficient.

Proposition 5: Let G be a connected almost claw-free graph and $x \in V(G)$. Then there is a 2-walk C such that $v(x, C) = 1$ if and only if x is not a cutvertex of G.

Proof: Clearly if $v(x, C) = 1$ for some 2-walk, then x is not a cutvertex. So suppose that x is not a cutvertex and $v(x, C) = 2$ for every 2-walk. Let C be a 2-walk. Denote by P, Q the two branches of C at x and by p_1, p_2, q_1, q_2 their endvertices, respectively.

If $p_i \sim q_j$ for some $i, j \in \{1, 2\}$, for example $p_2 \sim q_1$, then the walk $C' = xp_1P\widehat{p_2q_1}Qq_2x$ obtained from C by deleting the edges xp_2, xq_1 and adding $\widehat{p_2q_1}$ satisfies $v(x, C') = 1$; thus vertices p_i and q_j are not related $\forall i, j \in \{1, 2\}$.

If $p_1 \sim p_2$ then we observe that, as x is not a cutvertex of G, there are vertices $a \in P$ and $b \in Q$ such that $a \sim b$. If we put $C' = p_1P\widehat{ab}Qq_2xq_1Q\widehat{ba}P\widehat{p_2p_1}$ (deleting from C the edges xp_1, xp_2 and adding $\widehat{p_2p_1}$ and twice $\widehat{ab}$), then $v(x, C') = 1$ and, by Corollary 4, there is a 2-walk that visits x exactly once. Consequently, p_1 and p_2 are not related and similarly q_1 and q_2 are not related.

Hence no two of the vertices p_1, p_2, q_1, q_2 are related and thus $x \in A$. As G is almost claw-free , $\langle N(x) \rangle$ is 2-dominated. Let $\{d_1, d_2\}$ be a dominating set of $\langle N(x) \rangle$.

Suppose first that d_1 is adjacent to some p_i and q_j, $i, j \in \{1, 2\}$, for example p_2 and q_1. In this case, for the walk $C' = xp_1Pp_2d_1q_1Qq_2x$, obtained from C by deleting the edges xp_2, xq_1 and adding d_1p_2 and d_1q_1, we have $v(x, C') = 1$ and (using, if necessary, Corollary 4), we have a contradiction.

Thus some of the d_i's (say d_1) dominates p_1 and p_2 and the other one, d_2, dominates q_1 and q_2. We again find vertices $a \in P$ and $b \in Q$ such that $a \sim b$ (note that the edge ab may be one of the edges d_1q_i or d_2p_j). If we put $C' = xp_1P\widehat{ab}Qq_2d_2q_1Q\widehat{ba}Pp_2x$ (deleting in C the edges xq_1, xq_2 and adding d_2q_1, d_2q_2 and twice $\widehat{ab}$) then $v(x, C') = 1$ and, by Corollary 4, we can again construct a 2-walk that visits x exactly once. This contradiction achieves the proof of Proposition 5. $\square$

We now turn our attention to shortest covering walks, i.e., to 2-walks with as few edges as possible. The number of vertices which are visited twice by such a 2-walk is as small as possible. The following lemma studies the structure of the neighborhood of the vertices which are visited twice by a shortest 2-walk.

Lemma 6: Let G be a connected almost claw-free graph, C a shortest covering walk of G and x a vertex of G which is visited twice by C. Let P and Q be the two branches of C at x with endvertices p_1, p_2, q_1, q_2, respectively. Then

(i) $\forall i, j \in \{1, 2\}$, p_i and q_j are not related

(ii) If moreover $x \notin A$, then $p_1 \sim p_2$, $q_1 \sim q_2$ and $\forall i, j \in \{1, 2\}$, every vertex of $N(p_i) \cap N(q_j) \cap N(x)$ centres a claw.

Proof: Assume that G, C, x, P and Q fulfil the hypotheses of Lemma 6.

(i) If $p_1 \sim q_1$, then the covering walk $C' = p_1 P p_2 x q_2 \overleftarrow{Q} \widehat{q_1 p_1}$ is shorter than C.

(ii) Since $x \notin A$, $\langle x, p_1, p_2, q_1 \rangle$ is not a claw and by (i), $p_1 \sim p_2$. Similarly, $q_1 \sim q_2$. Let a be a vertex of $N(p_i) \cap N(q_j) \cap N(x)$; we can suppose without loss of generality that $a \in P$ and $i = j = 1$. If $a^- \sim a^+$, then the walk $p_1 P \widehat{a^- a^+} P p_2 x q_2 \overleftarrow{Q} q_1 a p_1$ is shorter than C. If $a^- x \in E(G)$ or $a^+ x \in E(G)$, then the walk $a^- x q_2 \overleftarrow{Q} q_1 a P \widehat{p_2 p_1} P a^-$ or $a^+ x q_2 \overleftarrow{Q} q_1 a \overleftarrow{P} \widehat{p_1 p_2} \overleftarrow{P} a^+$ is shorter than C, respectively. Therefore a centres the claw $\langle a, x, a^-, a^+ \rangle$. $\square$

In the 2-connected almost claw-free graph of Figure 1, the length of a shortest 2-walk is 17 (e.g. $bcdefnoghijaxm\ell xkb$). In accordance with Proposition 5, the vertex x is visited once by some 2-walk (e.g. by $abcdecbkx\ell mnofghija$ of length 18) but it is visited twice by every shortest 2-walk.

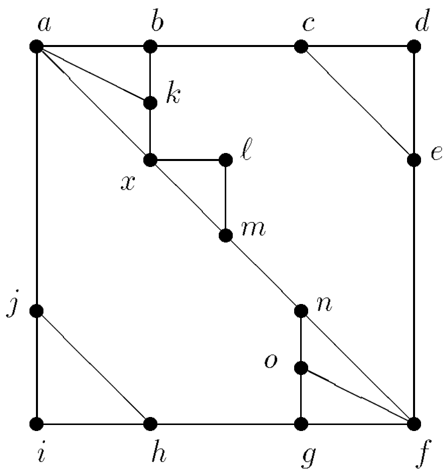

Figure 1

This example shows that, in the statement of Proposition 5, "2-walk" cannot be replaced by "shortest 2-walk". This, however, becomes possible with a good choice of the vertex x. Note that, in our counterexample, $\langle N(x) \rangle$ is not connected. Motivated by Therem A, we strengthen in Theorems 7 and 10 the result of Proposition 5 for the locally connected vertices of G, first restricting our considerations to those ones which do not centre a claw.

Theorem 7: Let G be a connected almost claw-free graph and put

$$B = \{x \in V(G) \setminus A; \langle N(x) \rangle \text{ connected}\}.$$

Then there is a shortest 2-walk C such that $v(x, C) = 1$ for every $x \in B$.

Proof: Suppose, on the contrary, that for every shortest 2-walk C the set $M_C = \{x \in B | v(x, C) = 2\}$ is not empty. Let C be a shortest 2-walk for which the cardinality of M_C is minimum and x a vertex of M_C. Let P and Q be the branches of C at x with endvertices p_1, p_2 and q_1, q_2 respectively. Since $\langle N(x) \rangle$ is connected, there is a shortest path R in $\langle N(x) \rangle$ which joins one of p_1, p_2 to one of q_1, q_2. Assume that C is chosen so that, among all shortest 2-walks with $x \in M_C$ and $|M_C|$ minimum, R is shortest possible. We can assume without loss of generality that R is a (p_1, q_1)-path and none of the vertices p_2, q_2 is on R. Let $p_1, x_1, ..., x_k, q_1$ be the vertices of R. Since $x \notin A$ and R is a shortest path, $k \leq 2$ (otherwise $\langle x, p_1, x_2, q_1 \rangle$ is a claw). By Lemma 6(i), $k \geq 1$. If $k = 1$, then, by Lemma 6(ii), the vertex x_1 belongs to A and the walk $p_1 P p_2 x q_2 \overleftarrow{Q} q_1 x_1 p_1$ contradicts the minimality of M_C. Therefore $k = 2$. Again by Lemma 6(ii), $p_1 \sim p_2$ and $q_1 \sim q_2$. Suppose now that $x_2 \in P$. If $x_2 \in A$, then for the walk $C' = x q_2 \overleftarrow{Q} q_1 x_2 P \widehat{p_2 p_1} P x_2 x$ we have $l(C') = l(C)$ and $|M_{C'}| < |M_C|$. Hence $x_2 \notin A$. We consider the induced subgraph $\langle x_2, x_2^-, x_2^+, x \rangle$. If $x_2^- \sim x_2^+$, then the walk $x p_1 P \widehat{x_2^- x_2^+} P x x_2 q_1 Q q_2 x$ is shorter than C or contradicts the minimality of R. If $x_2^- x \in E(G)$, then the walk $C' = x q_2 \overleftarrow{Q} q_1 x_2 P \widehat{p_2 p_1} P x_2^- x$ is shorter than C and the same holds if $x_2^+ x \in E(G)$ for the walk $C' = x x_2^+ P \widehat{p_2 p_1} P x_2 q_1 Q q_2 x$. Thus $\langle x_2, x_2^-, x_2^+, x \rangle$ is a claw, which contradicts the fact that $x_2 \notin A$. Hence necessarily $x_2 \notin P$ and similarly $x_1 \notin Q$. As C is a covering walk, we have $x_1 \in P$ and $x_2 \in Q$. Since $x_1 x_2 \in E(G)$, at least one of x_1, x_2 (say, x_1) is not in A. We now consider $\langle x_1, x_1^-, x_1^+, x \rangle$. If $x_1^- x_1^+ \in E(G)$, then the walk $x x_1 p_1 P x_1^- x_1^+ P p_2 x q_1 Q q_2 x$ contradicts the minimality of R. The same holds for the walk $x x_1^- \overleftarrow{P} \widehat{p_1 p_2} P x_1 x q_1 Q q_2 x$ or $x x_1^+ P \widehat{p_2 p_1} P x_1 x q_1 Q q_2 x$ if $x_1^- x \in E(G)$ or $x_1^+ x \in E(G)$, respectively. Thus $\langle x_1, x_1^-, x_1^+, x \rangle$ is a claw, a contradiction. $\square$

We notice that Theorem 7 is a common extension of Theorems A and B and it also admits the following corollary.

Corollary 8: Let G be a connected almost claw-free graph and put

$$B = \{x \in V(G) \setminus A; \langle N(x) \rangle \text{ connected}\}.$$

Then G can be vertex-covered by at most $n - |B| + 1$ elementary cycles (where a closed walk of length 2 is considered as an elementary cycle of length 2).

Proof: By Theorem 7, there is a shortest 2-walk C that visits every vertex of B exactly once. We may obtain elementary cycles by cutting the 2-walk at any vertex x with $v(x, C) = 2$ in such a way that if P and Q are the two branches at x, then we cut the walk into xPx and xQx. There are at most $n - |B| + 1$ such elementary cycles for we have at most $n - |B|$ vertices with $v(x, C) = 2$. The graph G is vertex-covered by these cycles since their union gives a covering walk.

In the study of locally connected vertices which belong to A we use the following lemma.

Lemma 9: Let G be a connected almost claw-free graph, x a non-separating vertex which is visited twice by every shortest covering walk of G and C a shortest covering walk. Then, with the same notation as in Lemma 6:

(i) $p_1 \neq p_2$ and $q_1 \neq q_2$,

(ii) $\forall i, j \in \{1, 2\}, \quad N(p_i) \cap N(q_j) = \{x\}$.

Proof. Assume that G, x and C fulfil the hypotheses of Lemma 9.

(i) Since x is not a cutvertex of G, there are vertices $a \in P$ and $b \in Q$ such that $a \sim b$. If, e.g., $p_1 = p_2$, then for the walk $C' = xq_1Q\widehat{ba}Pp_2(= p_1)P\widehat{ab}Qq_2x$, which is obtained from C by deleting the edges xp_1, xp_2 and adding twice $\widehat{ab}$, we have $l(C') \leq l(C)$ and $v(x, C') = 1$, a contradiction.

(ii) If, e.g., $a \in N(p_1) \cap N(q_1)$ and $a \neq x$, then the walk $C' = xp_2 \overleftarrow{P} p_1aq_1Qq_2x$ also yields a contradiction.

Theorem 10: Let G be a connected almost claw-free graph and $x \in V(G)$. If $\langle N(x) \rangle$ is connected then there is a shortest 2-walk C such that $v(x, C) = 1$.

Proof. Let $x \in V(G)$ be such that $v(x, C) = 2$ for every shortest 2-walk C. By Theorem 7, it is sufficient to consider the case $x \in A$. Choose C in such a way that, among all shortest 2-walks C with branches P and Q at x and endvertices p_1, p_2 and q_1, q_2, respectively, the p_1, q_1-path R in $\langle N(x) \rangle$ is shortest possible (i.e., there is no shortest 2-walk C' containing a path in $\langle N(x) \rangle$ between the disjoint sets $\{p_1, p_2\}$ and $\{q_1, q_2\}$ that is shorter than R). Let $p_1, x_1, ..., x_k, q_1$ be the vertices of R. Note that, since $x \in A$ and G is almost claw-free, no vertex of $N(x)$ centres a claw.

We first show that $k = 2$.

Let $\{d_1, d_2\}$ be a dominating set of $\langle N(x) \rangle$. By Lemma 9(ii), one of d_1, d_2 (say, d_1) dominates p_1 and p_2 and the other, i.e. d_2, dominates q_1 and q_2.

Suppose now that $x_k \in P$ and consider $\langle x_k, x_k^-, x_k^+, x \rangle$. If $x_k^- \sim x_k^+$, then the walk $xp_1Px_k^-\widehat{x_k^-x_k^+}Pp_2xx_kq_1Qq_2x$ is shorter than C or contradicts the minimality of R. In the case $x_k^-x \in E(G)$ the walk $C' = xp_1Px_k^-x \overleftarrow{P} x_kq_1Qq_2x$ and in the case $x_k^+x \in E(G)$ the walk $C' = xp_1Px_kq_1Qq_2xx_k^+Pp_2x$ contradicts Lemma 6(i) or Lemma 9(ii) since p_1 and p_2 are endvertices of two different branches of C' that are dominated by the same vertex d_1. Thus $\langle x_k, x_k^-, x_k^+, x \rangle$ is a claw, which is a contradiction. Hence, as C is a covering walk, necessarily $x_k \in Q$ and, by symmetry, $x_1 \in P$. This implies that x_k^- and x_k^+ are not related for otherwise $xx_kq_1Qx_k^-\widehat{x_k^-x_k^+}Qq_2xp_1Pp_2x$ is shorter than C or contradicts the minimality of R. Similarly, x_1^- and x_1^+ are not related.

We show that $p_1p_2 \notin E(G)$ and $q_1q_2 \notin E(G)$. Let, e.g. $q_1q_2 \in E(G)$ and consider $\langle x_k, x_k^-, x_k^+, x \rangle$. If $x_k^-x \in E(G)$ or $x_k^+x \in E(G)$, then the walk $xx_kQq_2q_1Qx_k^-xp_1Pp_2x$ or $xx_k^+Qq_2q_1Qx_kxp_1Pp_2x$, respectively, contradicts the minimality of R. Thus $\langle x_k, x_k^-, x_k^+, x \rangle$

is a claw, a contradiction which implies $q_1q_2 \notin E(G)$ and, by symmetry, $p_1p_2 \notin E(G)$.

Necessarily, d_1 is not in R since otherwise, as R is shortest possible, d_1 would centre a claw. Moreover, x_1 is the only vertex among $x_1, x_2, ..., x_k$ which can be adjacent to d_1 (if $x_i,\ i > 1$ was adjacent to d_1, then, from the hypothesis on R, $\langle d_1, p_1, p_2, x_i\rangle$ would be a claw). Similarly, x_k is the only vertex among $x_1, x_2, ..., x_k$ which can be adjacent to d_2 and hence $k = 2$.

We now consider the induced subgraphs $\langle x_1, x_1^-, x_1^+, x_2\rangle$ and $\langle x_2, x_2^-, x_2^+, x_1\rangle$. We know from above that neither x_1^- and x_1^+ nor x_2^- and x_2^+ are related. As $x \in A$, neither x_1 nor x_2 can centre a claw and then we have, up to symmetry, the following three possibilities.

Case	The walk C'
$x_1^-x_2 \in E(G)$ and $x_2^-x_1 \in E(G)$	$xp_1Px_1^-x_2Qq_2x \overleftarrow{P} x_1x_2^- \overleftarrow{Q} q_1x.$
$x_1^+x_2 \in E(G)$ and $x_2^+x_1 \in E(G)$	$xp_1Px_1x_2^+Qq_2x \overleftarrow{P} x_1^+x_2 \overleftarrow{Q} q_1x.$
$x_1^+x_2 \in E(G)$ and $x_2^-x_1 \in E(G)$	$xp_1Px_1x_2^- \overleftarrow{Q} q_1xp_2 \overleftarrow{P} x_1^+x_2Qq_2x.$

In each of these cases, p_1 and p_2 are endvertices of different branches of C' at x, which contradicts Lemma 9. This contradiction completes the proof of Theorem 10. $\square$

Corollary 11: Let G be a connected locally connected almost claw-free graph. Then for every $x \in V(G)$ there is a shortest 2-walk C such that $v(x, C) = 1$.

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