2-Factors and Hamiltonicity

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Abstract

We prove the following generalization of a result of Faudree and van den Heuvel.

Let G be a 2-connected graph with a 2-factor. If $d(u)+d(v) \geq n-2$ for all pairs of non-adjacent vertices u, v contained in an induced $K_{1,3}$, in an induced $K_{1,3}+e$ or as end-vertices in an induced P_4, then G is hamiltonian.

Keywords: 2-factor, Hamilton cycle, induced subgraph, degree condition.

1 Terminology and notation

We use [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph on n vertices. We say that G is *hamiltonian* if G has a *Hamilton cycle*, i.e. a cycle containing all vertices of G. If X is a graph, we say that G is *X-free* if G does not contain an induced subgraph isomorphic to X. In this paper we use $K_{1,3}$, $Z_1 \simeq K_{1,3}+e$ and P_4 to denote the graphs of Figure 1. According to the labeling of the vertices we will write $\langle a,b,c,d\rangle \simeq K_{1,3}$, $\langle a,b,c,d\rangle \simeq Z_1$ and $\langle a,b,c,d\rangle \simeq P_4$, respectively.

We will use $\omega(G)$ to denote the number of components of G. A graph G is said to be *t-tough* (cf. [3]) if $t \cdot \omega(G-S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. If $v \in V(G)$, then $N(v)$ denotes the set of vertices adjacent to v (the *neighborhood* of v) and $d(v) = |N(v)|$ denotes the degree of v. If we restrict $N(v)$ and $d(v)$ to a subgraph

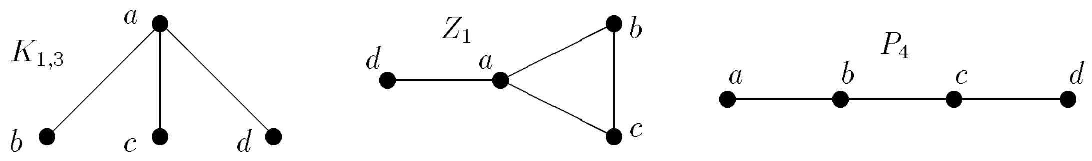

Figure 1.

$F \subset G$, then we will use $N_F(v)$ and $d_F(v)$, respectively. We say that a subgraph $H \subset G$ is a *2-factor* of G if H is a spanning subgraph of G and $d_H(v) = 2$ for every $v \in V(G)$.

Let C be a cycle of G. If an orientation of C is fixed and $u, v \in V(C)$, then by $u \overrightarrow{C} v$ we denote the consecutive vertices on C from u to v in the orientation specified by the orientation of C. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. If $C \subset G$ is a cycle with a fixed orientation and $v \in V(C)$, then v^+ and v^- denotes the successor and predecessor of v on C, respectively.

2 Main result

Our research was motivated by the following famous conjecture by Chvátal.

Conjecture [3]. *Every 2-tough graph is hamiltonian.*

For the class of 2-tough graphs Enomoto, Jackson, Katerinis and Saito proved the following result.

Theorem 1 [5]. *Every 2-tough graph has a 2-factor.*

Obviously, having a 2-factor is a necessary condition for a graph to be hamiltonian. Moreover, it can be decided in polynomial time whether a given graph G has a 2-factor (see [1]).

The first result for hamiltonicity of graphs having a 2-factor is due to Hoede.

Theorem 2 [7]. *Let G be a connected graph with a 2-factor and let $G_1, \ldots, G_{11}$ be the graphs shown in Fig. 2. If G is $G_1, \ldots, G_{11}$-free, then G is hamiltonian.*

We now turn our attention to degree conditions. The following result by Faudree and van den Heuvel shows that Ore's [8] and Dirac's [4] degree conditions for hamiltonicity can be relaxed under the additional assumption that G has a 2-factor.

Theorem 3 [6]. *Let G be a 2-connected graph with a 2-factor. If $d(u) + d(v) \geq n - 2$ for all pairs of non-adjacent vertices $u, v \in V(G)$, then G is hamiltonian.*

Motivated by Theorem 2, we got the impression that it might be sufficient to require the condition $d(u) + d(v) \geq n - 2$ for all pairs of non-adjacent vertices u, v which are

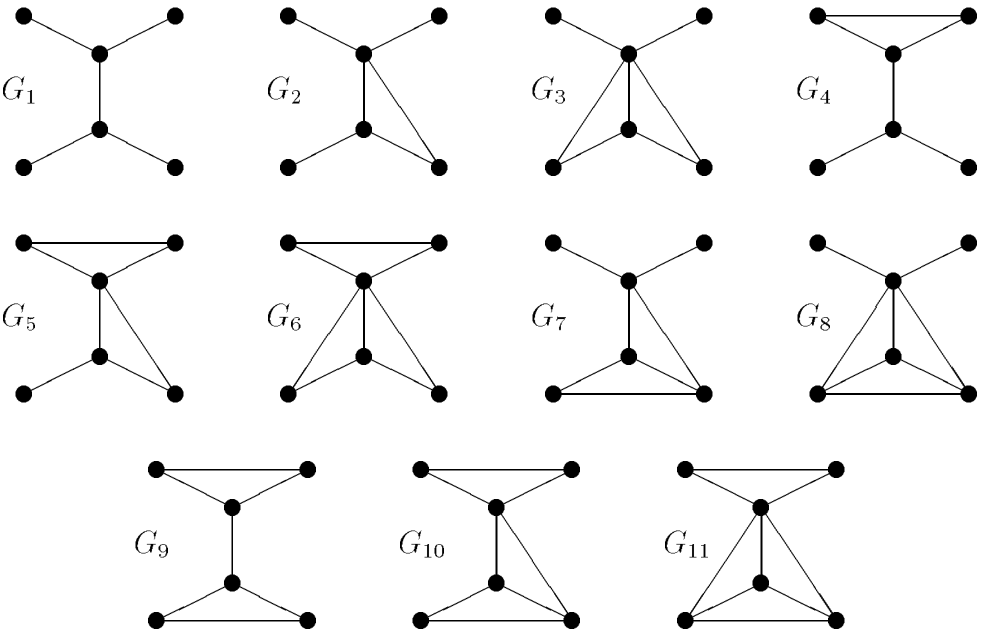

Figure 2.

contained in an induced P_4 or Z_1 (cf. G_1 and G_2 in Fig. 2). However, examples can be given showing that this is not the case even with the requirement $d(u) + d(v) \geq n - 1$. A class of such graphs can be obtained by joining two additional vertices u, v to two prescribed vertices of an arbitrary clique on at least 5 vertices (notice that u and v are contained in an induced $K_{1,3}$ and have $d(u) + d(v) = 4 \leq n - 3$). Thus, the degree condition required for the induced claw is necessary.

Next consider the class of graphs $G_{p,q,r}$ which consist of three complete graphs K_p, K_q, K_r for $p \geq q \geq r \geq 3$ and the additional edges u_iv_i, u_iw_i, v_iw_i for $i = 1, 2$ and vertices $u_1, u_2 \in V(K_p)$, $v_1, v_2 \in V(K_q)$ and $w_1, w_2 \in V(K_r)$. These graphs are 2-connected, claw-free with a 2-factor, but the degree condition is not satisfied for all induced P_4 and induced Z_1.

Finally, the complete bipartite graph $K_{p,q}$ with $p = \lfloor \frac{n-1}{2} \rfloor$ and $q = \lceil \frac{n+1}{2} \rceil$ for $n \geq 5$ is 2-connected, satisfies $d(u) + d(v) \geq n - 2$ for every pair of nonadjacent vertices u, v, but it has no 2-factor.

These examples show that all the assumptions of the following theorem are, in some sense, best possible.

Theorem 4. *Let G be a 2-connected graph with a 2-factor. If $d(u) + d(v) \geq n - 2$ for all pairs of non-adjacent vertices u, v contained in a $K_{1,3}$, in a Z_1 or as endvertices in a P_4, then G is hamiltonian.*

Example. Let i_0, i_1, i_2, i_3, i_4 be integers such that $i_0, i_4 \geq 1$, $i_2 \geq 2$, $i_1 \geq i_2 + i_4 - 1$, $i_3 \geq i_0 + i_2 - 1$. Let G be the graph obtained by taking vertex-disjoint graphs H_0, H_1, H_2, H_3, H_4, where $H_j \simeq K_{i_j}$ for $j = 0, 1, 3, 4$ and $H_2 \simeq \overline{K_{i_2}}$, and by adding all edges xy for $x \in V(H_i)$, $y \in V(H_{i+1})$, $i = 0, 1, 2, 3$. Then the graph G satisfies the assumptions of Theorem 4, but not of Theorem 3. Note that G has diameter $diam(G) = 4$ while the assumptions of Theorem 3 imply $diam(G) \leq 3$.

3 Proofs

We first prove some lemmas which will be useful for the proof of Theorem 4.

Lemma 1. *Let C_p, C_q and C be three vertex-disjoint cycles with $V(C_p) = \{u_1, \ldots, u_p\}$ and $V(C_q) = \{v_1, \ldots, v_q\}$. If $u_p v_q \in E(G)$ and $d_C(u_1) + d_C(v_1) \geq |V(C)| + 1$, then there is a cycle C' such that $V(C') = V(C_p) \cup V(C_q) \cup V(C)$.*

Proof. Since $d_C(u_1) + d_C(v_1) \geq |V(C)| + 1$, there exists a pair of consecutive vertices $w_1, w_2 \in V(C)$ such that $u_1 w_1, v_1 w_2 \in E(G)$ or $u_1 w_2, v_1 w_1 \in E(G)$ and we can easily construct the desired cycle C'. ■

Lemma 2. *Let C_p and C_q be vertex-disjoint cycles with vertices labeled $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$. Suppose $u_p v_q \in E(G)$; $u_p v_1, u_1 v_q, u_1 v_1 \notin E(G)$. If $d_{C_p \cup C_q}(u_1) + d_{C_p \cup C_q}(v_1) \geq p + q - 1$, then there is a cycle C such that $V(C) = V(C_p) \cup V(C_q)$.*

Proof. Suppose there is no such cycle. Then $v_1 u_{p-1}, v_{q-1} u_1 \notin E(G)$. Let

$$S = \{i |\ v_1 u_i \in E(G), 2 \leq i \leq p-2\},\ T = \{i |\ u_1 u_{i+1} \in E(G), 1 \leq i \leq p-2\}.$$

If there is some $i \in T \cap S$, then $C = v_1 u_i \overleftarrow{C_p} u_1 u_{i+1} \overrightarrow{C_p} u_p v_q \overleftarrow{C_q} v_1$ would be the desired cycle. Hence we can assume that $S \cap T = \emptyset$. Now $d_{C_p}(v_1) = |S|$ and $d_{C_p}(u_1) = |T| + 1$, from which $d_{C_p}(u_1) + d_{C_p}(v_1) = |S| + |T| + 1 = |S \cup T| + 1 \leq p - 1$. By the same argument we obtain $d_{C_q}(u_1) + d_{C_q}(v_1) \leq q - 1$ and thus $d_{C_p \cup C_q}(u_1) + d_{C_p \cup C_q}(v_1) \leq p + q - 2$, a contradiction. ■

Let C^1, C^2 be two vertex-disjoint cycles. We say that a vertex $v \in V(C^1)$ is C^2-*universal*, if v is adjacent to all vertices of C^2.

Assume now that there are two vertex-disjoint cycles C^1, C^2 and a C^2-universal vertex $v \in V(C^1)$. If v^- or v^+ has a neighbor on C^2, then we can again easily construct a cycle C such that $V(C) = V(C^1) \cup V(C^2)$.

Lemma 3. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k \geq 2$ cycles C^1, $C^2, \ldots,$ C^k, where k is minimal. Then for every pair of cycles C^i, C^j, $1 \leq i < j \leq k$, and every C^j-universal vertex $v \in V(C^i)$, neither v^- nor v^+ has a neighbor on C^j.*

Corollary 4. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k \geq 2$ cycles C^1, $C^2, \ldots,$ C^k, where k is minimal. Then for every pair of cycles C^i, C^j, $1 \leq i < j \leq k$, all C^j-universal vertices of $V(C^i)$ are pairwise non-consecutive.*

Corollary 5. *Let G be a non-hamiltonian graph with a 2-factor consisting of $k \geq 2$ cycles C^1, $C^2, \ldots,$ C^k, where k is minimal. Then there is no pair of cycles C^i, C^j, $1 \leq i < j \leq k$, such that there is both a C^j-universal vertex $v_i \in V(C^i)$ and a C^i-universal vertex $v_j \in V(C^j)$.*

We will also use the following simple lemma.

Lemma 6. *Let C be a cycle in a graph G and let x, $y \in V(C)$ be such that there is no x, y-path P with $V(P) = V(C)$. Then $x^+y^+ \notin E(G)$ and $d_C(x^+) + d_C(y^+) \leq |V(C)|$.*

Proof. If $x^+y^+ \in E(G)$, then $P = x \overleftarrow{C} y^+x^+ \overrightarrow{C} y$ is a x, y-path with $V(P) = V(C)$. Hence $x^+y^+ \notin E(G)$. Put $M = \{z \in V(C) | \ zx^+ \in E(G)\}$ and $N = \{z \in x^{++} \overrightarrow{C} y^+ | \ z^-y^+ \in E(G)\} \cup \{z \in y^{++} \overrightarrow{C} x | \ z^+y^+ \in E(G)\}$. Then $|M| = d_C(x^+)$, $|N| = d_C(y^+) - 1$ and $x^+ \notin M \cup N$. Thus, if $d_C(x^+) + d_C(y^+) \geq |V(C)| + 1$, there is a vertex $z \in M \cap N$, but then the path $x \overleftarrow{C} y^+z^- \overleftarrow{C} x^+z \overrightarrow{C} y$ (if $z \in x^+ \overrightarrow{C} y^+$) or $x \overleftarrow{C} z^+y^+ \overrightarrow{C} zx^+ \overrightarrow{C} y$ (if $z \in y^+ \overrightarrow{C} x^+$) yields a contradiction. Hence $d_C(x^+) + d_C(y^+) \leq |V(C)|$. ■

Proof of Theorem 4. Assume G is not hamiltonian and choose a 2-factor of G with $k \geq 2$ cycles C^1, $C^2, \ldots, C^k$ such that k is minimal. We distinguish the following cases.

> <u>Case 1</u>. There are two cycles C^{t_1}, C^{t_2}, $1 \leq t_1 < t_2 \leq k$, which are connected by two vertex-disjoint edges.
>
> > <u>Subcase A</u>. There is an edge xy such that $x \in V(C^{t_1})$, $y \in V(C^{t_2})$ and neither x is C^{t_2}-universal nor y is C^{t_1}-universal.
> >
> > <u>Subcase B</u>. Every vertex $x \in V(C^{t_1})$ with $N(x) \cap V(C^{t_2}) \neq \emptyset$ is C^{t_2}-universal.
>
> <u>Case 2</u>. No pair of cycles C^i, C^j, $1 \leq i < j \leq k$, is connected by two vertex-disjoint edges.

By Corollary 5, no other possibilities can occur.

Throughout the proof, we denote $n_i = |V(C^i)|$, $1 \leq i \leq k$. For convenience we set $p = n_1$ and $q = n_2$.

<u>Case 1</u>. We can without loss of generality suppose that $C^{t_1} = C^1 \simeq C_p$ with vertices labeled $u_1, \ldots, u_p$, $C^{t_2} = C^2 \simeq C_q$ with vertices labeled $v_1, \ldots, v_q$, $u_pv_q \in E(G)$ and $u_iv_j \in E(G)$ for some i, j with $1 \leq i \leq p-1$, $1 \leq j \leq q-1$.

<u>Subcase A</u>. Suppose (without loss of generality) that $u_pv_1, u_1v_q, u_1v_1 \notin E(G)$. Thus $\langle u_1, u_p, v_q, v_1 \rangle \simeq P_4$, from which $d(u_1)+d(v_1) \geq n-2$. Since k is minimal, by Lemma 1 and Lemma 2 we have $d_{C^1}(u_1)+d_{C^1}(v_1) = p-1$, $d_{C^2}(u_1)+d_{C^2}(v_1) = q-1$. If $u_1u_{i+1}, v_1v_{j+1} \in$

$E(G)$, then the cycle $u_1u_{i+1} \overrightarrow{C^1} u_pv_q \overleftarrow{C^2} v_{j+1}v_1 \overrightarrow{C^2} v_ju_i \overleftarrow{C^1} u_1$ contradicts the minimality of k. Hence we can without loss of generality assume that $u_1u_{i+1} \notin E(G)$. Since equality holds in Lemma 2, this implies $v_1u_i \in E(G)$ and thus $2 \leq i \leq p-2$. Moreover, since $v_1u_{p-1} \notin E(G)$, there exists $r > i$ such that $u_{r-1}v_1, u_{r+1}u_1 \in E(G)$ and $u_1u_r, v_1u_r \notin E(G)$. Since there is no cycle C such that $V(C) = V(C^1) \cup V(C^2)$, we have $u_rv_2, u_rv_q \notin E(G)$. By symmetry and since $u_rv_1 \notin E(G)$, we conclude $u_rv_{q-1} \notin E(G)$. Now $C = v_1u_{r-1} \overleftarrow{C^1} u_1u_{r+1} \overrightarrow{C^1} u_pv_q \overleftarrow{C^2} v_1$ is a cycle such that $V(C) = V(C^1) \cup V(C^2) \setminus \{u_r\}$. If $u_ru_i, u_ru_{i+1} \in E(G)$ for some i with $2 \leq i \leq r-2$ or $r+1 \leq i \leq p-1$, then u_r can be *inserted* into the cycle C by replacing the edge u_iu_{i+1} by the path $u_iu_ru_{i+1}$. Hence we conclude that $u_ru_{r-2}, u_ru_{r+2} \notin E(G)$ and $d_{C^1}(u_r) \leq p/2$. Likewise u_r can be inserted if $u_rv_i, u_rv_{i+1} \in E(G)$ for some i with $2 \leq i \leq q-3$. Hence $d_{C^2}(u_r) \leq (q-4+1)/2 = (q-3)/2$. For any other cycle C^j, $3 \leq j \leq k$, if $u_rw_1, u_rw_2 \in E(G)$ for two consecutive vertices w_1, w_2 on C^j, then u_r can be inserted into C^j, contradicting the minimality of k. Hence $d_{C^j}(u_r) \leq n_j/2$ and thus $d(u_r) \leq p/2 + (q-3)/2 + \sum_{j=3}^{k} n_j/2 = (n-3)/2$. Now $\langle u_{r-1}, u_{r-2}, u_r, v_1\rangle$ and $\langle u_{r+1}, u_r, u_{r+2}, u_1\rangle$ are isomorphic to $K_{1,3}$ or Z_1 implying $d(v_1) \geq (n-1)/2$ and $d(u_1) \geq (n-1)/2$. Altogether we obtain $n-1 \leq d(u_1)+d(v_1) \leq p+q-2+\sum_{j=3}^{k} n_j = n-2$, a contradiction.

<u>Subcase B</u>. Let $M = \{x \in V(C^1) |\ N_{C^2}(x) \neq \emptyset\}$. Then, by the assumptions of Case 1, $|M| \geq 2$, $u_p \in M$ and (recall Corollary 5 and Corollary 4), no two vertices in M are consecutive on C^1. Suppose first that there are $x, y \in M$, $x \neq y$, such that both $x^-x^+ \notin E(G)$ and $y^-y^+ \notin E(G)$. Then, since (by Lemma 3) both $\langle x, x^-, x^+, v_q\rangle \simeq K_{1,3}$ and $\langle y, y^-, y^+, v_q\rangle \simeq K_{1,3}$, we have $d(x^-)+d(x^+)+d(y^-)+d(y^+) \geq 2(n-2) \geq 2(p+q-2+n-p-q) \geq 2(p+1)+2(n-p-q)$. On the other hand, by the minimality of k, there is no hamiltonian x,y-path in $G[V(C^1)]$ and hence, by Lemma 6, $d_{C^1}(x^+)+d_{C^1}(y^+)+d_{C^1}(x^-)+d_{C^1}(y^-) \leq 2p$. Together we obtain $2(p+1)+2(n-p-q) \leq d(x^+)+d(y^+)+d(x^-)+d(y^-) \leq 2p+2(n-p-q)$, which is a contradiction.

Hence we can suppose that $x^-x^+ \in E(G)$ for every $x \in M$, $x \neq u_p$. But then, for any $x \in M$, $x \neq u_p$, we have $u_1x \notin E(G)$ and $u_1x^{++} \notin E(G)$ (otherwise the cycles $u_1xv_1 \overrightarrow{C^2} v_qu_p \overleftarrow{C^1} x^+x^- \overleftarrow{C^1} u_1$ and $u_1x^{++} \overrightarrow{C^1} u_pv_q \overleftarrow{C^2} v_1xx^+x^- \overleftarrow{C^1} u_1$ contradict the minimality of k). Now $x^{++} \notin M$, since $x^{++} \overrightarrow{C^1} x^-x^+x$ is a hamiltonian path in $G[V(C^1)]$. Since also (by Lemma 6) $u_1x^+ \notin E(G)$ and, by Lemma 3, $d_{C^2}(u_1) = 0$, we have $d_{C^1\cup C^2}(u_1) \leq p-1-3(|M|-1)$. Since every vertex in M is C^2-universal, we have $d_{C^1\cup C^2}(v_q) \leq q-1+|M|$. If there is a cycle C^i, $3 \leq i \leq k$, such that u_1 and v_q have consecutive neighbors on C^i, then we easily construct a cycle C' with $V(C') = V(C^1) \cup V(C^2) \cup V(C^i)$, contradicting the minimality of k; hence $d_{C^3\cup\ldots\cup C^k}(u_1) + d_{C^3\cup\ldots\cup C^k}(v_q) \leq |V(C^3) \cup \ldots \cup V(C^k)| = n-p-q$. Since $\langle u_p, v_q, v_1, u_1\rangle \simeq Z_1$, we have $d(u_1)+d(v_q) \geq n-2$. Altogether we obtain $n-2 \leq d(u_1)+d(v_q) \leq p-1-3(|M|-1)+q-1+|M|+n-p-q$, from which $|M| \leq 3/2$, a contradiction.

<u>Case 2</u>. Since G is 2-connected, there are m cycles, $3 \leq m \leq k$, say, $C^1, C^2, \ldots, C^m$, with vertices labeled $v_1^i, \ldots, v_{n_i}^i$, and pairs of vertices $v_{r_i}^i, v_{s_i}^i \in V(C^i)$ such that $v_{s_i}^i v_{r_{i+1}}^{i+1} \in E(G)$ (modulo m). If $s_i = r_i \pm 1$ for all $1 \leq i \leq m$, then there is a cycle C such that

$V(C) = \cup_{i=1}^m V(C^i)$, e.g. $C = v_{s_1}^1 v_{r_2}^2 \overrightarrow{C^2} v_{s_2}^2 v_{r_3}^3 \dots v_{s_m}^m v_{r_1}^1 \overrightarrow{C^1} v_{s_1}^1$, if $s_i = r_i + 1$ for $1 \le i \le m$, which contradicts the minimality of k.

Now suppose without loss of generality that $s_1 \neq r_1 \pm 1$. Thus $n_1 \geq 4$. If $v_{r_1+1}^1 v_{s_1+1}^1 \in E(G)$ or $d_{C^1}(v_{r_1+1}^1) + d_{C^1}(v_{s_1+1}^1) \geq n_1 + 1$, then, by Lemma 6, there is a hamiltonian path in $G[V(C^1)]$ with endvertices $v_{r_1}^1$, $v_{s_1}^1$.

Suppose such a path does not exist. With a repeat of previous arguments we will show that $v_{s_1}^1, v_{r_1}^1$ are both universal vertices and that $n_1 = 4$. Suppose first that $v_{s_1}^1$ is not universal. Then there is a vertex $x \in V(C^2)$ such that $v_{s_1}^1 x \in E(G)$, but $v_{s_1}^1 x^+ \notin E(G)$. As in Subcase A we obtain this time $d(v_{s_1+1}^1) + d(x^+) \leq (n_1 - 2) + (n_2 - 1) + \sum_{j=3}^k n_j < n - 2$, a contradiction. The same argument holds for $v_{r_1}^1$. Thus both $v_{s_1}^1$ and $v_{r_1}^1$ are universal vertices. Suppose next that $n_1 \geq 5$. By Lemma 6 we have $d_{C^1}(v_{s_1+1}^1) + d_{C^1}(v_{r_1+1}^1) \leq n_1$. Hence we may assume that $d_{C^1}(v_{s_1+1}^1) \leq n_1/2$. But then $\langle v_{s_1}^1, x, x^+, v_{s_1+1}^1 \rangle \simeq Z_1$ for any pair of consecutive vertices $x, x^+ \in V(C^2)$ and $d_{C^1 \cup C^2}(v_{s_1+1}^1) + d_{C^1 \cup C^2}(x^+) \leq n_1/2 + (n_2 - 1 + 1) + \sum_{j=3}^k n_j < (n_1 - 2) + n_2 + \sum_{j=3}^k n_j \leq n - 2$, a contradiction. Hence $n_1 = 4$.

Let $\{s_1, r_1\} = \{2, 4\}$. Then $d_{C^1}(v_1^1) = d_{C^1}(v_3^1) = 2$ and both v_2^1 and v_4^1 are contained in an induced Z_1, say, $\langle v_2^1, v_1^1, v_{n_m}^m, v_1^m \rangle$ and $\langle v_4^1, v_3^1, v_{n_2}^2, v_1^2 \rangle$. Since $N_{C^m}(v_3^1) = \emptyset$, $N_{C^m}(v_1^1) = \emptyset$, $N_{C^2}(v_3^1) = \emptyset$, $N_{C^2}(v_1^1) = \emptyset$, we have $d_{C^1 \cup C^2 \cup C^3}(v_1^1) + d_{C^1 \cup C^2 \cup C^3}(v_3^1) = 4$, where $n_1 + n_2 + n_3 \geq 4 + 3 + 3 = 10$. Since $d(v_1^1) + d(v_3^1) \geq n - 2$, we have $k \geq 4$ and $\sum_{j=4}^k d_{C^j}(v_1^1) + d_{C^j}(v_3^1) \geq \sum_{j=4}^k n_j + 4$. Hence there exists a cycle C^j and two consecutive vertices w_1, w_2 on C^j such that (without loss of generality) $v_1^1 w_1, v_3^1 w_2 \in E(G)$. Then $C^a = v_4^1 v_1^2 \overrightarrow{C^2} v_{n_2}^2 v_4^1$ and $C^b = v_1^1 v_2^1 v_3^1 w_2 \overrightarrow{C^j} w_1 v_1^1$ are two cycles such that $V(C^a) \cup V(C^b) = V(C^1) \cup V(C^2) \cup V(C^j)$, which contradicts the minimality of k.

This shows that, for each cycle C^i, the vertices $v_{r_i}^i$ and $v_{s_i}^i$ are connected by a hamiltonian path in $G[V(C^i)]$, $1 \leq i \leq m$. But then there is a cycle C such that $V(C) = \cup_{j=1}^m V(C^j)$, contradicting again the minimality of k. This contradiction completes the proof of Theorem 4. ■

References

[1] Anstee, R.P.: An algorithmic proof of Tutte's f-factor theorem. J. Algorithms 6 (1985), 112-131.

[2] Bondy, J.A.; Murty, U.S.R.: Graph theory with applications. Macmillan, London and Elsevier, New York, 1976.

[3] Chvátal, V.: Tough graphs and hamiltonian circuits. Discrete Math. 5(1973), 215-228.

[4] Dirac, G.A.: Some theorems on abstract graphs. Proc. London Math. Soc. (3) 2 (1952) 69-81.

[5] Enomoto, H.; Jackson, B.; Katerinis, P.; Saito, A.: Toughness and the existence of k-factors. J. Graph Theory 9(1985), 87-95.

[6] Faudree, R.; van den Heuvel, J.: Degree sums, k-factors and Hamilton cycles in graphs. Graphs and Combinatorics 11(1995), 21-28.

[7] Hoede, C.: A comparison of some conditions for non-hamiltonicity of graphs. Ars Combinatoria (to appear).

[8] Ore, O.: Note on hamiltonian circuits. Amer. Math. Monthly 67 (1960), 55.