Matching extension in $K_{1,r}$-free graphs with independent claw centers

Zdeněk Ryjáček
Department of Mathematics
University of West Bohemia
Americká 42
306 14 Plzeň
Czech Republic

March 10, 1995

Abstract

We say that a graph G is k-extendable if every set of k independent edges of G can be extended to a perfect matching. In the paper it is proved that if G is an even $(2k+1)$-connected $K_{1,k+3}$-free graph such that the set of all centers of claws is independent, then G is k-extendable. As a corollary we obtain an analogous result for almost claw-free graphs and for claw-free graphs, thus extending a result by M.D. Plummer.

1. INTRODUCTION

In this paper, a *graph* will be a finite undirected graph $G = (V(G), E(G))$ without loops and multiple edges. We say that a graph G is *odd* or *even* if G has an odd or even number of vertices, respectively. For any $A \subset V(G)$ we denote by $\langle A \rangle$ the subgraph of G induced on A, $G - A$ stands for $\langle V(G) \setminus A \rangle$, $c(G-A)$ denotes the number of components of $G - A$ and $c_o(G-A)$ denotes the number of odd components of $G - A$. A set $A \subset V(G)$ such that $c(G-A) > 1$ will be called a *cutset*. For any $A, B \subset V(G)$ we put $N_A(B) = \{x \in A |\ xy \in E(G) \text{ for some } y \in B\}$. If H is a subgraph of G and $B \subset V(G)$, then we put $N_H(B) = N_{V(H)}(B)$ and $N_B(H) = N_B(V(H))$; we simply denote $N(B) = N_G(B)$.

A set $A \subset V(G)$ is *independent* if $A \cap N(A) = \emptyset$. The size of a maximum independent set in G will be called the *independence number* of G and will be denoted by $\alpha(G)$. A set $B \subset V(G)$ is a *dominating set* if $B \cup N(B) = V(G)$. The size of a minimum dominating set in G will be called the *domination number* of G and will be denoted by $\gamma(G)$. If $\gamma(G) \leq k$, we say that G is *k-dominated.* A 1-factor of G will be referred to as a *perfect matching.* For other notation and terminology not defined here we refer to [1].

G is said to be $K_{1,r}$-free ($r \geq 3$) if G does not contain an induced subgraph which is isomorphic to the star $K_{1,r}$. In the special case $r = 3$ we say that G is *claw-free.* If G is not claw-free, then any induced subgraph H of G isomorphic to $K_{1,3}$ will be called a *claw.* A vertex $v \in V(G)$ is a *claw center* if there is a claw H in G such that $v \in V(H)$ and $|N_H(v)| = 3$.

Claw-free graphs are known to have many interesting properties. The following theorem appeared in [2] and [8].

Theorem A [2], [8]. Every even connected claw-free graph has a perfect matching.

In [9], Sumner further extended Theorem A showing the following.

Theorem B [9]. If G is an even connected graph that does not have a perfect matching, then there is a set $S \subset V(G)$ such that $c_o(G - S) > |S|$ and every vertex of S is adjacent to vertices of at least three odd components of $G - S$.

The set S introduced in Theorem B will be called an *antifactor set* in G. Clearly, every vertex of an antifactor set is a claw center.

Sumner [9] also showed that Theorem A can be strengthened for graphs with higher connectivity.

Theorem C [9]. Let G be an even k-connected $K_{1,k+1}$-free graph. Then G has a perfect matching.

In [6], the class of claw-free graphs was extended in the following way. We say that a graph G is *almost claw-free* if there is a (not necessarily nonempty) independent set $A \subset V(G)$ such that $\alpha(\langle N(x) \rangle) \leq 2$ for $x \notin A$ and $\gamma(\langle N(x) \rangle) \leq 2 < \alpha(\langle N(x) \rangle)$ for $x \in A$. Equivalently, G is almost claw-free if the neighborhood of every vertex is 2-dominated and the set of all claw centers is independent.

Since G is claw-free if and only if $\alpha(\langle N(x) \rangle) \leq 2$ for every $x \in V(G)$ and since $\gamma(H) \leq \alpha(H)$ for every graph H, every claw-free graph is almost claw-free.

The following properties of almost claw-free graphs were proved in [6].

Theorem D [6]. Let G be a connected almost claw-free graph. Then

(i) G is $K_{1,5}$-free,

(ii) if a vertex $x \in V(G)$ centers an induced claw, then $\gamma(\langle N(x)\rangle) = 2$,

(iii) if G is even, then G has a perfect matching.

We say that an even graph G is *k-extendable* ($0 \le k < |V(G)|/2$) if every set of k independent edges of G can be extended to (i.e., is a subset of) a perfect matching (for $k = 0$, G is 0-extendable if G has a perfect matching). For a survey paper on matching extension see [5].

The following result was proved by Plummer in [4].

Theorem E [4]. Every even $(2k+1)$-connected claw-free graph is k-extendable.

In this paper we extend Theorem E in a way similar to the way that Theorems C and D(iii) extend Theorem A.

2. MAIN RESULT

Theorem 1. Let G be an even $(2k+1)$-connected $K_{1,k+3}$-free graph such that the set of claw centers is independent. Then G is k-extendable.

We first prove the following auxiliary assertion.

Lemma. Let $T \subset V(G)$ be a cutset in a graph G such that no vertex of T is a claw center and $c(G-T) \ge |T|+2$. Then G is disconnected.

Proof. We prove the lemma by induction on $t = |T|$.

(i) If $t = 0$, then $T = \emptyset$ and clearly G is disconnected.

(ii) Suppose that the lemma holds for any graph having a cutset with at most $t-1$ vertices with the described properties and let T be a cutset in a connected graph G such that $|T| = t$, $c(G-T) \ge t+2$ and no vertex of T is a claw center. Then every vertex of T has neighbors in at most two components of $G-T$ and, since $c(G-T) \ge t+2$, there is a component H of $G-T$ such that $|N_T(H)| \le 1$. If $|N_T(H)| = 0$, then G is disconnected; thus let $N_T(H) = \{u\} \subset T$. Put $G' = G - (V(H) \cup \{u\})$. Then the set $T' = T \setminus \{u\}$ is a cutset of G' such that $c(G'-T') \ge |T'|+2$, no vertex of T' is a claw center and $|T'| = t-1$. By the induction assumption, G' is disconnected. Let G'_1 be a component of G' and put $G'_2 = G' - G'_1$. Since no vertex of H has a neighbor in G'_1 or in G'_2, $N_T(H) = \{u\}$ and since G is connected, we have $N_{G'_1}(u) \ne \emptyset$ and $N_{G'_2}(u) \ne \emptyset$ which implies that for any vertex $v_1 \in N_{G'_1}(u), v_2 \in N_{G'_2}(u)$ and $v_3 \in N_H(u)$, $\langle\{u, v_1, v_2, v_3\}\rangle$ is an induced claw. This contradiction proves the lemma. ■

Proof of Theorem 1. For $k = 0$ the theorem follows immediately from Theorem A and hence we can suppose that $k \geq 1$. Suppose, on the contrary, that G satisfies the assumptions of the theorem and there is a set H of k independent edges that cannot be extended to a perfect matching. Put $H = \{e_1, \ldots, e_k\}$ and denote $e_i = x_i y_i$, $i = 1, \ldots, k$. Since no claw centers are adjacent, we can assume that no y_i ($i = 1, \ldots, k$) is a claw center. Put $M_1 = \{x_1, \ldots, x_k\}$, $M_2 = \{y_1, \ldots, y_k\}$ and $M = M_1 \cup M_2$ (i.e., $|M_1| = |M_2| = k$). By the assumption, the graph $G_1 = G - M$ has no perfect matching. Let $S = \{z_1, \ldots, z_s\}$ be an antifactor set in G_1, put $C = V(G_1 - S)$ and let $C_1, \ldots, C_c$ be the components of $\langle C \rangle$. Since G is even, by Theorem B and by parity we have $c_o(G_1 - S) \geq s + 2$ and hence also $c \geq s + 2$.

Suppose first that $s \leq k$. Choose a subset $M_2^1 \subset M_2$ with $|M_2^1| = k - s$ and put $M_2^2 = M_2 \setminus M_2^1$. Then $|M_2^2| = s$ and the graph $G_1^1 = G - (S \cup M_1 \cup M_2^1)$ is induced by $V(C_1) \cup \ldots \cup V(C_c) \cup M_2^2$. By the lemma (with $T = M_2^2$ and $G = G_1^1$), the graph G_1^1 is disconnected. Hence $S \cup M_1 \cup M_2^1$ is a cutset of G. Since apparently $|S \cup M_1 \cup M_2^1| = 2k$, we have a contradiction with the assumption that G is $(2k+1)$-connected. Hence $s \geq k + 1$.

Let $C(G)$ be a graph obtained from G by contracting every component C_i to a vertex c_i and by deleting multiple edges that might be created in doing these contractions. For each subset $A \subset M \cup S$ and for any $i = 1, \ldots, c$ denote $e(C_i, A) = |\{c_i x \in E(G) |\ x \in A\}|$ and let $e(C, A) = \sum_{i=1}^{c} e(C_i, A)$. (Note that $e(C_i, A)$ is equal to the number of vertices of attachment of the component C_i in A.)

Since G is $(2k+1)$-connected, we have $|N_{M \cup S}(C_i)| \geq 2k + 1$ for every C_i, $i = 1, \ldots, c$. From this and from the fact that G is $K_{1,k+3}$-free and no y_i ($i = 1, \ldots, k$) is a claw center, we have

$$(2k+1)c \leq \sum_{i=1}^{c} N_{M \cup S}(C_i) = e(C, S) + e(C, M) \leq (k+2)s + (k+2)k + 2k,$$

from which, since $c \geq s + 2$,

$$(2k+1)(s+2) \leq (k+2)s + (k+2)k + 2k,$$

or, equivalently,

$$(k-1)s \leq k^2 - 2.$$

Since $s \geq k + 1$, we further have

$$(k-1)(k+1) \leq k^2 - 2,$$

which implies $-1 \leq -2$, a contradiction. ■

Example 2. For any $k \geq 1$, we construct a graph G_k by the following construction.

Let $k \geq 1$ and put $s = k+1$ and $c = k+3$. Choose an odd number $p \geq k^2 + 2k + 2$ and let $C_1, \ldots, C_c$ be c vertex disjoint copies of the complete graph K_p. Let H be a set of k independent edges $e_i = x_i y_i$ $(i = 1, \ldots, k)$ and let $S = \{z_1, \ldots, z_{k+1}\}$ be a set of $k+1$ independent vertices. For every ℓ, $1 \leq \ell \leq c$, choose in $V(C_\ell)$ a set of $2(k+1)$ vertices $a^\ell_{i,j}$, $i = 1, \ldots, k+1$, $j = 1, 2$ and a set of k^2 vertices $b^\ell_{i,j}$, $i, j = 1, \ldots, k$. Let

$$N(z_i) = \{a^\ell_{i,j} |\ j = 1, 2,\ \ell = 1, \ldots, c\}, i = 1, \ldots, k+1$$

and

$$N(x_i) = \{b^\ell_{i,j} |\ j = 1, \ldots, k,\ \ell = 1, \ldots, c\}, i = 1, \ldots, k.$$

Finally, for every $i = 1, \ldots, k$ choose two integers ℓ^1_i, ℓ^2_i such that $1 \leq \ell^1_i, \ell^2_i \leq c$ and put

$$N(y_i) = \{b^{\ell^1_i}_{i,j} |\ j = 1, \ldots, k\} \cup \{b^{\ell^2_i}_{i,j} |\ j = 1, \ldots, k\}, i = 1, \ldots, k$$

(for $k = 1$, see Figure 1).

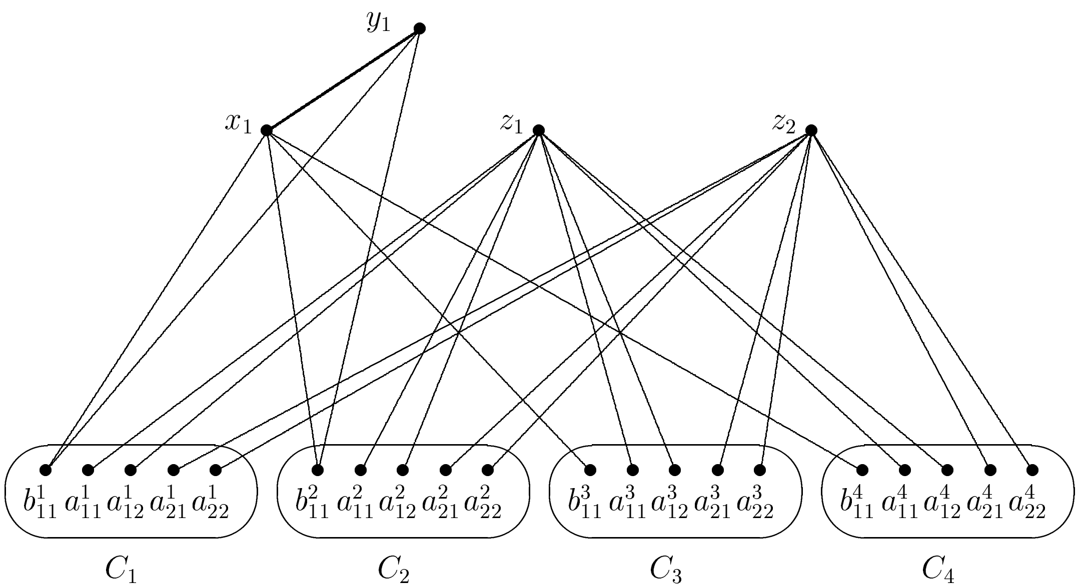

Figure 1

Then G_k is an even $(2k+1)$-connected $K_{1,k+4}$-free graph with independent claw centers. However, G_k is not $K_{1,k+3}$-free and the set of edges H cannot be extended to a perfect matching. This example shows that the assumption that G is $K_{1,k+3}$-free is sharp.

It is easy to construct an example showing that also the connectivity assumption of Theorem E (and hence also of Theorem 1) is sharp.

In the special case of almost claw-free graphs, we have the following analogous result on k-extendability.

Theorem 3. If G is an even $(2k+1)$-connected almost claw-free graph, then G is k-extendable.

Proof. For $k=0$ the theorem follows immediately from part (iii) of Theorem D and for $k \geq 2$ it follows from Theorem 1 and from part (i) of Theorem D. Hence it remains to consider the case $k=1$.

Suppose again that G satisfies the assumptions but is not 1-extendable and let G_1, M, S and C be the same as in the proof of Theorem 1. For every $v \in V(G)$ that is a claw center choose a 2-element (cf. part (ii) of Theorem D) dominating set $D(v)$ in $\langle N(v) \rangle$.

Suppose now that some $z_i \in S$ has neighbors in at least four components of $\langle C \rangle$ (say, $z_i u_j \in E(G)$ for some $u_j \in V(C_j), j = 1,2,3,4$). Then, as there is no edge with vertices in two different C_j's, $\langle \{z_i, u_1, u_2, u_3, u_4\} \rangle$ is an induced $K_{1,4}$ (with center at z_i). Since every vertex in $D(z_i)$ is adjacent to exactly two u_j's (otherwise we have two adjacent claw centers), $D(z_i) \cap C = \emptyset$. Clearly also $D(z_i) \cap S = \emptyset$ (otherwise we again have two adjacent claw centers since, by Theorem B, every vertex in S is a claw center) and hence $D(z_i) = \{x_1, y_1\}$. We can suppose that $\{u_1, u_2\} \subset N(x_1)$ and $\{u_3, u_4\} \subset N(y_1)$. Since $\langle \{x_1, u_1, u_2, y_1\} \rangle$ cannot be an induced claw and $u_1 u_2 \notin E(G)$, we have $u_1 y_1 \in E(G)$ or $u_2 y_1 \in E(G)$, but then $\langle \{y_1, u_1, u_3, u_4\} \rangle$ or $\langle \{y_1, u_2, u_3, u_4\} \rangle$ is an induced claw - a contradiction. Hence every $z_i \in S$ has neighbors in at most three different components of $\langle C \rangle$.

By a similar argument, x_1 has also neighbors in at most three different components of $\langle C \rangle$ (otherwise at least one vertex of $D(x_1)$ is in S or in C which both yields an immediate contradiction). Finally, recall that y_1 is not a claw center. Hence we have

$$3c \leq e(C,S) + e(C,M) \leq 3s + 3 + 2,$$

from which

$$3c \leq 3s + 5.$$

From this and from $c \geq s+2$ we further have

$$3(s+2) \leq 3s + 5,$$

which implies $6 \leq 5$, a contradiction. ■

Since every claw-free graph is almost claw-free, Theorem 3 immediately implies Theorem E.

Remark. A vertex $x \in V(G)$ is a center of an induced $K_{1,r}$ if and only if $\langle N(x) \rangle$ contains an independent set I with $|I| = r$. Thus, if we put

$$\alpha_L(G) = \max\{\alpha(\langle N(x) \rangle) |\ x \in V(G)\},$$

then the smallest integer r for which G is $K_{1,r}$-free satisfies $r = \alpha_L + 1$. If G is a graph with independent claw centers, then $\langle N(x) \rangle$ is a claw-free graph for any $x \in V(G)$ (otherwise we have two adjacent claw centers). Since the determination of the independence number is polynomial in claw-free graphs (see [3], [7]), $\alpha_L(G)$ can be computed in polynomial time for any graph with independent claw centers (although it is NP-complete in general). It is easy to see that also the assumption of independent claw centers can be verified in polynomial time. Thus, it is convenient to restate Theorem 1 in the following way.

Corollary 4. Let G be a graph of connectivity $\kappa(G)$ with independent claw centers and let

$$\alpha_L(G) = \max\{\alpha(\langle N(x) \rangle) |\ x \in V(G)\}.$$

Suppose that

$$\alpha_L(G) \leq \frac{\kappa(G) + 3}{2}.$$

Then G is $\lfloor \frac{\kappa(G)-1}{2} \rfloor$-extendable.

References

[1] Bondy, J.A.; Murty, U.S.R.: Graph theory with applications. Macmillan, London and Elsevier, New York, 1976.

[2] Las Vergnas, M.: A note on matchings in graphs. Cahiers Centre Etudes Rech. Opér. 17(1975), 257-260.

[3] Minty, G.J.: On maximal independent sets of vertices in claw-free graphs. J. Combin. Theory Ser. B28(1980) 284-304.

[4] Plummer, M.D.: Extending matchings in claw-free graphs. Proc. 13th British Combin. Conference, Guildford, 1991; Discrete Math. 125(1994) 301-308.

[5] Plummer, M.D: Extending matchings in graphs: A survey. Discrete Math. 127(1994) 277-292.

[6] Ryjáček, Z.: Almost claw-free graphs. J. of Graph Theory 18(1994) 469-477.

[7] Sbihi, N.: Algorithmes de recherche d'un stable de cardinalite maximum dans un graphe sans etoile. Discrete Math. 29(1980) 53-76.

[8] Sumner, D.P.: Graphs with 1-factors. Proc. Amer. Math. Soc. 42(1974), 8-12.

[9] Sumner, D.P.: 1-factors and antifactor sets. J. London Math. Soc.(2) 13(1976), 351-359.