On a closure concept in claw-free graphs

Zdeněk Ryjáček
Department of Mathematics
University of West Bohemia
Univerzitní 22
306 14 Plzeň, Czech Republic

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Abstract

If G is a claw-free graph, then there is a graph $cl(G)$ such that

(i) G is a spanning subgraph of $cl(G)$,

(ii) $cl(G)$ is a line graph of a triangle-free graph, and

(iii) the length of a longest cycle in G and in $cl(G)$ is the same.

A sufficient condition for hamiltonicity in claw-free graphs, the equivalence of some conjectures on hamiltonicity in 2-tough graphs and the hamiltonicity of 7-connected claw-free graphs are obtained as corollaries.

1 Introduction

In this paper, a *graph* will be a finite undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For terminology and notation not defined here we refer to [1]. For any set $A \subset V(G)$ we denote by $\langle A \rangle$ the induced subgraph on A, $G - A$ stands for $\langle V(G) \setminus A \rangle$ and $\omega(G - A)$ denotes the number of components of $G - A$. The (vertex) connectivity of G will be denoted by $\kappa(G)$ and the circumference of G (i.e., the length of a longest cycle in G) will be denoted by $c(G)$. The line graph of a graph G will be denoted by $L(G)$. By a *clique* we mean a (not necessarily maximal) complete subgraph of G.

If H is a graph, then we say that a graph G is H*-free* if G contains no copy of H as an induced subgraph. Specifically, the four-vertex star $K_{1,3}$ will be also called the *claw* and in this case we say that G is *claw-free*. Whenever vertices of a claw are listed, its *center* (i.e., the only vertex of degree 3) will be always the first vertex of the list. It is well known (and can be easily checked) that every line graph is claw-free.

For a vertex $x \in V(G)$, the set $N_G(x) = \{y \in V(G) : \ xy \in E(G)\}$ is called the *neighborhood* of x in G. We say that x is a *locally connected vertex* if $\langle N_G(x)\rangle$ is a connected graph. The set of all locally connected vertices of G will be denoted by $M_{loc}(G)$. If $M_{loc}(G) = V(G)$, then we say that G is *locally connected*.

Oberly and Sumner [7] proved that every connected, locally connected claw-free graph on at least three vertices is hamiltonian. This result was later on strengthened in many directions; from one of these results (see [4]) it e.g. follows that a claw-free graph G is hamiltonian if $M_{loc}(G)$ is a dominating set (i.e., every vertex in $V(G) \setminus M_{loc}(G)$ has a neighbor in $M_{loc}(G)$) and $\langle M_{loc}(G)\rangle$ is connected.

A graph G is said to be *t-tough* (where $t \geq 0$ is a real number) if $|S| \geq t \cdot \omega(G - S)$ for every set $S \subset V(G)$ with $\omega(G - S) > 1$. The *toughness* $\tau(G)$ of G is the maximum value of t for which G is t-tough ($\tau(K_n) = \infty$ for every $n \geq 1$). In [2], Chvátal conjectured the existence of an integer t_0 such that every t_0-tough graph is hamiltonian. Since it is known [3] that for every $\varepsilon > 0$ there is a $(2 - \varepsilon)$-tough graph on at least 3 vertices containing no 2-factor, the smallest such value of t_0 can be 2. The following conjecture is usually attributed to Chvátal.

Conjecture A. *Every 2-tough graph on at least three vertices is hamiltonian.*

It is easy to observe that, for any graph G, $\tau(G) \leq \kappa(G)/2$. Matthews and Sumner [6] proved that if G is claw-free, then $\tau(G) = \kappa(G)/2$. Conjecture A, if true, therefore implies the following conjecture (by Matthews and Sumner).

Conjecture B [6]. *Every 4-connected claw-free graph is hamiltonian.*

Since every line graph is claw-free, the following conjecture by Thomassen [9] is a special case of Conjecture B.

Conjecture C [9]. *Every 4-connected line graph is hamiltonian.*

The following conjecture is a special case of Conjecture C.

Conjecture D. *Every 4-connected line graph of a triangle-free graph is hamiltonian.*

Zhan [10] and independently Jackson [5] proved the analogue of Conjectures C and D in the case of 7-connected line graphs.

Theorem E [10], [5]. *Every 7-connected line graph is hamiltonian.*

In the main result of this paper we show that for every claw-free graph G there is a graph $cl(G)$ such that G is a spanning subgraph of $cl(G)$, $cl(G)$ is a line graph of a triangle-free graph and $c(G) = c(cl(G))$. In Section 3 we prove the following corollary of this result.

Theorem.

(i) Conjectures B, C and D are equivalent.
(ii) Every 7-connected claw-free graph is hamiltonian.

2 Main result

We begin with the following simple lemma.

Lemma 1. *Let G be a graph such that, for every $x \in V(G)$, $\langle N(x)\rangle$ is either a clique or a disjoint union of two cliques. Then there is a triangle-free graph H such that $G = L(H)$.*

Proof. Suppose G satisfies the assumptions of the lemma. We can suppose without loss of generality that G is connected (otherwise we apply the proof to every component of G) and $E(G) \neq \emptyset$ (otherwise apparently $G = L(K_2)$). Let $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ be the collection of maximal cliques in G. By the assumptions of the lemma, $|V(K_i) \cap V(K_j)| \leq 1$ for every $i \neq j$, and $|\{K_i :\ x \in V(K_i)\}| \leq 2$ for every $x \in V(G)$. Let $X = \{x_1, \ldots, x_k\}$ be the set of vertices of G which lie in exactly one clique of $\mathcal{K}$ and let H be the graph with vertex set $V(H) = X \cup \mathcal{K}$ and with edge set $E(H) = \{K_iK_j :\ |V(K_i) \cap V(K_j)| = 1,\ i \neq j\} \cup \{x_mK_j :\ x_m \in V(K_j)\}$. It is straightforward to check that $G = L(H)$.

Suppose that H contains a triangle T. Then, by the definition of H, $V(T) \subset \mathcal{K}$ (since vertices from X have degree 1 in H). Let $V(T) = \{K_{i_1}, K_{i_2}, K_{i_3}\}$ and let $v \in V(K_{i_1}) \cup V(K_{i_2})$. Then v is locally connected (in G) and thus, by the assumptions of the lemma, $\langle N_G(v)\rangle$ is a clique. This implies $K_{i_1} = K_{i_2}$, a contradiction. ■

The following proposition shows that replacing the neighborhood of a locally connected vertex of a claw-free graph G by a clique affects neither the claw-freeness nor the circumference of G.

Proposition 2. *Let G be a claw-free graph and let x be a locally connected vertex of G such that $\langle N_G(x)\rangle$ is not complete. Let $N' = \{uv :\ u, v \in N_G(x), uv \notin E(G)\}$ and let G' be the graph with vertex set $V(G') = V(G)$ and with edge set $E(G') = E(G) \cup N'$. Then*
(i) the graph G' is claw-free,
(ii) $c(G') = c(G)$.

Proof. *(i)* Suppose G' is not claw-free and let H be a claw in G'. Since G is claw-free, $|E(H) \cap N'| \geq 1$; since $\langle N_{G'}(x)\rangle$ is a clique, $|E(H) \cap N'| \leq 1$. Denote $H = \langle\{z, y_1, y_2, y_3\}\rangle$, where $zy_1 \in N'$. Then $xy_2 \notin E(G)$ (since otherwise $y_2 \in N_G(x)$ and, by the construction of G', $y_1y_2 \in E(G')$), and similarly $xy_3 \notin E(G)$. But then $\langle\{z, x, y_2, y_3\}\rangle$ is a claw in G, which is a contradiction. Hence G' is claw-free.

(ii) We show that $c(G) = c(G')$. Since obviously $c(G) \leq c(G')$, it is sufficient to prove that for every longest cycle C' in G' there is a cycle C in G such that $V(C') =$

$V(C)$. Let, on the contrary, C' be a longest cycle in G' such that there is no cycle C in G with $V(C') = V(C)$. Then $E(C') \cap N' \neq \emptyset$ (since otherwise C' is a cycle in G) and $N_G(x) \cup \{x\} \subset V(C')$ (since $\langle N_{G'}(x) \rangle$ is complete and C' is a longest cycle in G'). Denote by $P_1, \ldots, P_k$ the components of the graph $(V(C') \setminus \{x\}, E(C' - x) \setminus N')$ and put $\mathcal{P}(C') = \{P_1, \ldots, P_k\}$. Then each P_i is a (trivial or nontrivial) path in G with endvertices in $N_G(x)$. Suppose that the cycle C' is chosen such that, among all cycles in G' with vertex set $V(C')$, $k = |\mathcal{P}(C')|$ is minimum.

Choose one of the orientations of C' and, for any $v \in V(C')$, denote by v^- and v^+ the predecessor and successor of v on C', respectively. For any $v_1, v_2 \in V(C')$, denote by $v_1 C' v_2$ or $v_1 \overleftarrow{C'} v_2$ the consecutive vertices on C' from v_1 to v_2 in the same or opposite orientation with respect to the orientation of C' (if $v_1 = v_2$, then we define both $v_1 C' v_2$ and $v_1 \overleftarrow{C'} v_2$ as a single vertex). Denote by y_i^1, y_i^2 the endvertices of P_i, $i = 1, \ldots, k$ (not excluding the possible case $y_i^1 = y_i^2$) and let the numbering of the paths P_i and of their endvertices be chosen such that $y_1^1 = x^+$, $y_k^2 = x^-$ and $y_{i+1}^1 = (y_i^2)^+, i = 1, \ldots, k-1$.

We show that $y_i^r y_j^s \notin E(G)$ for every $r, s \in \{1, 2\}$ and $i, j \in \{1, \ldots, k\}, i \neq j$. Indeed, if e.g. $y_1^2 y_2^1 \in E(G)$, then, replacing in $\mathcal{P}(C')$ the paths P_1, P_2 by the path $y_1^1 P_1 y_1^2 y_2^1 P_2 y_2^2$, we have a contradiction with the minimality of k. Other cases are similar.

Now, if $k \geq 3$, then $\langle \{x, y_1^1, y_2^1, y_3^1\} \rangle$ is a claw in G and if $k = 1$, then $C = x y_1^1 P_1 y_1^2 x$ is a cycle in G with $V(C) = V(C')$. Hence $k = 2$ and $\mathcal{P}(C') = \{P_1, P_2\}$.

If $y_1^1 \neq y_1^2$ and $y_1^1 y_1^2 \notin E(G)$, then $\langle \{x, y_1^1, y_1^2, y_2^1\} \rangle$ is a claw in G. Hence either $y_1^1 = y_1^2$ or $y_1^1 y_1^2 \in E(G)$ and, by symmetry, $y_2^1 = y_2^2$ or $y_2^1 y_2^2 \in E(G)$.

Since $\langle N_G(x) \rangle$ is connected, there is a path $Q(C')$ in $\langle N_G(x) \rangle$ joining one of y_1^1, y_1^2 to one of y_2^1, y_2^2. Suppose that C' is chosen such that, among all cycles in G' with vertex set $V(C')$ and with $k = |\mathcal{P}(C')| = 2$, $Q(C')$ is shortest possible and assume without loss of generality that $Q(C')$ is a y_1^2, y_2^1-path (otherwise we can modify the cycle C' in $\langle N_{G'}(x) \rangle$ in an obvious way). Let $y_1^2 = x_0, x_1, \ldots, x_\ell = y_2^1$ be the vertices of $Q(C')$. Since $y_1^2 y_2^1 \notin E(G)$, $\ell \geq 2$. Note that, since $\langle N_G(x) \rangle \subset V(C')$, $V(Q(C')) \subset V(C')$.

We now consider $\langle \{x_1, x_1^-, x_1^+, x\} \rangle$. Suppose first that $x_1^- x \in E(G)$ or $x_1^+ x \in E(G)$. If $x_1 \in V(P_2)$, then $|V(P_2)| \geq 3$ (since we already know that $x_1 \notin \{y_2^1, y_2^2\}$) and the cycle $C = x y_1^1 C' y_1^2 x_1 C' y_2^2 y_2^1 C' x_1^- x$ (if $x_1^- x \in E(G)$) or $C = x y_1^1 C' y_1^2 x_1 \overleftarrow{C'} y_2^1 y_2^2 \overleftarrow{C'} x_1^+ x$ (if $x_1^+ x \in E(G)$) is a cycle in G with $V(C) = V(C')$, which is a contradiction. Hence $x_1 \in V(P_1)$, but then similarly $|V(P_1)| \geq 3$ and the cycle $C'' = x x_1^- \overleftarrow{C'} y_1^1 y_1^2 \overleftarrow{C'} x_1 y_2^1 C' y_2^2 x$ (if $x_1^- x \in E(G)$) or $C'' = x x_1^+ C' y_1^2 y_1^1 C' x_1 y_2^1 C' y_2^2 x$ (if $x_1^+ x \in E(G)$) is a cycle in G' with $V(C'') = V(C')$, $|\mathcal{P}(C'')| = 2$ and such that $|V(Q(C''))| = |V(Q(C'))| - 1$, which contradicts the minimality of $Q(C')$. Hence $x_1^- x \notin E(G)$ and $x_1^+ x \notin E(G)$. Finally, if $x_1^- x_1^+ \in E(G)$, then similarly the path P_i containing x_1 has at least 3 vertices and $C'' = x y_1^1 C' x_1^- x_1^+ C' y_1^2 x_1 y_2^1 C' y_2^2 x$ (if $x_1 \in V(P_1)$) or $C'' = x y_1^1 C' y_1^2 x_1 y_2^1 C' x_1^- x_1^+ C' y_2^2 x$ (if $x_1 \in V(P_2)$) is again a cycle in G' with $V(C'') = V(C')$, $|\mathcal{P}(C'')| = 2$ and such that $|V(Q(C''))| = |V(Q(C'))| - 1$. Hence $\langle \{x_1, x_1^-, x_1^+, x\} \rangle$ is a claw. This contradiction proves Proposition 2. ■

We can now proceed to the definition of the main concept of this paper.

Let G be a claw-free graph. We say that a graph H is a closure *of G, denoted $H = cl(G)$, if*

(i) there is a sequence of graphs $G_1, \ldots, G_t$ such that $G_1 = G$, $G_t = H$, $V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \cup \{uv : \ u, v \in N_{G_i}(x_i), uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ with connected noncomplete $\langle N_{G_i}(x_i) \rangle$, $i = 1, \ldots, t-1$,

(ii) no vertex of H has a connected noncomplete neighborhood.

(Equivalently, $cl(G)$ is obtained from G by recursively repeating the construction described in Proposition 2, as long as this is possible).

Theorem 3. *Let G be a claw-free graph. Then*

(i) the closure $cl(G)$ is well-defined,

(ii) there is a triangle-free graph H such that $cl(G) = L(H)$,

(iii) $c(G) = c(cl(G))$.

Proof. *(i)* Let H_1, H_2 be two closures of G, suppose that $E(H_1) \setminus E(H_2) \neq \emptyset$ and let $G_1, \ldots, G_t$ be the sequence of graphs that yields H_1. Let j be the smallest integer for which $E(G_j) \setminus E(H_2) \neq \emptyset$ and let $e = uv \in E(G_j) \setminus E(H_2)$. Then, since $e \in E(G_j)$, the vertices u, v have a common neighbor x in G_{j-1} such that $\langle N_{G_{j-1}}(x) \rangle$ is connected. But then, since $E(\langle N_{G_{j-1}}(x) \rangle) \subset E(G_{j-1}) \subset E(H_2)$, $\langle N_{H_2}(x) \rangle$ is connected, hence $e = uv \in E(H_2)$ – a contradiction.

(ii) By part *(i)* of Proposition 2, $cl(G)$ is claw-free and hence, by the construction, the neighborhood of every vertex of $cl(G)$ is either a clique or is a disjoint union of two cliques. By Lemma 1, $cl(G)$ is a line graph of a triangle-free graph.

(iii) $c(G) = c(cl(G))$ immediately by part *(ii)* of Proposition 2. ■

3 Corollaries

Corollary 4. *Let G be a claw-free graph. Then G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

The following corollary shows that Conjectures B, C and D are equivalent.

Corollary 5. *The following statements are equivalent.*

(i) Every 4-connected claw-free graph is hamiltonian.

(ii) Every 4-connected line graph is hamiltonian.

(iii) Every 4-connected line graph of a triangle-free graph is hamiltonian.

Proof. Obviously *(i)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)*. If G is a counterexample to *(i)*, then, by Theorem 3 and by Corollary 4, $cl(G)$ is a counterexample to *(iii)* and hence also *(iii)* $\Rightarrow$ *(i)*. ■

Corollary 6. *Every 7-connected claw-free graph is hamiltonian.*

Proof. If G is a 7-connected nonhamiltonian claw-free graph, then $cl(G)$ is a 7-connected nonhamiltonian line graph, which contradicts Theorem E. ■

Corollary 7. *Let G be a claw-free graph on at least three vertices.*

(i) If $cl(G)$ is a complete graph, then G is hamiltonian.

(ii) [4] If $M_{loc}(G)$ is dominating and $\langle M_{loc}(G)\rangle$ is connected, then G is hamiltonian.

(iii) [7] If G is connected and locally connected, then G is hamiltonian.

Proof. The statement *(i)* follows immediately from Corollary 4; if G satisfies the assumptions of *(ii)* or of *(iii)*, then $cl(G)$ is complete and G is hamiltonian by *(i)*. ■

Example. The graph in Figure 1 is an example of a claw-free graph with a complete closure which satisfies the assumptions of neither part *(ii)* nor part *(iii)* of Corollary 7.

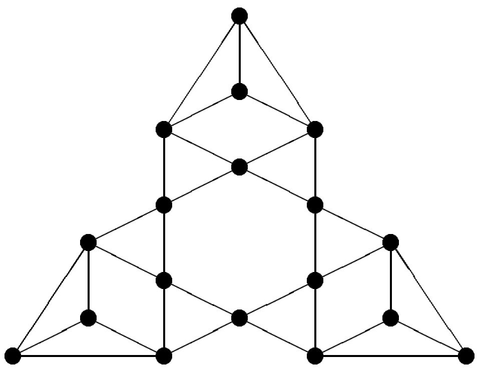

Figure 1: A claw-free graph with a complete closure

Remarks. 1. It is easy to see that $cl(G)$ can be equivalently characterized as the minimum $(K_4 - e)$-free graph on $V(G)$ containing G.

2. If a claw-free graph G is k-connected, or satisfies some of the degree conditions (expressed as a lower bound on $\delta(G)$ or on $\sigma_i(G) = \min\{\sum_{x\in S} d(x) : S \subset V(G)$ independent, $|S| = i\}$ in terms of a function of $n = |V(G)|$), or is N_2-locally connected (see [8]) etc., then apparently so does the closure $cl(G)$. Proofs of many known sufficient conditions for hamiltonicity in claw-free graphs can be therefore simplified by considering $cl(G)$.

3. It would be of interest to observe the behavior of some graph parameters (such as e.g. the independence number) during the process of constructing $cl(G)$. Such observations could possibly yield extensions of some line graph algorithms to the class of claw-free graphs.

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