UNIFYING RESULTS ON HAMILTONIAN CLAW-FREE GRAPHS

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Abstract

This work was motivated by many (recent) papers on hamiltonicity of claw-free graphs, i.e. graphs that do not contain $K_{1,3}$ as an induced subgraph. By combining ideas from these papers with some new observations, we unify several of the existing sufficiency results, using a new sufficient condition consisting of seven subconditions. If each pair of vertices at distance two of a 2-connected claw-free graph G satisfies at least one of these subconditions, then G is hamiltonian. We also present infinite classes of examples of graphs showing that these subconditions are, in some sense, independent.

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1. TERMINOLOGY AND NOTATION

We use [1] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. We say that G is *hamiltonian* if G has a *Hamilton cycle*, i.e. a cycle containing all vertices of G. If X is a graph, we say that G is *X-free* if G does not contain an induced subgraph isomorphic to X. Instead of $K_{1,3}$-free, we use the more common term *claw-free.* We denote by $\langle S\rangle$ the subgraph of G induced by a set $S \subseteq V(G)$. We use B (of *bull*), D (of *deer*), and H (of *hourglass*) to denote the graphs of Figure 1, and P_7 for a path on 7 vertices. Let u and v be two distinct vertices of G. Then $\{u,v\}$ is called an *i-pair* if the distance $d(u,v)$ between u and v is i $(i = 1, 2, \ldots)$; $\{u,v\}$ is a *B-pair* (respectively *D-pair*, *H-pair* or *P_7-pair*) if u and v are the circled vertices of Figure 1 in an induced subgraph of G isomorphic to B (respectively D, H or P_7).

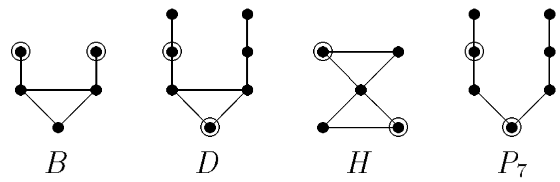


Figure 1

If $v \in V(G)$, then $N(v)$ denotes the set of vertices adjacent to v (the *neighborhood* of v). A vertex $v \in V(G)$ is *locally-connected* if $\langle N(v)\rangle$ is connected, and the graph G is *locally-connected* if all vertices of G are locally-connected.

2. INTRODUCTION

During the last two decades many results on hamiltonian properties of claw-free graphs have appeared. We refer the reader to [4] for a recent survey. Most of these results involve sufficient conditions in terms of degrees, neighborhoods, forbidden subgraphs or (local) connectivity. These conditions are "significantly weaker" than the corresponding sufficient conditions for general graphs. Nevertheless, these conditions are still "strong" in the sense that many vertices (or vertex pairs, triples, ...) are required to have the same property. Moreover, recent results also show that these conditions sometimes can be relaxed significantly if some classes of nonhamiltonian claw-free graphs are excluded.

Motivated by these observations, and in an attempt to unify existing results, we considered a choice of several different types of conditions, in order to relax the requirements for pairs of vertices.

We present our main result and its corollaries in the next section. The proof of the main result is postponed to Section 5; it is self-contained but uses ideas from older work together with some new observations. The corollaries should give the reader a good indication as to where the ideas originate.

In Section 4 we present infinite classes of examples of graphs showing that the subconditions in the main result are, in some sense, independent.

3. MAIN RESULT AND COROLLARIES

In the sequel we let G denote a 2-connected claw-free graph on n vertices.

Theorem 1. If each 2-pair $\{u, v\} \subseteq V(G)$ satisfies at least one of the following conditions, then G is hamiltonian.

(1) $\min\{d(u), d(v)\} \geq \frac{1}{3}(n-2)$;

(2) $|N(u) \cap N(v)| \geq 2$;

(3) $\{u, v\}$ is a 3-pair in $G-w$, and there exists a path ux_1x_2v of length 3 in $G-w$ such that $wx_2 \in E(G)$, where $w \in N(u) \cap N(v)$;

(4) $\{u, v\}$ is neither a D-pair nor a P_7-pair in G;

(5) $\{u, v\}$ is neither an H-pair nor a P_7-pair in G;

(6) $\{u, v\}$ is a 3-pair in $G-w$, and not a B-pair in $G-w$, where $w \in N(u) \cap N(v)$;

(7) $\{u, v\}$ is a 3-pair in $G-w$, and there are at least two internally-disjoint (u, v)-paths of length 3 in $G-w$, where $w \in N(u) \cap N(v)$.

Most of the corollaries that follow have been generalized or extended, sometimes in several directions. We refer to [4] for more information. The corollaries that are stated without proof are immediate.

Corollary 1.1. ([6]).
If $\delta(G) \geq \frac{1}{3}(n-2)$, then G is hamiltonian.

Corollary 1.2. ([8]).
If $|N(u) \cap N(v)| \geq 2$ for all 2-pairs $\{u, v\} \subseteq V(G)$, then G is hamiltonian.

Corollary 1.3. ([7]).
If G is locally-connected, then G is hamiltonian.

Proof. Consider a 2-pair $\{u, v\}$ of G with $w \in N(u) \cap N(v)$. If $|N(u) \cap N(v)| \geq 2$, then $\{u, v\}$ satisfies condition (2) of Theorem 1. Assuming $N(u) \cap N(v) = \{w\}$, consider a shortest (u, v)-path P in $\langle N(w) \rangle$. If P has length at least 4, then G contains an induced $K_{1,3}$ (induced by u, v, w, and a vertex of P at distance at least 2 from u and v on P). Hence P has length 3, and $\{u, v\}$ satisfies condition (3) of Theorem 1. ∎

The next corollary is implicit in the proof of Theorem 8 of [3].

Corollary 1.4.
If G is D-free and P_7-free, then G is hamiltonian.

Corollary 1.5. ([5]).
If G is H-free and P_7-free, then G is hamiltonian.

4. EXAMPLES

In this section we present eight infinite classes of graphs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_8$. For $i = 1, \ldots, 7$, the class $\mathcal{C}_i$ consists of (hamiltonian) graphs satisfying the hypothesis of Theorem 1, and in which at least one pair of vertices satisfies condition (i) of Theorem 1 only. This shows that none of the conditions (1)–(7) is superfluous. The class $\mathcal{C}_8$ consists of nonhamiltonian claw-free graphs showing that, in some sense, we cannot relax the conditions (1)–(7) of Theorem 1.

The graphs of $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_7$ are sketched in Figure 2; those of $\mathcal{C}_8$ in Figure 3. In the figures, the vertices are represented by black circles and the other circles represent complete subgraphs. Whenever the circles overlap, the number of common vertices of the corresponding subgraphs is indicated in the overlapping region; the subgraphs contain at least as many vertices as shown in the corresponding circles (but possibly more in non-overlapping regions). Arrows indicate a repetition of the suggested pattern in the obvious way. In each of the sketched graphs corresponding to $\mathcal{C}_i$

$(i = 1, \ldots, 7)$ a 2-pair $\{u, v\}$ satisfying only condition (i) of Theorem 1 is indicated. In $\mathcal{C}_1$ the subgraphs should be chosen in such a way that $\min\{d(u), d(v)\} \geq \frac{1}{3}(n-2)$, in $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6$ and $\mathcal{C}_7$ such that $\min\{d(u), d(v)\} \leq \frac{1}{3}(n-3)$. We leave the details concerning $\mathcal{C}_1, \ldots, \mathcal{C}_7$ to the reader.

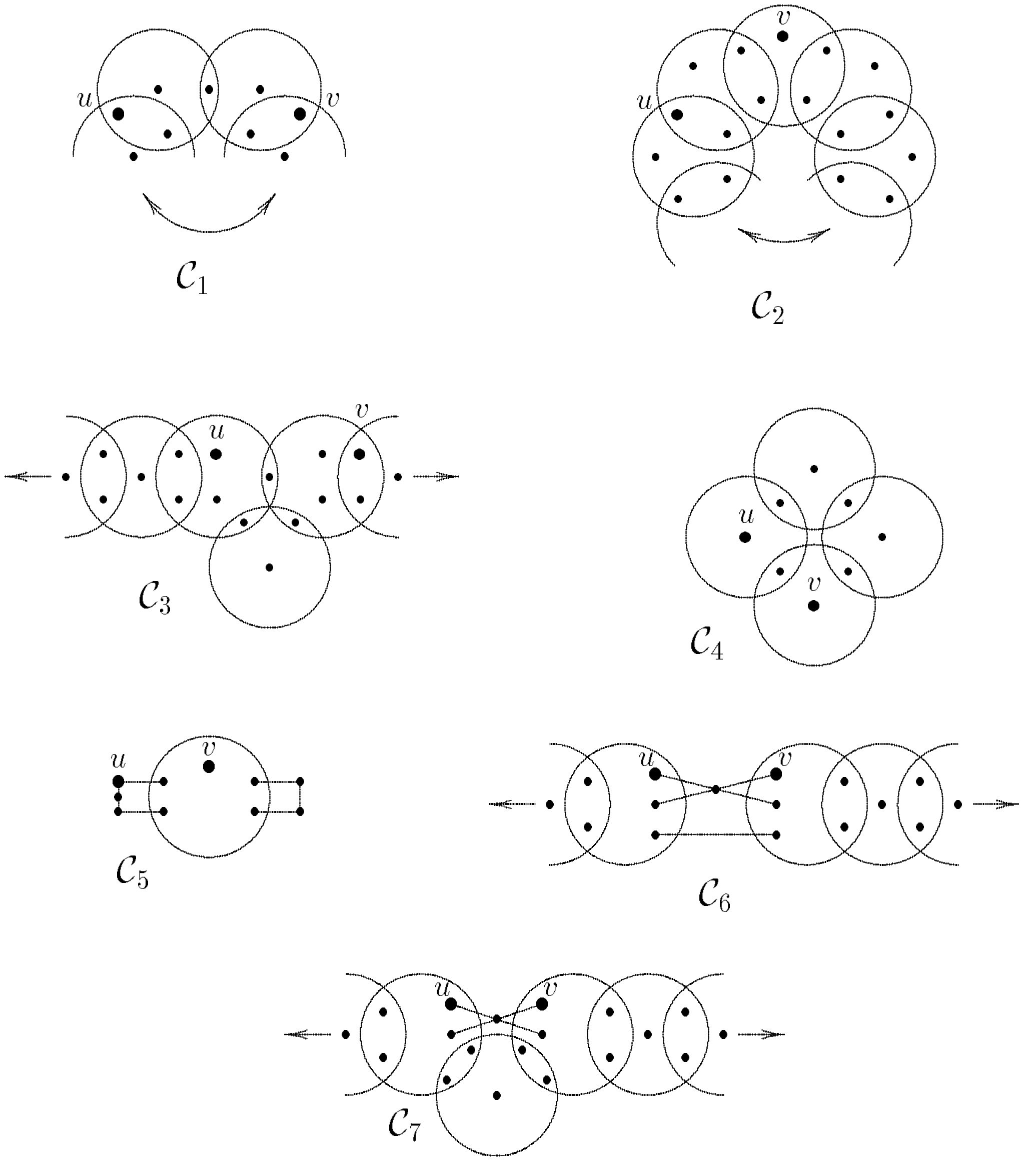
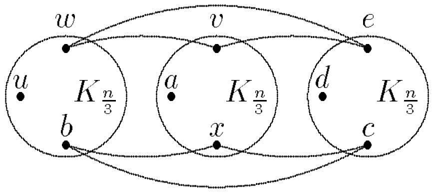


Figure 2: the classes $\mathcal{C}_1, \ldots, \mathcal{C}_7$

For $\mathcal{C}_8$ (see Figure 3) consider the 2-pair $\{u, v\}$. Then it is easy to check that $\min\{d(u), d(v)\} = d(u) = \frac{1}{3}(n-3), N(u) \cap N(v) = \{w\}$, there is precisely one (u, v)-path of length 3 in $G - w$, it contains $x \notin N(w), \langle\{a, b, c, d, u, v, w\}\rangle \cong D, \langle\{b, e, u, v, w\}\rangle \cong H$, and $\langle\{b, c, u, v, x\}\rangle \cong B$.



$\mathcal{C}_8$

Figure 3: the class $\mathcal{C}_8$

5. PROOF

Before we present a proof of the main result, we first introduce some additional notation.

Let G be a graph and let C be a cycle of G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u \overrightarrow{C} v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. We will consider $u \overrightarrow{C} v$ and $v \overleftarrow{C} u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor.

We use the following result which is a consequence of Theorem 5 of [2].

Lemma 5.1. A 2-connected claw-free graph G on n vertices has a cycle containing all vertices of degree at least $\frac{n-2}{3}$.

Proof of Theorem 1.
Let G be a 2-connected claw-free graph on n vertices and suppose G is nonhamiltonian. By Lemma 5.1, G has a cycle containing all vertices of degree at least $\frac{n-2}{3}$. Among all cycles with this property, choose a longest cycle C. By the assumption, G has a vertex $u \in V(G) \setminus V(C)$ with a neighbor $w \in V(C)$. Fix an orientation on C. Let v be the first vertex occurring on $w^+ \overrightarrow{C} w^-$ such that $wv^+ \notin E(G)$. We may assume without loss of generality that $v = w^+$. (If this is not the case, we can replace C by the cycle $v \overrightarrow{C} w^- w^+ \overrightarrow{C} v^- wv$.) Obviously, for all such triples $\{u, v, w\}$, the choice of C implies $d(u, v) = 2$ and $w^- w^+ \in E(G)$ (using the claw-freeness of G). Assume $\{u, v\}$ satisfies one of the conditions (1)–(7) of Theorem 1.

Clearly, the choice of C implies

$$d(u) < \frac{n-2}{3}. \qquad (1)$$

Suppose $|N(u) \cap N(v)| \geq 2$, and let $x \in (N(u) \cap N(v)) \setminus \{w\}$. Then, by the choice of $C, x \in V(C)$ and $wx \notin E(C)$. Also by the choice of C, $ux^- \notin E(G)$ and $ux^+ \notin E(G)$. Since G is claw-free, $x^-x^+ \in E(G)$. Now the cycle $v \overrightarrow{C} x^-x^+ \overrightarrow{C} wuxv$ contradicts the choice of C. Hence

$$|N(u) \cap N(v)| = 1. \qquad (2)$$

Before we proceed, we first prove the following lemma.

Lemma 5.2 If $P = ux_1x_2v$ is a path of length 3 in $G - w$, then $x_1, x_2 \in V(C)$, $x_2 \in v^+ \overrightarrow{C} x_1^-$, and $wx_2 \notin E(G)$.

Proof of Lemma 5.2

Let $P = ux_1x_2v$ be a path of length 3 in $G - w$. Then clearly the choice of C implies $\{x_1, x_2\} \cap V(C) \neq \emptyset$. Suppose first that $x_1 \notin V(C)$. Then $x_2 \in V(C)$ and the choice of C implies $wx_2 \notin E(C)$. By similar arguments as before, the choice of C and the claw-freeness of G imply $x_2^-x_2^+ \in E(G)$. Now the cycle $v \overrightarrow{C} x_2^-x_2^+ \overrightarrow{C} wux_1x_2v$ contradicts the choice of C. Hence $x_1 \in V(C)$.

Next suppose $x_2 \notin V(C)$. Then, by similar arguments as before, the choice of C and the claw-freeness of G yields a cycle $v \overrightarrow{C} x_1^-x_1^+ \overrightarrow{C} wux_1x_2v$ contradicting the choice of C. Hence $x_1, x_2 \in V(C)$.

Suppose $x_2 \notin v^+ \overrightarrow{C} x_1^-$. Then $x_2 \in x_1^+ \overrightarrow{C} w^-$. By standard arguments, $x_1x_2 \notin E(C)$. If $vx_2^+ \in E(G)$, then the cycle $v \overrightarrow{C} x_1^-x_1^+ \overrightarrow{C} x_2x_1uw \overleftarrow{C} x_2^+v$ contradicts the choice of C. Hence $vx_2^+ \notin E(G)$. Also $vx_1 \notin E(G)$ by (2). Considering $\langle\{v, x_1, x_2, x_2^+\}\rangle$, the claw-freeness of G implies $x_1x_2^+ \in E(G)$. If $ux_2^+ \in E(G)$, then the cycle $v \overrightarrow{C} x_1^-x_1^+ \overrightarrow{C} x_2x_1ux_2^+ \overrightarrow{C} v$ contradicts the choice of C. If $x_1^+x_2^+ \in E(G)$, then the cycle $v \overrightarrow{C} x_1uw \overleftarrow{C} x_2^+x_1^+ \overrightarrow{C} x_2v$ contradicts the choice of C. Considering $\langle\{u, x_1, x_1^+, x_2^+\}\rangle$, the claw-freeness of G implies $ux_1^+ \in E(G)$, an obvious contradiction with the choice of C. Hence $x_1, x_2 \in V(C)$ and $x_2 \in v^+ \overrightarrow{C} x_1^-$.

Supposing $wx_2 \in E(G)$ and considering $\langle\{w, x_2^-, x_2, x_2^+\}\rangle$, we obtain that at least one of the pairs $\{w, x_2^-\}, \{w, x_2^+\}$ and $\{x_2^-, x_2^+\}$ is joined by an edge. This leads, respectively, to the following cycles contradicting the choice of C:

$$v \overrightarrow{C} x_2^-wux_1x_2 \overrightarrow{C} x_1^-x_1^+ \overrightarrow{C} w^-v,$$
$$v \overrightarrow{C} x_2x_1uwx_2^+ \overrightarrow{C} x_1^-x_1^+ \overrightarrow{C} w^-v, \text{ and}$$
$$v \overrightarrow{C} x_2^-x_2^+ \overrightarrow{C} x_1^-x_1^+ \overrightarrow{C} wux_1x_2v.$$

This completes the proof of Lemma 5.2. ∎

We proceed with the proof of Theorem 1. From Lemma 5.2. we deduce that

$$\text{there is no path } P = ux_1x_2v \text{ of length 3 in } G - w \text{ such that } wx_2 \in E(G). \quad (3)$$

Since G is 2-connected, u is connected by a path to another vertex ($\neq w$) of C. Consider a shortest path $Q = wux_1 \dots x_r$ with $x_r \in V(C) \setminus \{w\}$ and internally-disjoint with C. Let y be the first vertex occurring on $x_r^+ \overrightarrow{C}\, w^-$ such that $x_r y^+ \notin E(G)$. As with the choice of v, similar arguments show we may assume $y = x_r^+$. If $r = 1$, set $S = \{u, v, v^+, w, x_1, y, y^+\}$. It is easy to check that the choice of C implies that $\langle S \rangle$ is isomorphic to P_7 or D: if, e.g., $v^+y^+ \in E(G)$, then the cycle $v^+ \overrightarrow{C}\, x_1^- y x_1 u w v w^- \overleftarrow{C}\, y^+ v^+$ contradicts the choice of C. We leave the other cases to the reader. If $r \geq 2$, then the choice of Q implies $ux_r \notin E(G)$. If $wx_r \in E(G)$, then, considering $\langle \{w^-, w, u, x_r\} \rangle$, we conclude that $w^- x_r \in E(G)$. But then the cycle $v \overrightarrow{C}\, x_r^- x_r^+ \overrightarrow{C}\, w^- x_r x_{r-1} \dots x_1 u w v$ contradicts the choice of C. Hence $wx_r \notin E(G)$. For $r = 2$, and $r \geq 3$, respectively, set $S = \{u, v, v^+, w, x_1, x_2, y\}$ and $S = \{u, v, v^+, w, x_1, x_2, x_3\}$. It is again easy to check that the choice of C implies $\langle S \rangle \cong P_7$. Hence in all cases we conclude that

$$\{u, v\} \text{ is a } P_7\text{-pair or a } D\text{-pair.} \quad (4)$$

If $\{u, v\}$ is not a P_7-pair, then by (4) $\{u, v\}$ is a D-pair, and from the above observations we get that $r = 1$ and $wx_1 \in E(G)$. Now it is easy to check that $\langle \{u, v, w^-, w, x_1\} \rangle \cong H$, and that $\{u, v\}$ is an H-pair. Hence

$$\{u, v\} \text{ is a } P_7\text{-pair or an } H\text{-pair.} \quad (5)$$

By (2), $|N(u) \cap N(v)| = 1$, so $d(u, v) \geq 3$ in $G - w$. Suppose $d(u, v) = 3$ in $G - w$ and let $R = ux_1x_2v$ be a (u, v)-path in $G - w$ of length 3. By Lemma 5.2, $x_1, x_2 \in V(C), x_2 \in v^+ \overrightarrow{C}\, x_1^-$, and $wx_2 \notin E(G)$. Considering $\langle \{u, x_1, x_1^+, x_2\} \rangle$, we obtain $x_1^+ x_2 \in E(G)$. It is easy to check that $\langle \{u, v, x_1, x_1^+, x_2\} \rangle \cong B$, implying that $\{u, v\}$ is a B-pair in $G - w$. Hence

$$\text{if } \{u, v\} \text{ is a 3-pair in } G - w, \text{ then } \{u, v\} \text{ is a } B\text{-pair in } G - w. \quad (6)$$

From (1)–(6), and the assumptions, we conclude that $\{u, v\}$ satisfies condition (7) of Theorem 1. Let R (as above) and $R' = uy_1y_2v$ be two internally-disjoint (u, v)-paths in $G - w$ of length 3. By Lemma 5.2, $x_1, x_2, y_1, y_2 \in V(C)$, $x_2 \in v^+ \overrightarrow{C}\, x_1^-, y_2 \in v^+ \overrightarrow{C}\, y_1^-, wx_2 \notin E(G), wy_2 \notin E(G)$, and, by the previous observations for R, $\{x_1^+ x_2, y_1^+ y_2\} \subseteq E(G)$. Clearly, $x_2 \neq v^+$, since otherwise the cycle $v^+ \overrightarrow{C}\, x_1 u w v w^- \overleftarrow{C}\, x_1^+ v^+$ contradicts the choice of C. Consider $\langle \{v, x_1, x_2^-, x_2\} \rangle$. As before,

$vx_1 \notin E(G)$. If $x_1x_2^- \in E(G)$, then the cycle $v \overrightarrow{C} x_2^- x_1 uw \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} x_2 v$ contradicts the choice of C. Hence $vx_2^- \in E(G)$, and similarly $vy_2^- \in E(G)$.

We assume without loss of generality that $x_1 \in y_1 \overrightarrow{C} w$. If $w^- x_2^- \in E(G)$, then the cycle $v \overrightarrow{C} x_2^- w^- \overleftarrow{C} x_1^+ x_1^- \overrightarrow{C} x_2 x_1 uwv$ contradicts the choice of C. Hence $w^- x_2^- \notin E(G)$, and similarly $w^- y_2^- \notin E(G)$. We complete the proof by showing that in all possible cases $x_2^- y_2^- \notin E(G)$, implying that $\langle \{v, w^-, x_2^-, y_2^-\} \rangle \cong K_{1,3}$, our final contradiction.

If $x_2 \in v \overrightarrow{C} y_2$ and $x_2^- y_2^- \in E(G)$, then the cycle $v \overrightarrow{C} x_2^- y_2^- \overleftarrow{C} x_2 x_1 uw \overleftarrow{C} x_1^+ x_1^- \overleftarrow{C} y_2 v$ contradicts the choice of C.

If $x_2 \in y_2 \overrightarrow{C} y_1$ and $x_2^- y_2^- \in E(G)$, then the cycle $v \overrightarrow{C} y_2^- x_2^- \overleftarrow{C} y_2 y_1 uw \overleftarrow{C} y_1^+ y_1^- \overleftarrow{C} x_2 v$ contradicts the choice of C.

If $x_2 \in y_1 \overrightarrow{C} w$ and $x_2^- y_2^- \in E(G)$, then the cycle $v \overrightarrow{C} y_2^- x_2^- \overleftarrow{C} y_1^+ y_1^- \overleftarrow{C} y_2 y_1 uw \overleftarrow{C} x_2 v$ contradicts the choice of C. Note that $x_2^- \neq y_1$, since $vx_2^- \in E(G)$ and $vy_1 \notin E(G)$ (because $d(u,v) = 3$ in $G - w$). ∎

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