Forbidden subgraphs, hamiltonicity and closure in claw-free graphs

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Abstract

We study the stability of some classes of graphs defined in terms of forbidden subgraphs under the closure operation introduced by the second author. Using these results, we prove that every 2-connected claw-free and P_7-free, or claw-free and Z_4-free, or claw-free and eiffel-free graph is either hamiltonian or belongs to a certain class of exceptions (all of them having connectivity 2).

1 Introduction

In this paper we consider only finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges. For terminology and notation not defined here we refer to [3].

If $H_1, \ldots, H_k (k \geq 1)$ are graphs, then we say that a graph G is $H_1 \ldots H_k$-*free* if G contains no copy of any of the graphs $H_1, \ldots, H_k$ as an induced subgraph; the graphs $H_1, \ldots, H_k$ will be also referred to in this context as *forbidden subgraphs*. Specifically, the four-vertex star $K_{1,3}$ will be also denoted by C and called the *claw* and in this case we say that G is *claw-free*. Whenever we list vertices of an induced claw, its *center*, (i.e. its only vertex of degree 3) is always the first vertex of the list. Further graphs that will be often considered as forbidden subgraphs are shown in Fig. 1.

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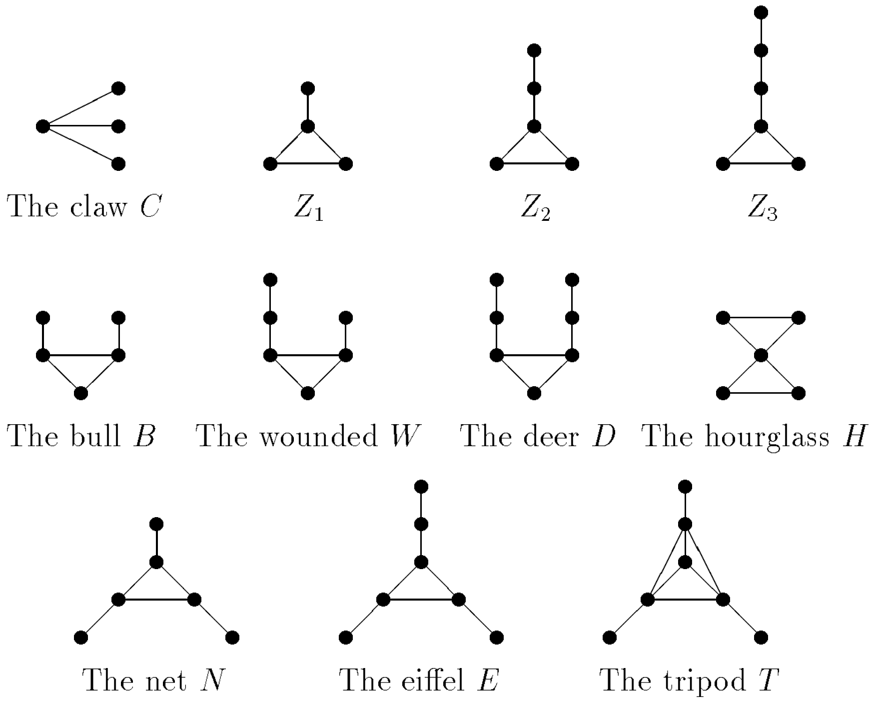


Figure 1

If $A \subset V(G)$, then the induced subgraph on A in G will be denoted by $\langle A \rangle_G$ (or simply by $\langle A \rangle$). A 2-element cutset of G will be called a *biarticulation* of G and, if $A \subset V(G)$ is a biarticulation of G, then the components of the graph $\langle V(G) \setminus A \rangle$ will be called the *bicomponents* of G. By a *clique* we mean a (not necessarily maximal) complete subgraph of G. We denote by P_k ($k \geq 2$) the path on k vertices, i.e. of length $k-1$. For $A, B \subset V(G)$, a path in G having one endvertex in A and the other in B will be referred to as an (A, B)-path. The *circumference* of G (i.e. the length of a longest cycle in G) is denoted by $c(G)$ and the *independence number* of G (i.e. the size of a largest independent set in G) is denoted by $\alpha(G)$.

One of the first results on forbidden subgraphs and hamiltonicity is by Goodman and Hedetniemi [12].

Theorem A [12]. *Every 2-connected CZ_1-free graph is hamiltonian.*

This result was extended to the larger class of CN-free graphs by Duffus, Gould and Jacobson [7].

Theorem B [7]. *Every 2-connected CN-free graph is hamiltonian.*

Concerning other pairs and triples of forbidden subgraphs, the following results were proved in [13], [5] and [11].

Theorem C.

(i) [13] Every 2-connected CZ_2-free graph is hamiltonian.

(ii) [5] Every 2-connected CP_6-free graph is hamiltonian.

(iii) [13] Every 2-connected CHZ_3-free graph is hamiltonian.

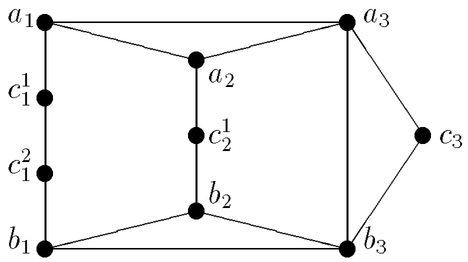
(iv) [5] Every 2-connected CDP_7-free graph is hamiltonian.

(v) [11] Every 2-connected CHP_7-free graph is hamiltonian.

Bedrossian [1] characterized all pairs of connected forbidden subgraphs X, Y such that every 2-connected X, Y-free graph is hamiltonian.

Theorem D [1]. *Let X and Y be connected graphs with $X,\ Y \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then, G being XY-free implies G is hamiltonian if and only if (up to symmetry) $X = C$ and $Y = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W.*

Following [6], we denote by $\mathcal{P}$ the class of all graphs that are obtained by taking two vertex-disjoint triangles $\langle\{a_1, a_2, a_3\}\rangle$, $\langle\{b_1, b_2, b_3\}\rangle$ and by joining every pair of vertices $\{a_i, b_i\}$ by a copy of a path $P_{k_i} = a_i c_i^1 c_i^2 \dots c_i^{k_i-2} b_i$ for $k_i \geq 3$ or by a triangle $\langle\{a_i, b_i, c_i\}\rangle$. We denote a graph from $\mathcal{P}$ by P_{x_1,x_2,x_3}, where $x_i = k_i$ if a_i, b_i are joined by a copy of P_{k_i}, and $x_i = T$, if a_i, b_i are joined by a triangle (see Fig. 2).



$P_{4,3,T} \in \mathcal{P}$

Figure 2

Since, as shown in [9], $P_{T,T,T}$ and $P_{3,T,T}$ are the only two 2-connected nonhamiltonian CZ_3-free graphs, Theorem D was extended by Faudree and Gould [10] in the following way (where the proof of the "only if" part of Theorem E is now based on infinite families of nonhamiltonian graphs).

Theorem E [10]. *Let X and Y be connected graphs with $X,\ Y \neq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then, G being XY-free implies G is hamiltonian if an only if (up to symmetry) $X = C$ and $Y = P_4, P_5, P_6, C_3, Z_1, Z_2, Z_3, B, N$ or W.*

The following theorem was proved in [6].

Theorem F [6]. *Every nonhamiltonian 2-connected claw-free graph contains an induced subgraph $H \in \mathcal{P}$.*

Note that Theorem F implies the "if" part of Theorem E as an immediate corollary.

For any $x \in V(G)$ and any $i \geq 1$, the set $N_G^i(x) = \{y \in V(G) |\ \mathrm{dist}(x,y) = i\}$ (where $\mathrm{dist}(x,y)$ denotes the distance of x and y) is called the *neighborhood of x at distance i*. The neighborhood of x at distance 1 will be simply called *neighborhood of x* and denoted by $N_G(x)$.

It is easy to see that a graph G is claw-free if and only if $\alpha(\langle N_G(x)\rangle) \leq 2$ for every $x \in V(G)$. Shepherd [15] introduced the following concept.

A graph G is said to be *distance claw-free* if $\alpha(\langle N_G^i(x)\rangle) \leq 2$ for every $x \in V(G)$ and $i \geq 1$. The following theorem was proved in [15].

Theorem G [15].
(i) A graph G is distance claw-free if and only if G is CET-free.
(ii) Every 2-connected distance claw-free graph is traceable.
(iii) Every 3-connected distance claw-free graph is hamiltonian.

We say that a vertex $x \in V(G)$ is *locally connected* (*eligible, simplicial, locally disconnected*) if the subgraph $\langle N_G(x)\rangle$ is connected (connected noncomplete, complete, disconnected). The set of all locally connected (eligible, simplicial, locally disconnected) vertices of G will be denoted by $V_{LC}(G)$ $(V_{EL}(G), V_{SI}(G), V_{LD}(G))$, respectively. Thus, the sets $V_{EL}(G), V_{SI}(G), V_{LD}(G)$ are pairwise disjoint, $V_{EL}(G) \cup V_{SI}(G) = V_{LC}(G)$ and $V_{LC}(G) \cup V_{LD}(G) = V(G)$.

For an eligible vertex $x \in V_{EL}(G)$ set $B_x = \{uv|\ u,v \in N_G(x), uv \notin E(G)\}$ and let G'_x be the graph with vertex set $V(G'_x) = V(G)$ and with edge set $E(G'_x) = E(G) \cup B_x$ (i.e., G'_x is obtained from G by adding to $\langle N_G(x)\rangle_G$ the set B_x of all missing edges). The graph G'_x is called the *local completion of G at x*. The following statement was proved in [14].

Proposition H [14]. *Let G be a claw-free graph and let $x \in V_{EL}(G)$ be an eligible vertex of G. Then*
(i) the graph G'_x is claw-free,
(ii) $c(G'_x) = c(G)$.

The following concept was introduced in [14].

Let G be a claw-free graph. We say that a graph H is a closure *of G, denoted $H = \mathrm{cl}(G)$, if*
(i) there is a sequence of graphs $G_1, \ldots, G_t$ and vertices $x_1, \ldots, x_{t-1}$ such that $G_1 = G$, $G_t = H$, $x_i \in V_{EL}(G_i)$ and $G_{i+1} = (G_i)'_{x_i}$, $i = 1, \ldots, t-1$,
(ii) $V_{EL}(H) = \emptyset$.

(Equivalently, $\mathrm{cl}(G)$ is obtained from G by recursively repeating the operation of local completion, as long as this is possible).

Theorem K [14]. *Let G be a claw-free graph. Then*
(i) the closure $\mathrm{cl}(G)$ is well-defined,
(ii) there is a triangle-free graph H such that $\mathrm{cl}(G)$ is the line graph of H,
(iii) $c(G) = c(\mathrm{cl}(G))$.

Remarks. 1. Specifically, part (*i*) of Theorem K implies that $\mathrm{cl}(G)$ does not depend on the order of eligible vertices used during the construction of $\mathrm{cl}(G)$.

2. It is easy to see that $\mathrm{cl}(G)$ can be equivalently characterized as the minimum $(K_4 - e)$-free graph on $V(G)$ containing G.

3. If in some step G_i of the closure process, a vertex z has a complete neighborhood $\langle N(z)\rangle_{G_i}$, then at the end of the process, its neighborhood in $\mathrm{cl}(G)$ is also complete. In particular, if $z \in V_{EL}(G_i)$ for some i, $i \leq i \leq t-1$, then $z \in V_{SI}(\mathrm{cl(G)})$ (since otherwise the closure process could be continued by a local completion at z).

We say that a claw-free graph G is *closed* if $G = \mathrm{cl}(G)$. Thus, G is closed if and only if $V_{EL}(G) = \emptyset$ (i.e., $V(G) = V_{SI}(G) \cup V_{LD}(G)$). By Theorem K(*ii*), if G is a closed claw-free graph, then every simplicial vertex of G belongs to exactly one maximal clique of G, and every locally disconnected vertex $x \in V_{LD}(G)$ belongs to exactly two maximal cliques $K^1(x)$ and $K^2(x)$ such that $V(K^1(x)) \cap V(K^2(x)) = \{x\}$ and there are no edges between $V(K^1(x) \setminus \{x\})$ and $V(K^2(x) \setminus \{x\})$.

Let $\mathcal{C}$ be a subclass of the class of claw-free graphs. Following [4], we say that the class $\mathcal{C}$ is *stable under the closure* (or simply *stable*) if $\mathrm{cl}(G) \in \mathcal{C}$ for every $G \in \mathcal{C}$ (equivalently, the class $\mathcal{C}$ is stable if the closure operation is internal on $\mathcal{C}$).

Specifically, $\mathcal{C}$ is stable if $G'_x \in \mathcal{C}$ for every $G \in \mathcal{C}$ and every $x \in V(G)$. Thus, the class of k-connected claw-free graphs is an example of a stable class for any $k \geq 1$ and, by Theorem K, both the class of hamiltonian claw-free graphs and the class of 2-connected nonhamiltonian claw-free graphs are also stable. However, in Theorem 3 we will see that this sufficient condition is, in general, not necessary.

In this paper we first observe the stability of some classes of graphs defined in terms of forbidden induced subgraphs and then, using these results and making use of the special structure of closed claw-free graphs (= line graphs of triangle-free graphs), we extend Theorems B, C and G(*ii*), (*iii*).

2 Main results

We first consider the stability of some classes defined in terms of forbidden induced subgraphs. We denote by (see also Fig. 3):

Z_i $(i \geq 1)$ – the graph which is obtained by identifying a vertex of a triangle with an endvertex of a path of length i,

$B_{i,j}$ $(j \geq i \geq 1)$ – the *generalized* (i,j)*-bull*, i.e. the graph which is obtained by identifying each of some two distinct vertices of a triangle with an endvertex of one of two vertex-disjoint paths of lengths i, j,

$N_{i,j,k}$ $(k \geq j \geq i \geq 1)$ – the *generalized* (i,j,k)*-net*, i.e. the graph which is obtained by identifying each vertex of a triangle with an endvertex of one of three vertex-disjoint paths of lengths i, j, k.

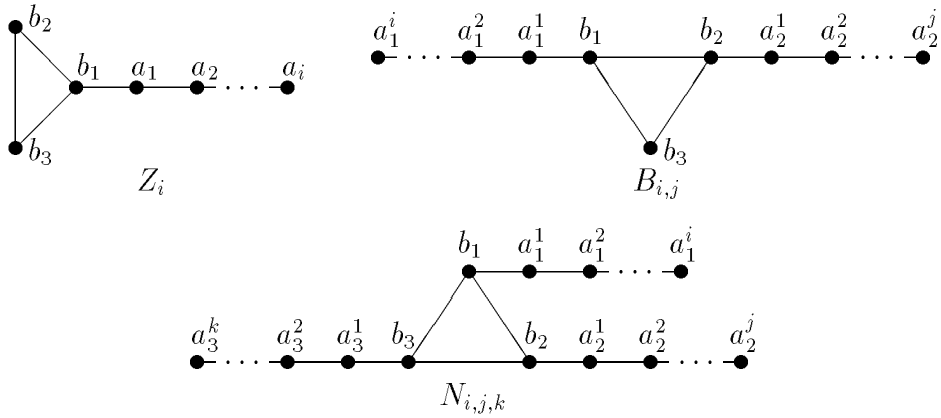


Figure 3

Thus, $B_{1,1} \simeq B$, $B_{1,2} \simeq W$, $B_{2,2} \simeq D$, $N_{1,1,1} \simeq N$ and $N_{1,1,2} \simeq E$. We will always keep the labelling of the vertices of the graphs Z_i, $B_{i,j}$ and $N_{i,j,k}$ as shown in Figure 3.

Theorem 1. *Let G be a CP_i-free graph ($i \geq 1$) and let $x \in V_{EL}(G)$. Then the graph G'_x is CP_i-free.*

Proof. If G is CP_i-free, then, by Proposition H, G'_x is claw-free. Suppose that $H = \langle\{a_1, \ldots, a_i\}\rangle_{G'_x}$ is an induced path in G'_x and let $B_x = E(G'_x) \setminus E(G)$. Then, since G is P_i-free, $|E(H) \cap B_x| \geq 1$. Since $\langle N_G(x)\rangle_{G'_x}$ is a clique and H is triangle-free, $|E(H) \cap B_x| \leq 1$. Let thus $E(H) \cap B_x = a_s a_{s+1}$ ($1 \leq s \leq i-1$). Since H is an induced path, $x \notin V(H)$. If $xa_t \in E(G)$ for some $t \neq s, s+1$, then $a_s a_t, a_{s+1} a_t \in E(H)$, which again contradicts the fact that H is an induced path; hence $N_G(x) \cap H = \{a_s, a_{s+1}\}$. But then $\langle\{a_1, \ldots, a_s, x, a_{s+1}, \ldots, a_i\}\rangle_G$ is an induced path of length i in G, which contradicts the fact that G is P_i-free. ∎

Corollary 2. *The class of CP_i-free graphs is a stable class for any $i \geq 3$.*

We now turn our attention to the class of CZ_i-free graphs ($i \geq 1$). Consider the graph G_i shown in Fig. 4. When $i \geq 3$, the graph G_i is clearly CZ_i-free, while $\langle\{b_1, b_2, x, a_1, \ldots, a_i\}\rangle_{G'_x} \simeq Z_i$. This example shows that for $i \geq 3$, the analogue of Theorem 1 for the class of CZ_i-free graphs fails. Nevertheless, we can still prove the analogue of Corollary 2 in this case.

Theorem 3. *The class of CZ_i-free graphs is a stable class for any $i \geq 1$.*

Proof of Theorem 3 will be given in Section 3.

The following proposition is an analogue of Proposition 1 in the case of $CN_{i,j,k}$-free graphs.

Theorem 4. *Let G be a $CN_{i,j,k}$-free graph ($k \geq j \geq i \geq 1$) and let $x \in V_{EL}(G)$. Then the graph G'_x is $CN_{i,j,k}$-free.*

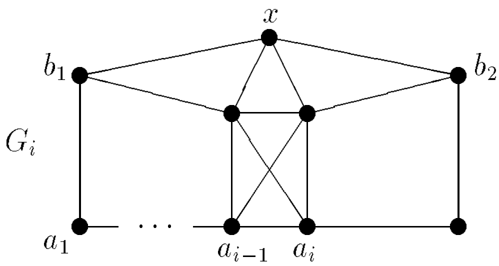


Figure 4

Proof. Suppose that $N_{i,j,k} \simeq H = \langle\{b_1, b_2, b_3, a_1^1, \ldots, a_1^i, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}\rangle_{G'_x} \subset G'_x$. Similarly as in the proof of Theorem 1 we can show that neither any of the edges $b_s a_s^1$ ($s = 1, 2, 3$) nor any of the edges $a_s^r a_s^{r+1}$ ($s = 1$ and $1 \leq r \leq i-1$, or $s = 2$ and $1 \leq r \leq j-1$, or $s = 3$ and $1 \leq r \leq k-1$) can be in B_x (since if e.g. $a_1^r a_1^{r+1} \in B_x$ for some r, $1 \leq r \leq i-1$, then obviously $x \notin V(H)$ and $\langle\{b_1, b_2, b_3, a_1^1, \ldots, a_1^r, x, a_1^{r+1}, \ldots, a_1^{i-1}, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}\rangle_G \simeq N_{i,j,k}$ – a contradiction). Hence $B_x^H = B_x \cap E(H) \subset \{b_1b_2, b_1b_3, b_2b_3\}$.

If $|B_x^H| = 3$, then $\langle\{x, b_1, b_2, b_3\}\rangle_G \simeq C$; hence $|B_x^H| \leq 2$. On the other hand, if $|B_x^H| = 1$, then e.g. for $|B_x^H| = \{b_1b_2\}$ we have $\langle\{b_3, b_1, b_2, a_3^1\}\rangle_G \simeq C$; other cases are similar. Hence $|B_x^H| = 2$. Suppose without loss of generality that $|B_x^H| = \{b_1b_2, b_1b_3\}$. Then evidently $x \notin V(H)$ (otherwise $xb_1, xb_2 \in E(G)$, which is impossible), and $N_G(x) \cap V(H) = \{b_1, b_2, b_3\}$ (since if e.g. $xa_s^r \in E(G)$, then $a_s^r b_1, a_s^r b_2 \in E(G'_x)$, which is impossible). But then $\langle\{x, b_2, b_3, b_1, a_1^1, \ldots, a_1^{i-1}, a_2^1, \ldots, a_2^j, a_3^1, \ldots, a_3^k\}\rangle_G \simeq N_{i,j,k}$ – a contradiction. ■

Corollary 5. *The class of $CN_{i,j,k}$-free graphs is a stable class for any i, j, k, $k \geq j \geq i \geq 1$.*

If G is claw-free and triangle-free, then G is a disjoint union of paths and cycles and hence G is closed. This implies that the class of claw-free and triangle-free graphs is also (trivially) stable. In the list given in Theorem E, it thus remains to consider the classes of CB-free and CW-free graphs. The following statement shows that, surprisingly, none of these classes is stable.

Proposition 6. *The class of $CB_{i,j}$-free graphs is not stable for any i, j, $j \geq i \geq 1$.*

Proof. Let $i, j \geq 1$ and let $G_{i,j,k}$ be the graph obtained by identifying each of the two vertices of a copy of a diamond $K_4 - e$ with one endvertex of a path P_k with $k \geq i + j + 3$ and let x be one of the two eligible vertices of $G_{i,j,k}$ (for $i = j = 2$ and $k = 7$ see Fig. 5). Then G is $CB_{i,j}$-free while G'_x is closed (hence $G'_x = \mathrm{cl}(G)$) and contains an induced subgraph isomorphic to $B_{i,j}$. ■

Now, suppose e.g. that G is a 2-connected nonhamiltonian CP_7-free graph. By Theorems D, E such graphs G exist; by Theorem K and by Corollary 2, $\mathrm{cl}(G)$ is also a 2-connected nonhamiltonian CP_7-free graph. By Theorem F, $\mathrm{cl}(G)$ contains an induced subgraph $H \in \mathcal{P}$ and, using the properties of the closure, it is possible to describe the structure of $\mathrm{cl}(G)$. This basic idea, applied to the classes of CP_7-free, CZ_4-free and $CN_{1,2,2}N_{1,1,3}$-free graphs, yields the following Theorems 7 – 9, extending Theorems B, C and G(ii), (iii). Proofs of Theorems 7 – 9 and of Corollary 10 are given in Section 3.

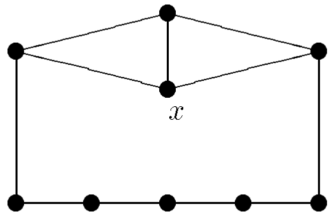


Figure 5

Denote by $\mathcal{F}_1, \ldots, \mathcal{F}_6$ the classes of graphs shown in Fig. 6 (where the elliptical parts represent cliques of size at least 3 and the remark "odd" above the dots indicates that the total number of maximal cliques in $\mathcal{F}_1$ and $\mathcal{F}_2$ is odd).

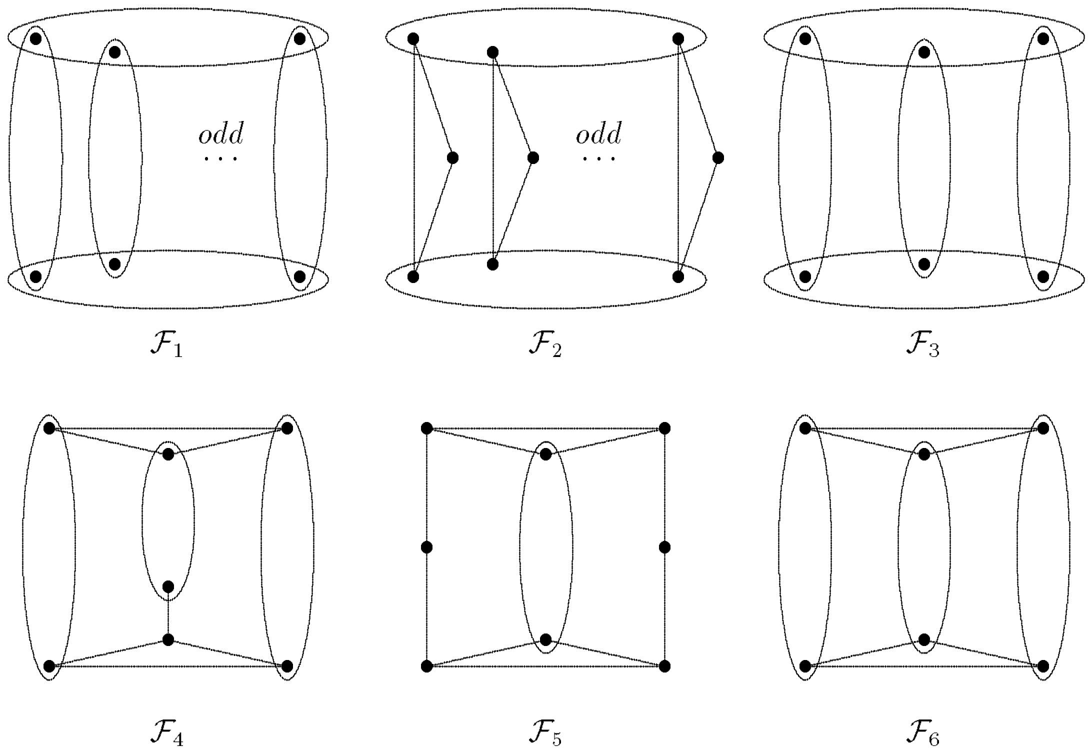


Figure 6

Theorem 7. *Let G be a 2-connected CP_7-free graph. Then either G is hamiltonian or* $\mathrm{cl}(G) \in \mathcal{F}_1$.

Theorem 8. *Let G be a 2-connected CZ_4-free graph. Then either G is hamiltonian, or* $G \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$, *or* $\mathrm{cl}(G) \in \mathcal{F}_2$.

Theorem 9. *Let G be a 2-connected $CN_{1,2,2}N_{1,1,3}$-free graph. Then either G is hamiltonian, or $G \simeq P_{3,3,3}$, or $\mathrm{cl}(G) \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$.*

Since the eiffel $E = N_{1,1,2}$ is an induced subgraph of both $N_{1,2,2}$ and $N_{1,1,3}$, the following statement is a special case of Theorem 9 for the class of CE-free graphs.

Corollary 10. *Let G be a 2-connected CE-free graph. Then either G is hamiltonian or $G \in \mathcal{F}_6$.*

Since all the (nonhamiltonian) exceptional graphs in Theorems 7 – 9 and in Corollary 10 are of connectivity 2, we immediately obtain the following corollary.

Corollary 11.
(i) *Every 3-connected CP_7-free graph is hamiltonian.*
(ii) *Every 3-connected CZ_4-free graph is hamiltonian.*
(iii) *Every 3-connected $CN_{1,2,2}N_{1,1,3}$-free graph is hamiltonian.*
(iv) *Every 3-connected CE-free graph is hamiltonian.*

Remarks. 1. By the Shepherd's characterization of distance claw-free graphs (Theorem G(i)), Corollary 11(iv) extends Theorem G(iii).

2. The graph G in Fig. 7 (a) belongs to neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ while its closure $\mathrm{cl}(G)$ belongs to both $\mathcal{F}_1$ and $\mathcal{F}_2$. Since G is 2-connected, nonhamiltonian, claw-free and both P_7-free and Z_4-free, it shows that Theorems 7 and 8 fail if we replace the conclusion $\mathrm{cl}(G) \in \mathcal{F}_1$ (or $\mathrm{cl}(G) \in \mathcal{F}_2$) by $G \in \mathcal{F}_1$ (or $G \in \mathcal{F}_2$), respectively. The graph in Fig. 7 (b) gives a similar example for Theorem 9.

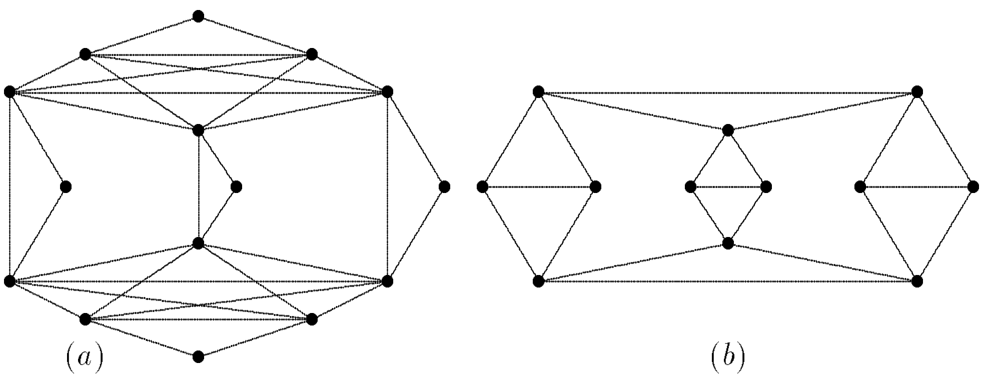


Figure 7

3. It is easy to see that the closure of a claw-free graph G is computable in polynomial time, and the classes $\mathcal{F}_1, \ldots, \mathcal{F}_6$ are, due to their simple structure, recognizable in polynomial time, too. Consequently, all the sufficient conditions for hamiltonicity given in Theorems 7 – 10 and in Corollary 11 can be checked in polynomial time. On the other hand, it is known that the decision whether G is hamiltonian is NP-complete even in line graphs (see [2], or, for more information on complexity results in claw-free graphs, Chapter 5 of the survey paper [8]).

4. In the proofs of Theorems 7 – 9, the fact that the classes considered are stable allows to assume that all graphs under consideration are closed (i.e., are line graphs of triangle-free graphs) and to use the structural information given by this fact to reduce the number of situations to be considered (see e.g. Lemma 12).

3 Proofs

Let G be a claw-free graph and let H be an induced subgraph of G. We say that H is a *permanent* (or *temporary*) *induced subgraph* of G if $\langle V(H)\rangle_{\mathrm{cl}(G)} \simeq H$ (or $\langle V(H)\rangle_{\mathrm{cl}(G)} \not\simeq H$), respectively.

Proof of Theorem 3. Let $i \geq 1$, let G be a CZ_i-free graph and suppose that $\mathrm{cl}(G)$ is not Z_i-free. Let $G_1, \ldots, G_t$ be the sequence of graphs that yields $\mathrm{cl}(G)$ (i.e. $G = G_1$, $\mathrm{cl}(G) = G_t$, $x_j \in V_{EL}(G_j)$ and $G_{j+1} = (G_j)'_{x_j}$, $j = 1, \ldots, t-1$) and let $r > 1$ be the smallest integer such that G_r contains a permanent induced subgraph isomorphic to Z_i. For each such subgraph H, denote the vertices of H as in Fig. 3 (for simplicity, put $a_0 = b_1$) and the path $a_0 a_1 \cdots a_i$, which is induced in G_r as in $\mathrm{cl}(G)$, by $P(H)$.

If for some $1 \leq k < r$, some edge $a_j a_{j+1}$ of $P(H)$, with $0 \leq j \leq i-1$, is missing in G_k, then G_k contains an $a_j a_{j+1}$-path P_j^k whose internal vertices, say $y_{j_1}, y_{j_2}, \cdots, y_{j_q}$, are some of the vertices $x_k, x_{k+1}, \cdots, x_{r-1}$. By Remark 3 (Section 1), the final neighborhood in $\mathrm{cl}(G)$ of each $y_{j_1}, y_{j_2}, \cdots, y_{j_q}$ is a clique containing all the y_{j_l}'s, a_j and a_{j+1}. Hence no y_{j_l} is equal or adjacent in G_k to any vertex of $V(H) \setminus \{a_i, a_{i+1}\}$ for otherwise this would contradict the property of H to be an induced subgraph of $\mathrm{cl}(G)$. By the same reason, no two interior vertices of two different paths $P_{j'}^k$, $P_{j''}^k$ $(0 \leq j' < j'' \leq i-1)$ can be adjacent in G_k. Thus, by concatenating these different induced paths with the edges of the path of H already existing in G_k, we can find for each $k < r$ an induced path $P^k = y_0 y_1 \ldots y_i \ldots y_l$ of length $l \geq i$ such that $y_0 = a_0$, $y_l = a_i$, and the vertices $y_1 \ldots y_l$ are adjacent in G_k to neither b_2 nor b_3.

Let $s(H)$ $(1 \leq s(H) \leq r)$ be the smallest integer for which the set $\{b_1, b_2, b_3\}$ induces a triangle in $G_{s(H)}$. We choose H such that $s(H)$ is smallest possible and we put $s = s(H)$. If $s = 1$ (i.e., the vertices b_1, b_2 and b_3 induce a triangle already in G), then, thanks to P^1, G contains an induced subgraph isomorphic to Z_i, contradicting the hypothesis. Hence $s \geq 2$. This implies that $\{b_1 b_2, b_1 b_3, b_2 b_3\} \cap B_{x_{s-1}} \neq \emptyset$ (i.e., some of the edges of the triangle $\langle\{b_1, b_2, b_3\}\rangle_{G_s}$ has been added during the step from G_{s-1} to G_s). If both $b_1 b_2 \in E(G_{s-1})$ and $b_1 b_3 \in E(G_{s-1})$, then $b_2 b_3 \notin E(G_{s-1})$, which implies $\langle\{b_1, b_2, b_3, y_1\}\rangle_{G_{s-1}} \simeq C$ (where y_1 is the second vertex of the path P^{s-1}), a contradiction. By the symmetry, we can suppose that $b_1b_2 \notin E(G_{s-1})$, i.e. $b_1b_2 \in B_{x_{s-1}}$. This implies $b_1, b_2 \in N_{G_{s-1}}(x_{s-1})$ (note that x_{s-1} is possibly equal to b_3). Let $b_1z_1 \dots z_pb_2$ be a shortest (b_1, b_2)-path in $\langle N_{G_{s-1}}(x_{s-1})\rangle_{G_{s-1}}$ (such a path exists since $x_{s-1} \in V_{EL}(G_{s-1})$). The vertex z_1 is eligible in G_{s-1} since z_1 has two nonadjacent neighbors b_1 and z_2 or b_2 lying in the same component of its neighborhood. By Remark 3 of Section 1, the neighborhoods of x_{s-1} and of z_1 in $\mathrm{cl}(G)$ are cliques containing b_1 and b_2. Hence, and since H is a permanent Z_1, x_{s-1} and z_1 are neither equal nor adjacent in $\mathrm{cl}(G)$, and a fortiori in G_r, to any a_j, $1 \leq j \leq i$. Therefore the graph $H' = \langle\{x_{s-1}, z_1, b_1, a_1, a_2, \cdots, a_i\}\rangle_{G_r}$ is isomorphic to a permanent Z_1. Since $s(H') \leq s-1$, we get a contradiction to the choice of s.

Therefore $\mathrm{cl}(G)$ is also Z_i-free. ■

Before proving Theorems 7 – 9, we first introduce some additional notation that will be kept throughout the rest of the paper.

Let G be a closed 2-connected nonhamiltonian claw-free graph and let (by Theorem F) $H = P_{x_1,x_2,x_3} \in \mathcal{P}$ be an induced subgraph of G. Recall that we keep the notation of vertices a_i, b_i, c_i^j as in Fig. 2. We denote by:

- K_a the largest clique in G containing the triangle $\langle\{a_1, a_2, a_3\}\rangle_G$,
- K_b the largest clique in G containing the triangle $\langle\{b_1, b_2, b_3\}\rangle_G$,
- for every $i \in \{1, 2, 3\}$ for which $x_i = T$, by K_i the largest clique in G containing the triangle $\langle\{a_i, c_i, b_i\}\rangle_G$,
- for every $i \in \{1, 2, 3\}$ for which $x_i \neq T$, by K_i the path $a_ic_i^1c_i^2 \dots c_i^{k_i-2}b_i$ and by K_i^j $(j = 1, \dots, k_i - 1)$ the largest clique in G containing the j-th edge of the path K_i,
- for every $i \in \{1, 2, 3\}$, $K_i^* = K_i$ if $x_i = T$, and $K_i^* = \langle \cup_{j=1}^{k_i-1} V(K_i^j)\rangle_G$, if $x_i \neq T$,
- $H^* = \langle V(K_a) \cup V(K_b) \cup (\cup_{i=1}^3 V(K_i^*))\rangle_G$.

Note that since G is closed, all these sets are well-defined.

The following lemma summarizes basic properties of H^*.

Lemma 12. *Let G be a closed 2-connected nonhamiltonian claw-free graph and let $H \in \mathcal{P}$ be an induced subgraph of G. Then the graph H^* has the following properties.*

(i) $|V(A_1) \cap V(A_2)| \leq 1$ *for every* $A_1, A_2 \in \{K_a, K_b\} \cup \{K_i |\ x_i = T\} \cup \{K_i^j |\ x_i \neq T, 1 \leq j \leq x_i - 1\}, A_1 \neq A_2$,

(ii) if $x_i = T$ *for some* $i \in \{1, 2, 3\}$, *then* $V(K_i) \cap V(A) = \emptyset$ *for every* $A \in \{K_j |\ x_j = T\} \cup \{K_j^1 |\ x_j \neq T\} \cup \{K_j^{x_j-1} |\ x_j \neq T\}$, $A \neq K_i$,

(iii) if $x_i \neq T$ *for some* $i \in \{1, 2, 3\}$, *then* $V(K_i^1) \cap V(K_j^1) = \emptyset$ *and* $V(K_i^{x_i-1}) \cap V(K_j^{x_j-1}) = \emptyset$ *for every* $j \in \{1, 2, 3\}$ *such that* $j \neq i$ *and* $x_j \neq T$,

(iv) if $x_i = T$ *for at least one* $i \in \{1, 2, 3\}$, *then* $V(K_a) \cap V(K_b) = \emptyset$,

(v) $a_i, b_i, c_i^\ell \in V_{LD}(G)$ *for* $1 \le \ell \le k_i - 2$ *and* $i = 1, 2, 3$,
(vi) $N_G(a_i) \subset V(K_a) \cup V(K_i^*)$, $N_G(b_i) \subset V(K_b) \cup V(K_i^*)$, $N_G(c_i^\ell) = V(K_i^\ell) \cup V(K_i^{\ell+1})$ *for* $1 \le \ell \le k_i - 2$ *and* $i = 1, 2, 3$.

Proof follows immediately from the claw-freeness of G and from the properties of the closure operation. ■

Proof of Theorem 7. By Theorem K and by Corollary 2, it is sufficient to prove that if G is a closed CP_7-free graph, then $G \in \mathcal{F}_1$. Let thus G be a closed CP_7-free nonhamiltonian graph. By Theorem F, G contains an induced subgraph $H \in \mathcal{P}$. It is straightforward to check that the only P_7-free graph in the class $\mathcal{P}$ is the graph $P_{T,T,T}$; hence $H = P_{T,T,T}$ and $K_i^* = K_i$, $i = 1, 2, 3$. Recall that by Lemma 12(ii), the cliques K_i are pairwise disjoint, and by Lemma 12(iv), $V(K_a) \cap V(K_b) = \emptyset$.

<u>Claim 1.</u> There is no edge $y_i y_j \in E(G)$ with $y_i \in V(K_i) \setminus \{a_i, b_i\}$, $y_j \in V(K_j) \setminus \{a_j, b_j\}$, $i \ne j$, $i, j \in \{1, 2, 3\}$.

<u>Proof of Claim 1.</u> Let, to the contrary, e.g. $y_1 y_2 \in E(G)$ with $y_1 \in V(K_1)$, $y_2 \in V(K_2)$ (other cases are symmetric). By Lemma 12(vi), $y_1, y_2 \notin V(K_a) \cup V(K_b)$. If $|V(K_a)| > 3$, then, for some $d \in V(K_a) \setminus \{a_1, a_2, a_3\}$, $dy_1 \notin E(G)$ (since otherwise the triangle $\langle\{d, a_1, y_1\}\rangle$ contradicts $a_1 \in V_{LD}(G)$) and similarly $dy_2 \notin E(G)$ and $dc_3 \notin E(G)$, but then $\langle\{d, a_1, y_1, y_2, b_2, b_3, c_3\}\rangle_G \simeq P_7$ – a contradiction. Hence $|V(K_a)| = 3$ and, by symmetry, $|V(K_b)| = 3$.

We show that $V(H^*) = V(G)$. Let thus, to the contrary, $z \in V(G) \setminus V(H^*)$ have a neighbor in $V(H^*)$. Since $|V(K_a)| = |V(K_b)| = 3$ and by Lemma 12(vi), $N_G(z) \cap (V(K_a) \cup V(K_b)) = \emptyset$.

If $zy_1 \in E(G)$, then from $\langle\{y_1, z, a_1, y_2\}\rangle_G \not\simeq C$ we get $zy_2 \in E(G)$ (since we already know that a_1 has no neighbors outside K_1 and K_a). Since a_1 is not adjacent to y_2, $y_1 \in V_{LD}(G)$ and thus z has no other neighbor in K_1. If $|V(K_1)| > 3$, then for some $d \in V(K_1) \setminus \{a_1, b_1, y_1\}$, $\langle\{d, b_1, b_3, a_3, a_2, y_2, z\}\rangle_G \simeq P_7$; hence $|V(K_1)| = 3$ and, by symmetry, also $|V(K_2)| = 3$. This implies $y_1 = c_1$ and $y_2 = c_2$, contradicting the fact that $c_1 c_2 \notin E(G)$. Hence $zy_1 \notin E(G)$ and, by symmetry, also $zy_2 \notin E(G)$.

Now, if $zd_3 \in E(G)$ for some $d_3 \in V(K_3)$, then (since obviously $d_3 \notin \{a_3, b_3\}$, $\langle\{z, d_3, a_3, a_2, y_2, y_1, b_1\}\rangle_G \not\simeq P_7$; hence $N(z) \cap V(K_3) = \emptyset$. Consequently, if $zd_1 \in E(G)$ for some $d_1 \in V(K_1)$, then $d_1 \notin \{a_1, b_1, y_1\}$ and $\langle\{z, d_1, y_1, y_2, a_2, a_3, c_3\}\rangle_G \simeq P_7$. This contradiction proves that there is no vertex $z \in V(G) \setminus V(H^*)$ and thus $V(H^*) = V(G)$.

Let P_1 (or P_2, or P_3) be a hamiltonian path in K_1 (or K_2, or K_3) with endvertices a_1, y_1 (or y_2, b_2, or b_3, a_3), respectively. Then $C = a_1 P_1 y_1 y_2 P_2 b_2 b_3 P_3 a_3 a_1$ is a hamiltonian cycle in $H^* = G$. This contradiction proves Claim 1. □

<u>Claim 2.</u> Every vertex $z \in V(G) \setminus V(H^*)$ satisfies $N_G(z) \cap (V(K_1) \cup V(K_2) \cup V(K_3)) = \emptyset$.

<u>Proof of Claim 2.</u> Let, to the contrary, $zc \in E(G)$ with $z \in V(G) \setminus V(H^*)$ and $c \in \cup_{i=1}^3 V(K_i)$. By symmetry, we can suppose that $c \in V(K_1)$, and obviously $c \notin \{a_1, b_1\}$. Since $\langle\{z, c, a_1, a_2, b_2, b_3, c_3\}\rangle_G \not\simeq P_7$ and, by Claim 1, $cc_3 \notin E(G)$, $zc_3 \in E(G)$. Similarly

we have $zc_2 \in E(G)$, since otherwise $\langle\{z, c, a_1, a_3, b_3, b_2, c_2\}\rangle_G \simeq P_7$. But then, by Claim 1, $\langle\{z, c, c_2, c_3\}\rangle_G \simeq C$, a contradiction. $\square$

Suppose now that there is a vertex $d \in V(K_a)$, having a neighbor in $V(G) \setminus V(H^*)$. Since G is closed, $d \in V_{LD}(G)$, and since G is claw-free, $d \notin \{a_1, a_2, a_3\}$. We can thus denote by K_d the clique containing d and all neighbors of d outside H^*. Let $y \in V(K_d) \setminus V(H^*)$. Since $d \in V_{LD}(G)$, $N_G(y) \cap (V(K_a) \setminus \{d\}) = \emptyset$. By Claim 2 also $N_G(y) \cap (\cup_{i=1}^3 V(K_i)) = \emptyset$. If y has a neighbor $z \in V(K_b)$, then $dz \notin E(G)$ (otherwise $z \in V(K_d)$), but then $\langle\{c_1, a_1, d, y, z, b_2, c_2\}\rangle_G \simeq P_7$. Finally, if y has a neighbor $z \in V(G) \setminus (V(H^*) \cup V(K_d))$, then $\langle\{z, y, d, a_1, b_1, b_2, c_2\}\rangle_G \simeq P_7$ (recall that $zc_2 \notin E(G)$ and $yc_2 \notin E(G)$ by Claim 2). Hence no vertex in $V(K_d) \setminus V(H^*)$ has a neighbor outside K_d.

Since G is 2-connected, d is not a cutvertex. Thus some other vertex of K_d except d (say, y) belongs to H^*. Since there is no edge between $K_a \setminus \{a_i\}$ and $K_i \setminus \{a_i\}$, $y \notin \cup_{i=1}^3 (V(K_i) \setminus \{a_i\})$. Since $d \in V_{LD}(G)$, $y \notin V(K_a)$. Hence $y \in V(K_b)$ and, since G is closed, $V(K_d) \cap V(K_b) = \{y\}$.

We have thus proved that every vertex $x \in V(G) \setminus V(H^*)$ is contained in a clique K_x such that $|V(K_x) \cap V(K_a)| = |V(K_x \cap V(K_b)| = 1$, i.e., there are cliques $K_1, \ldots, K_k$ such that $V(K_i) \cap V(K_j) = \emptyset$ for $i \neq j$, $|V(K_i) \cap V(K_a)| = |V(K_i \cap V(K_b)| = 1$ and $V(G) = V(K_a) \cup V(K_b) \cup (\cup_{i=1}^k V(K_i))$. It is straightforward to check that if G contains any edge having vertices in two different cliques, then G is hamiltonian. Similarly, since G is nonhamiltonian, k is odd. Thus, $G \in \mathcal{F}_1$. ∎

Proof of Theorem 8. By Theorems K and 3, it is sufficient to prove that if G is a closed CZ_4-free nonhamiltonian graph, then $G \in \{P_{T,T,T}, P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\} \cup \mathcal{F}_2$. Let thus G be a closed CZ_4-free nonhamiltonian graph. By Theorem F, G contains an induced subgraph $H \in \mathcal{P}$ and we can easily check that the only Z_4-free graphs in the class $\mathcal{P}$ are the graphs $P_{T,T,T}$, $P_{3,T,T}$, $P_{3,3,T}$, $P_{3,3,3}$ and $P_{4,T,T}$. When $K_i \simeq P_3$ we often denote c_i^1 by c_i.

<u>Claim 1.</u> If $a_ib_i \in E(G)$ for some $i \in \{1, 2, 3\}$, then $|V(K_i)| = 3$.

<u>Proof of Claim 1.</u> Let e.g. $a_3b_3 \in E(G)$ and $|V(K_3)| \geq 4$, and let $c_3, d_3 \in V(K_3) \setminus \{a_3, b_3\}$. Then $\langle\{b_3, c_3, d_3, b_2, a_2, a_1, c_1\}\rangle_G$ (if $a_2b_2 \in E(G)$) or $\langle\{b_3, c_3, d_3, b_2, c_2, a_2, a_1\}\rangle_G$ (if $a_2b_2 \notin E(G)$) is an induced Z_4. $\square$

<u>Case 1:</u> $H \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$

Let K_1 be a path P_4 if $H \simeq P_{4,T,T}$ and P_3 otherwise, and when $H \not\simeq P_{3,3,3}$ let K_3 be a triangle.

<u>Claim 2.</u> $|V(K_a)| = |V(K_b)| = 3$.

<u>Proof of Claim 2.</u> By symmetry, let e.g. $d \in V(K_a) \setminus \{a_1, a_2, a_3\}$. Then the graph $\langle\{a_1, a_2, d, c_1^1, c_1^2, b_1, b_3\}\rangle_G$ (if $K_1 \simeq P_4$) or $\langle\{a_1, a_2, d, c_1, b_1, b_3, c_3\}\rangle_G$ (if $K_1 \simeq P_3$) is an induced Z_4, unless $d \in V(K_b)$ (and then, by Lemma 12(iv), $x_1 = x_2 = x_3 = 3$, i.e.,

$H = P_{3,3,3}$), or $d \in V(K_1^{x_1-1})$ (and then, by the closure property, $x_1 = 4$ and thus $x_2 = x_3 = T$, i.e., $H = P_{4,T,T}$). We now consider these two subcases separately.

Let first $H = P_{3,3,3}$ and $d \in V(K_b)$. By Lemma 12(i) and by the symmetry, $|V(K_a)| = |V(K_b)| = 4$ and $V(K_a) \cap V(K_b) = \{d\}$. Evidently $d \in V_{LD}(G)$ and hence $N_G(d) \subset V(K_a) \cup V(K_b)$. We show that $V(G) = V(H^*)$. Let, to the contrary, $u \in V(G) \setminus V(H^*)$. By the connectivity, by Lemma 12(vi) and by the symmetry, we can suppose that $uv \in E(G)$ for some $v \in V(K_3^1) \setminus \{a_3, c_3\}$, but then $\langle\{d, a_1, a_2, b_3, c_3, v, u\}\rangle_G \simeq Z_4$. We thus have $V(G) = V(H^*)$, but then it is straightforward to check that G is hamiltonian. This contradiction shows that $d \notin V(K_b)$, i.e., $V(K_a) \cap V(K_b) = \emptyset$.

Let thus $H = P_{4,T,T}$ and $d \in V(K_1^3)$. Then $|V(K_1^3)| = 3$, for if there is a $z \in V(K_1^3) \setminus \{b_1, c_1^2, d\}$, then $\langle\{c_1^2, b_1, z, c_1^1, a_1, a_2, c_2\}\rangle_G \simeq Z_4$. We again show that $V(G) = V(H^*)$. Let thus $u \in V(G) \setminus V(H^*)$ and let $v \in V(H^*)$ be adjacent to u. Evidently $v \notin V(K_1^3)$ (since $|V(K_1^3)| = 3$ and, by the closure property, v cannot be any of d, c_1^2, b_1), $v \notin V(K_a)$ (since $|V(K_a)| = 4$ and v cannot be any of a_1, a_2, a_3, d) and, by the symmetry, $v \notin V(K_b)$. Hence $v \in V(K_1^1) \cup V(K_1^2) \cup V(K_2) \cup V(K_3)$.

If $v \in V(K_3)$, then, by Claim 1, $v = c_3$ and $\langle\{a_2, a_1, d, b_2, b_3, c_3, u\}\rangle_G \simeq Z_4$; hence $v \notin V(K_3)$. By symmetry, $v \notin V(K_2)$. Also easily $v \notin V(K_1^1)$, since otherwise $\langle\{b_1, b_2, b_3, c_1^2, c_1^1, v, u\}\rangle_G \simeq Z_4$. Finally, let $v \in V(K_1^2)$. Then $v \notin \{c_1^1, c_1^2\}$ (clearly), $va_2 \notin E(G)$ (since otherwise $\langle\{v, c_1^1, a_2, u\}\rangle_G \simeq C$) and similarly $vb_2 \notin E(G)$, but then $\langle\{b_2, b_1, b_3, a_2, a_1, c_1^1, v\}\rangle_G \simeq Z_4$. Hence $V(G) = V(H^*)$, implying that G is hamiltonian. This contradiction proves that $|V(K_a)| = 3$ and, by symmetry, $|V(K_b)| = 3$. □

Claim 3. If $a_i b_i \notin E(G)$, then $|V(K_i^j)| = 2$ for $1 \leq j \leq k_i - 1$ (i.e., the interior vertices of the path K_i have no neighbors outside K_i).

Proof of Claim 3. By symmetry, it is sufficient to suppose that there is a vertex $y \in V(G) \setminus V(K_1)$ such that $yc_1^1 \in E(G)$. By Lemma 12(vi), $y \in V(H^*)$ and thus if $ya_1 \notin E(G)$ then $yc_1^2 \in E(G)$ when $K_1 \simeq P_4$ and $yb_1 \in E(G)$ when $K_1 \simeq P_3$. Suppose first that y is adjacent to c_1^1 and a_1 (and thus, by the closure property, neither to c_1^2, when $K_1 \simeq P_4$, nor to b_1). By Claim 1 and by the closure property, y is adjacent to no vertex of $V(H) \setminus \{a_1, c_1^1\}$, except perhaps c_2 or c_3 in the case when K_2 or K_3 is a triangle. Since the subgraph H is induced, the set $\{c_1^1, c_2, c_3\}$ is independent and thus, since G is claw-free, y cannot be adjacent to both c_2 and c_3. We can thus suppose that $yc_2 \notin E(G)$ (if both K_2 and K_3 is a triangle). Then $\langle\{a_1, c_1^1, y, a_3, b_3, b_2, c_2\}\rangle_G$ (when $H \simeq P_{3,T,T}$ or $P_{4,T,T}$), or $\langle\{a_1, c_1^1, y, a_3, b_3, b_2, c_2^1\}\rangle_G$ (when $H \simeq P_{3,3,T}$), or $\langle\{a_1, c_1^1, y, a_3, c_3^1, b_3, b_2\}\rangle_G$ (when $H \simeq P_{3,3,3}$) is isomorphic to Z_4 – a contradiction. The cases when $K_1 \simeq P_3$ and y is adjacent to c_1^1 and to b_1, and $H \simeq P_{4,T,T}$ and when y is adjacent to c_1^2 and to b_1 are symmetrical.

Therefore it remains to consider the case when $H \simeq P_{4,T,T}$ and $yc_1^1 \in E(G)$, $yc_1^2 \in E(G)$ but $ya_1 \notin E(G)$ and $yb_1 \notin E(G)$. Since H is induced, y is different from c_2 and c_3 and by Lemma 12(vi), y has no neighbor in $\{a_2, a_3, b_2, b_3\}$. Hence $\langle\{c_1^1, y, c_1^2, a_1, a_2, b_2, b_3\}\rangle_G \simeq Z_4$, a contradiction. □

Claim 4. $G = H^*$.

Proof of Claim 4. By Claims 1, 2 and 3, the only vertices of G possibly having a neighbor

y not in H^* are c_2 and c_3, in the case where K_2 and K_3 are triangles. Since G is 2-connected, y is in a bicomponent B with biarticulation $\{c_2, c_3\}$ (by claw-freeness, such a bicomponent can be only one). Suppose that B contains a triangle and let T be a triangle in B whose distance from c_2 is minimum. Consider a shortest path P in B (possibly trivial) joining T with c_2. Then the graph $\langle V(T) \cup V(P) \cup \{a_2\} \cup V(K_1)\}\rangle_G$ contains an induced Z_4. Hence B is triangle-free and, by claw-freeness, B is a path. But then, since $G = \langle V(H^*) \cup V(B)\}\rangle_G$, G is hamiltonian. Therefore no such vertex y exists. □

Now, since (by Claims 1, 2, 3) $H^* = H$, we have $G = H \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$.

Case 2: $H = P_{T,T,T}$

If G contains an induced subgraph $H' \in \mathcal{P}$, $H' \not\simeq P_{T,T,T}$, then, by Case 1, $G \in \{P_{3,T,T}, P_{3,3,T}, P_{3,3,3}, P_{4,T,T}\}$, a contradiction. Hence every induced subgraph H' of G that belongs to $\mathcal{P}$ is isomorphic to $P_{T,T,T}$.

Claim 5. There is a sequence of cliques $K_1, \ldots, K_k$, $k \geq 3$, such that
(i) $V(K_i) \cap V(K_a) = \{a_i\}$, $V(K_i) \cap V(K_b) = \{b_i\}$, $i = 1, \ldots, k$,
(ii) $|V(K_i)| = 3$ for $i = 1, 2, 3$ and $|V(K_i)| \leq 3$ for $i = 4, \ldots, k$,
(iii) $N_G(a_i) \subset V(K_i) \cup V(K_a)$, $N_G(b_i) \subset V(K_i) \cup V(K_b)$, $i = 1, \ldots, k$,
(iv) there is no (K_a, K_b)-path in $\langle V(G) \setminus (\cup_{i=1}^{k} V(K_i))\rangle$.

Proof of Claim 5. If there is no (K_a, K_b)-path in $\langle\{V(G) \setminus (\cup_{i=1}^{3} V(K_i))\}\rangle$, put $k = 3$. Otherwise, let $P = y_0 y_1 \ldots y_\ell$ ($y_0 \in V(K_a), y_\ell \in V(K_b), \ell \geq 1$) be a shortest such path.

Suppose first that some of the vertices $y_0, \ldots, y_{\ell-1}$ is adjacent to some of the vertices c_1, c_2, c_3 (say, c_3) and let y_i be the first such vertex. Then, since P is shortest and by claw-freeness, we have also $y_{i+1}c_3 \in E(G)$. By the properties of the closure, $y_0 c_3 \notin E(G)$ and $y_\ell c_3 \notin E(G)$ (otherwise $c_3 \in V(K_a)$ or $c_3 \in V(K_b)$, respectively), but then $\langle\{y_i, y_{i+1}, c_3, y_{i-1}, \ldots, y_0, a_1, b_1, b_2\}\rangle$ contains an induced Z_4. Hence no inner vertex of P is adjacent to any c_j, but then, if $\ell \geq 2$, $\langle\{y_0, a_1, a_2, y_1, \ldots, y_\ell, b_3, c_3\}\rangle$ contains an induced Z_4. Hence $\ell = 1$ and P is an edge.

Denote $y_0 = a_4$, $y_1 = b_4$ and let $K_4 = \langle\{x \in V(G) \setminus V(H^*)|\ N(x) \cap \{a_4, b_4\} \neq \emptyset\} \cup \{a_4, b_4\}\rangle$. By the properties of the closure, K_4 is a clique, containing all neighbors of a_4 and b_4 outside K_a and K_b. If $|V(K_4)| \geq 4$, then, for some $c_4^1, c_4^2 \in V(K_4) \setminus \{a_4, b_4\}$, some of the vertices c_1, c_2, c_3 (say, c_3) is nonadjacent to both c_4^1 and c_4^2 (otherwise we have an induced claw centered at c_4^1 or at c_4^2), but then $\langle\{b_4, c_4^1, c_4^2, b_1, a_1, a_3, c_3\}\rangle \simeq Z_4$. Hence $|V(K_4)| \leq 3$.

Repeating this argument, we obtain a sequence of cliques $K_1, \ldots, K_k$ with the required properties. □

We put $H^{**} = \langle V(H^*) \cup V(K_4) \cup \ldots \cup V(K_k)\rangle$ and, if $|V(K_i)| = 3$, ($i \geq 4$), we denote the (only) vertex in $V(K_i) \setminus \{a_i, b_i\}$ by c_i.

Claim 6. Every nontrivial component of the graph $\overline{H} = \langle V(G) \setminus (V(K_a) \cup V(K_b))\rangle_G$ is a path.

Proof of Claim 6. Let B be a nontrivial component of the graph $\overline{H}$ and let $V(B) \cap V(H^{**}) = \{c_{j_1}, c_{j_2}, \ldots, c_{j_p}\} \subset \cup_{i=1}^{k} V(K_i) \setminus (V(K_a) \cup V(K_b))$. If B is not a path, then, since G is 2-connected, $p \geq 2$, and since G is claw-free, B contains a triangle. If some triangle T of B contains at least one vertex c_{j_ℓ}, $1 \leq \ell \leq p$, then since the subgraph H is induced, at most one vertex of $\{c_1, c_2, c_3\}$ belongs to T, say c_1 and c_2 are not in T. In this case $\langle V(T) \cup \{a_{j_\ell}, a_1, b_1, b_2\}\rangle$ contains an induced Z_4. Otherwise, let T be a triangle of B whose distance to c_{j_1} is minimum and let $P = y_0 y_1 \cdots y_\ell$ with $y_0 = c_{j_1}$, $y_\ell \in V(T)$ and $l \geq 1$ be a shortest path between c_{j_1} and T. Then $\langle V(T) \cup V(P) \cup \{a_{j_1}, a_{j_2}, b_{j_2}\}\rangle$ contains an induced Z_4. Hence B contains no triangle and thus B is a path joining two vertices c_{j_1} and c_{j_2}. $\square$

Claim 7. $G = H^{**}$.

Proof of Claim 7. Suppose that $V(G) \setminus V(H^{**}) \neq \emptyset$. Then, by Claim 5($iv$), by 2-connectedness and by symmetry, we can distinguish the following subcases.

Subcase a. There is a (K_b, K_b)-path P^1 with interior vertices outside H^{**}.

Subcase b. There is a (K_b, c_3)-path P^2 with interior vertices outside H^{**}.

Subcase c. Vertices in K_a and K_b have no neighbors outside H^{**} and there is a bicomponent B of G which is a (c_1, c_2)-path P^3 with interior vertices outside H^{**}.

Subcase a. Choose P^1 shortest possible and denote $P = d_1 x_1 \ldots x_\ell d_2$. Clearly $d_i \neq b_j$, $i = 1, 2$, $j = 1, \ldots, k$. By the properties of the closure, $\ell \geq 2$, but then $\langle\{a_1, a_2, a_3, b_1, d_1, x_1, x_2\}\rangle_G \simeq Z_4$.

Subcase b. Let again $P^2 = d_1 x_1 \ldots x_\ell c_3$ be shortest possible. By the properties of the closure, $\ell \geq 1$. If $\ell \geq 2$, then $\langle\{a_3, a_1, a_2, c_3, x_\ell, \ldots, x_1, d_1\}\rangle_G$ contains an induced Z_4. Thus $\ell = 1$. If $|V(K_a)| \geq 4$, then, for some $y \in V(K_a) \setminus \{a_1, a_2, a_3\}$, we have $\langle\{a_1, a_2, y, b_1, b_3, c_3, x_1\}\rangle_G \simeq Z_4$ (note that $yx_1 \notin E(G)$ since otherwise yx_1d_1 is a (K_a, K_b)-path and, by the construction of H^{**}, $yd_1 \in E(G)$ and $x_1 \in H^{**}$). Thus $|V(K_a)| = 3$.

If $|V(K_b)| \geq 5$, then, for some $y \in V(K_b) \setminus \{b_1, b_2, b_3, d_1\}$, $\langle\{b_1, b_2, y, a_1, a_3, c_3, x_1\}\rangle \simeq Z_4$ (recall that $x_1 y \notin E(G)$ by the properties of the closure). Thus $V(K_b) = \{b_1, b_2, b_3, d_1\}$. If x_1 has another neighbor $z \in V(G) \setminus V(H^{**})$, then, since $\langle\{x_1, c_3, d_1, z\}\rangle \not\simeq C$, we have $zc_3 \in E(G)$ or $zd_1 \in E(G)$, which implies $\langle\{x_1, c_3, z, d_1, b_2, a_2, a_1\}\rangle \simeq Z_4$ or $\langle\{x_1, z, d_1, c_3, a_3, a_1, c_1\}\rangle \simeq Z_4$. Therefore x_1 has no neighbors outside H^{**}, and thus, by the closure property, also both c_3 and d_1 have no neighbors outside H^{**}. Now, the only vertices which can have a neighbor outside $\langle V(H^{**}) \cup \{x_1\}\rangle$, are c_1 and c_2, Since e.g. $a_1 c_1 b_1 d_1 x_1 c_3 a_3 b_3 b_2 c_2 a_2 a_1$ is a hamiltonian cycle in $\langle V(H^{**}) \cup \{x_1\}\rangle$, there is a bicomponent B with biarticulation $\{c_1, c_2\}$ and with $V(B) \setminus (V(H^{**}) \cup \{x_1\}) \neq \emptyset$ (recall that, by claw-freeness, such a bicomponent can be only one and that this also implies that $x_1 c_1 \notin E(G)$ and $x_1 c_2 \notin E(G)$). By Claim 6, B is a path, which implies that $G = \langle V(H^{**}) \cup V(B) \cup \{x_1\}\rangle$ and it is straightforward to check that G is hamiltonian, which is a contradiction.

Subcase c. Let $c_1x_1\cdots x_\ell c_2$ be the path P^3 with $\ell \geq 1$. If $|V(K_a)| \geq 4$, let $d \in V(K_a) \setminus \{a_1,a_2,a_3\}$. Then $\langle\{a_3,d,a_2,b_3,b_1,c_1,x_1\}\rangle \simeq Z_4$. Therefore $V(K_a) = \{a_1,a_2,a_3\}$ and similarly $V(K_b) = \{b_1,b_2,b_3\}$. By the claw-freeness of G, there are no paths outside H^* between c_1 and c_3 or between c_2 and c_3. Hence $V(G) = V(H^*)$ and G is hamiltonian.

Thus there is no vertex $x \in V(G) \setminus V(H^{**})$ and hence $G = H^{**}$. This completes the proof of Claim 7. □

Since G is nonhamiltonian, necessarily k is odd, $|V(K_i)| = 3$ for every $i = 1,\ldots,k$ and $\{c_1,\ldots,c_k\}$ is an independent set. Hence $G \in \mathcal{F}_2$. ■

Proof of Theorem 9. First observe that, by Theorem K and by Corollary 5, it is sufficient to prove that every closed 2-connected nonhamiltonian $CN_{1,2,2}N_{1,1,3}$-free graph is either isomorphic to $P_{3,3,3}$ or is in $\mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$.

Let thus G be a closed 2-connected nonhamiltonian $CN_{1,2,2}N_{1,1,3}$-free graph and $H \in \mathcal{P}$ an induced subgraph of G. Immediately $H \in \{P_{T,T,T}, P_{3,T,T}, P_{3,3,T}, P_{3,3,3}\}$ (since otherwise H contains an induced $N_{1,1,3}$). We choose the notation such that $K_1 \simeq P_3$ if $H \neq P_{T,T,T}$ and K_3 is a triangle if $H \neq P_{3,3,3}$, and we often denote $c_i^1 = c_i$.

Claim 1. If $H \neq P_{T,T,T}$, then $|V(K_a)| = |V(K_b)| = 3$.

Proof of Claim 1. Let, to the contrary, e.g. $a_1b_1 \notin E(G)$ and $y \in V(K_a) \setminus \{a_1,a_2,a_3\}$ (other cases are symmetric). Then, since $\langle\{b_1,b_2,b_3,c_2,c_3,c_1,a_1,y\}\rangle_G \not\simeq N_{1,1,3}$, $yc_i \in E(G)$ for at least one i, $1 \leq i \leq 3$, contradicting Lemma 12(v). □

Claim 2. For any $z \in V(G) \setminus V(H^*)$, $N_G(z) \cap (V(K_a) \cup V(K_b)) = \emptyset$.

Proof of Claim 2. Let, to the contrary, $zy \in E(G)$ with $z \in V(G) \setminus V(H^*)$ and $y \in V(K_a) \cup V(K_b)$. By symmetry, we can suppose that $y \in V(K_a)$ and, by Lemma 12(vi), $y \in V(K_a) \setminus \{a_1,a_2,a_3\}$. By Claim 1, this implies $H = P_{T,T,T}$. If $|N_G(z) \cap \{c_1,c_2,c_3\}| \leq 1$ (say, $c_2z, c_3z \notin E(G)$), then $\langle\{b_1,b_2,b_3,c_2,c_3,a_1,y,z\}\rangle_G \simeq N_{1,1,3}$; hence $|N_G(z) \cap \{c_1,c_2,c_3\}| \geq 2$. By symmetry, let $c_1z \in E(G)$ and $c_2z \in E(G)$. Then, since $c_1c_2 \notin E(G)$ and $\langle\{z,y,c_1,c_2\}\rangle_G \not\simeq C$, we have $yc_1 \in E(G)$ or $yc_2 \in E(G)$, contradicting Lemma 12(v). □

Claim 3. $V(K_a) \cap V(K_b) = \emptyset$ and $V(K_i^*) \cap V(K_j^*) = \emptyset$ for $i,j \in \{1,2,3\}$, $i \neq j$.

Proof of Claim 3. $V(K_a) \cap V(K_b) = \emptyset$ immediately by Claim 1 and by Lemma 12(iv). Let thus $d \in V(K_i^*) \cap V(K_j^*)$ for some $i,j \in \{1,2,3\}$, $i \neq j$. By Lemma 12(ii),(iii) and by the symmetry we can without loss of generality suppose that $H = P_{3,3,T}$ or $H = P_{3,3,3}$ and that $d \in V(K_1^1) \cap V(K_2^2)$. We show that $V(G) = V(H^*)$. Let thus $u \in V(G) \setminus V(H^*)$ and let $v \in V(H^*)$ be adjacent to u. By Claim 1 and by Lemma 12(vi), $v \notin V(K_a) \cup V(K_b)$; since G is claw-free, $v \neq d$. Up to symmetry, it remains to consider the cases when $v \in V(K_1^1)$, $v \in V(K_1^2)$ and $v \in V(K_3^*)$. If $v \in V(K_1^1)$, then $\langle\{b_2,d,c_2,b_1,v,u,a_2,a_3\}\rangle_G \simeq N_{1,2,2}$. If $v \in V(K_3^*)$, then we can suppose that $va_3 \in E(G)$ (the case when $K_3 \simeq P_3$ and $vb_3 \in E(G)$ is symmetrical), and then $\langle\{d,c_1,a_1,c_2,b_1,a_3,v,u\}\rangle_G \simeq N_{1,1,3}$. Finally, if $v \in V(K_1^2)$, then $\langle\{d,c_1,a_1,b_2,v,u,a_3,c_3\}\rangle_G \simeq N_{1,2,2}$ (evidently $vc_3 \notin E(G)$ since otherwise $\langle\{v,u,c_1,c_3\}\rangle_G \simeq C$). Hence $V(G)=V(H^*)$, contradicting the fact that G is nonhamiltonian. □

Claim 4. There is no edge $y_iy_j \in E(G)$ with $y_i \in V(K_i^*)\setminus\{a_i,b_i\}$, $y_j \in V(K_j^*)\setminus\{a_j,b_j\}$, $1\le i<j\le 3$.

Proof of Claim 4. Suppose, to the contrary, that for some $i\ne j$ there is an edge $y_iy_j \in E(G)$ with $y_i \in V(K_i^*)\setminus\{a_i,b_i\}$ and $y_j \in V(K_j^*)\setminus\{a_j,b_j\}$, $1\le i<j\le 3$. By symmetry, we can suppose that $i=1$, $j=2$, and if $a_1b_1 \notin E(G)$, then $y_1 \in V(K_1^1)$.

First observe that if some vertex $y \in V(G)\setminus V(H^*)$ is adjacent to y_1, then, since $\langle\{y_1,y,a_1,y_2\}\rangle_G \not\simeq C$ and, by Lemma 12(vi), neither y nor y_2 is adjacent to a_1, we have $yy_2 \in E(G)$. Moreover, if y',y'' are two neighbors of y_1 in $V(G)\setminus V(H^*)$, then from $\langle\{y_1,y',y'',a_1\}\rangle_G \not\simeq C$ we get $y'y'' \in E(G)$. Hence, by symmetry, there is a clique K_y containing y_1,y_2 such that every vertex in $V(G)\setminus V(H^*)$ adjacent to y_1 or to y_2 is in K_y.

Put $H^{**}=\langle V(H^*)\cup V(K_y)\rangle_G$. We want to show that $V(H^{**})=V(G)$. Let thus $V(G)\setminus V(H^{**})\ne\emptyset$.

Case 1: There is a vertex $z \in V(G)\setminus V(H^{**})$ such that $N_G(z)\cap(V(K_1^*)\cup V(K_2^*))\ne\emptyset$.

By symmetry, we can suppose that z has a neighbor $u \in V(K_1^*)$ (note that $u\ne y_1$ since G is claw-free). Suppose first that $a_1b_1 \notin E(G)$. If $u \in V(K_1^1)$, then $y_2a_2 \in E(G)$ implies $\langle\{u,a_1,c_1,z,a_2,y_2,b_1,b_3\}\rangle_G \simeq N_{1,2,2}$ and $y_2b_2 \in E(G)$ implies $\langle\{u,a_1,c_1,z,a_3,b_1,b_2,y_2\}\rangle_G \simeq N_{1,1,3}$; the case $u \in V(K_1^2)$ is symmetric. Hence $a_1b_1 \in E(G)$ (and $K_1^*=K_1$). Now, again by symmetry, we can suppose that $y_2a_2 \in E(G)$ (i.e., $a_2b_2 \in E(G)$ or $y_2 \in K_2^1$). Let v be an arbitrary neighbor of b_3 in $V(K_3^*)\setminus\{a_3\}$. Since $\langle\{u,a_1,b_1,z,a_2,y_2,b_3,v\}\rangle_G \not\simeq N_{1,2,2}$, we get $zv \in E(G)$. Since G is closed and v is arbitrary, this implies that $v=c_3$, i.e., c_3 is the only neighbor of b_3 in $K_3^*\setminus\{a_3\}$ and $zc_3 \in E(G)$. Considering $\langle\{c_3,a_3,b_3,z\}\rangle_G \not\simeq C$ we now have $a_3b_3 \in E(G)$ and hence $|V(K_3^*)|=3$. Thus, every vertex $z \in V(G)\setminus V(H^{**})$, having a neighbor in $V(K_1^*)\cup V(K_2^*)$, must be adjacent to c_3. Let $K_z=\{z \in V(G)\setminus V(H^{**})|\ N_G(z)\cap(V(K_1^*)\cup V(K_2^*))\ne\emptyset\}$. If K_z contains two nonadjacent vertices z_1,z_2, then $\langle\{c_3,z_1,z_2,a_3\}\rangle_G \simeq C$; hence K_z is a clique.

Let $H^{***}=\langle V(H^{**})\cup V(K_z)\rangle_G$, suppose that there is a vertex $z' \in V(G)\setminus V(H^{***})$, having a neighbor z in $V(K_z)$ and let (by the definition of K_z) u be a neighbor of z in K_1. Then $\langle\{u,a_1,b_1,b_3,z,z',a_2,y_2\}\rangle_G \simeq N_{1,2,2}$. Hence no vertex outside H^{***} can have a neighbor in K_z. Thus, if $V(G)\setminus V(H^{***})\ne\emptyset$, then there is a vertex $w \in V(G)\setminus V(H^{***})$ such that $\emptyset\ne N_G(w)\cap V(H^{***})\subset V(K_y)$, but then, for any $y \in N_G(w)\cap V(K_y)$, $z \in V(K_z)$ and $u \in N_G(z)\cap V(K_1)$, $\langle\{y,y_1,y_2,w,u,z,a_2,a_3\}\rangle_G \simeq N_{1,2,2}$. Thus $V(G)=V(H^{***})$, but then it is straightforward to check that G is hamiltonian. This contradiction completes the proof in Case 1.

Case 2: No vertex in $V(G)\setminus V(H^{**})$ has a neighbor in $V(K_1^*)\cup V(K_2^*)$.

Let again $z \in V(G)\setminus V(H^{**})$ and $u \in N_G(z)\cap V(H^{**})$. Then, by Claim 2 and by the construction of K_y, $u \in V(K_3^*)\setminus\{a_3,b_3\}$ or $u \in V(K_y)\setminus\{y_1,y_2\}$. Suppose first that z has a neighbor $u \in V(K_3^*) \setminus \{a_3, b_3\}$. Recall that if $a_3b_3 \notin E(G)$, then $u \neq c_3$ (since otherwise $\langle\{c_3, z, a_3, b_3\}\rangle_G \simeq C$) and if $a_3b_3 \in E(G)$, then (since $y_1y_2 \in E(G)$ and H is an induced subgraph and since also $a_ib_i \in E(G)$ for $i = 1, 2$), we can (by symmetry) suppose that there is a vertex $v \in V(K_1^*)$ such that $vb_1 \in E(G)$ but $vy_2 \notin E(G)$.

We can distinguish, up to symmetry, the following 3 cases.

Case	Contradiction
$a_3b_3 \in E(G), y_2a_2 \in E(G)$	$\langle\{u, a_3, b_3, z, a_2, y_2, b_1, v\}\rangle_G \simeq N_{1,2,2}$
$a_3b_3 \notin E(G), a_3u \in E(G), y_2a_2 \in E(G)$	$\langle\{u, a_3, c_3, z, a_2, y_2, b_3, b_1\}\rangle_G \simeq N_{1,2,2}$
$a_3b_3 \notin E(G), a_3u \in E(G), y_2b_2 \in E(G)$	$\langle\{u, a_3, c_3, z, a_1, b_3, b_2, y_2\}\rangle_G \simeq N_{1,1,3}$

Hence $N_G(z) \subset V(K_y) \setminus \{y_1, y_2\}$. Let $u \in N_G(z) \cup V(K_y)$. We have, up to symmetry, the following 5 cases (recall that $a_ib_i \notin E(G)$ implies $y_i \neq c_i$, since otherwise $\langle\{c_i, a_i, b_i, y_{3-i}\}\rangle_G \simeq C$, $i = 1, 2$).

Case	Contradiction
$a_1b_1 \notin E(G), a_2b_2 \notin E(G); y_1a_1, y_2a_2 \in E(G)$	$\langle\{u, y_1, y_2, z, c_1, c_2, b_2, b_3\}\rangle_G \simeq N_{1,1,3}$
$a_1b_1 \notin E(G), a_2b_2 \notin E(G); y_1a_1, y_2b_2 \in E(G)$	$\langle\{u, y_1, y_2, z, c_1, c_2, a_2, a_3\}\rangle_G \simeq N_{1,1,3}$
$a_1b_1 \notin E(G), a_2b_2 \in E(G), y_1a_1 \in E(G)$	$\langle\{u, y_1, y_2, z, a_2, c_1, b_1, b_3\}\rangle_G \simeq N_{1,1,3}$
$a_1b_1 \in E(G), a_2b_2 \in E(G), a_3b_3 \notin E(G)$	$\langle\{u, y_1, y_2, z, a_1, b_2, b_3, c_3\}\rangle_G \simeq N_{1,1,3}$
$a_1b_1 \in E(G), a_2b_2 \in E(G), a_3b_3 \in E(G)$	$\langle\{u, y_1, y_2, z, c_1, b_2, b_3, a_3\}\rangle_G \simeq N_{1,1,3}$

(in the last subcase we use the fact that, since H is induced, $y_1 \neq c_1$ or $y_2 \neq c_2$ and thus, by the symmetry, we can suppose that $y_1 \neq c_1$). Hence $V(G) = V(H^{**})$, implying that G is hamiltonian. This contradiction completes the proof of Claim 4. $\square$

Claim 5. If $z \in V(G) \setminus V(H^*)$ and $N_G(z) \cap V(K_i^*) \neq \emptyset$ for some $i \in \{1, 2, 3\}$, then $a_ib_i \in E(G)$.

Proof of Claim 5. Let, by symmetry, $a_1b_1 \notin E(G)$ and $uz \in E(G)$ with $u \in V(K_1^1)$. Obviously $u \neq c_1$ (otherwise $\langle\{c_1, z, a_1, b_1\}\rangle_G \simeq C$). If both $zc_2 \in E(G)$ and $zc_3 \in E(G)$, then, by Claim 4, $\langle\{z, u, c_2, c_3\}\rangle_G \simeq C$. By symmetry, we can suppose $zc_3 \notin E(G)$, but then $\langle\{u, a_1, c_1, z, a_2, b_1, b_3, c_3\}\rangle_G \simeq N_{1,1,3}$. $\square$

Claim 6. If $z \in V(G) \setminus V(H^*)$ and $N_G(z) \cap V(H^*) \neq \emptyset$, then $|\{i| \; N_G(z) \cap V(K_i^*) \neq \emptyset\}| = 2$.

Proof of Claim 6. If $|\{i| \; N_G(z) \cap V(K_i^*) \neq \emptyset\}| = 3$, then, for $u_i \in N_G(z) \cap V(K_i^*)$, $i = 1, 2, 3$, by Claim 4 we have $\langle\{z, u_1, u_2, u_3\}\rangle_G \simeq C$. If $|\{i| \; N_G(z) \cap V(K_i^*) \neq \emptyset\}| = 1$, then, by symmetry, we can suppose that $zu \in E(G)$ for some $u \in V(K_1^*)$; by Claim 5, we then get $a_1b_1 \in E(G)$, implying that $\langle\{u, a_1, b_1, z, a_2, c_2, b_3, c_3\}\rangle_G \simeq N_{1,2,2}$. $\square$

Claim 7. There is at most one vertex $z \in V(G) \setminus V(H^*)$ with $N_G(z) \cap V(H^*) \neq \emptyset$.

Proof of Claim 7. Suppose that $z_1, z_2 \in V(G) \setminus V(H^*)$, $z_1 \neq z_2$, have a neighbor in H^*. By Claim 6, by symmetry and by the pigeonhole principle, we can suppose without loss of generality that there are vertices $u_1^1 \in N_G(z_1) \cap V(K_1^*)$, $u_1^2 \in N_G(z_1) \cap V(K_2^*)$, $u_2^1 \in N_G(z_2) \cap V(K_1^*)$ and $u_2^2 \in N_G(z_2) \cap (V(K_2^*) \cup V(K_3^*))$. By Claim 5, $a_ib_i \in E(G)$ for $i = 1, 2$ and if $u_2^2 \in V(K_3^*)$, then also $a_3b_3 \in E(G)$. By Claim 2 and since G is closed,

z_1, z_2 have no other neighbors in H^*.

Suppose first that $z_1z_2 \notin E(G)$. This immediately implies that $u_1^1 \neq u_2^1$ (since otherwise $\langle\{u_1^1, z_1, z_2, a_1\}\rangle_G \simeq C$), but then $\langle\{u_1^1, u_2^1, a_1, z_1, z_2, a_2, b_2, b_3\}\rangle_G \simeq N_{1,1,3}$ (note that $z_1u_2^1 \notin E(G)$ and $z_2u_1^1 \notin E(G)$ since G is closed). Hence $z_1z_2 \in E(G)$.

If $u_1^1 \neq u_2^1$, then, since $z_2u_1^1 \notin E(G)$, $u_1^1u_1^2 \notin E(G)$ and $\langle\{z_1, z_2, u_1^1, u_1^2\}\rangle_G \not\simeq C$, we get $z_2u_1^2 \in E(G)$. By Claim 6 thus $u_2^2 \in V(K_2^*)$, but then $\langle\{a_1, b_1, u_1^1, a_2, b_3, c_3, z_1, z_2\}\rangle_G \simeq N_{1,2,2}$. Hence $u_1^1 = u_2^1$.

Now, if $u_2^2 = u_1^2$, then $u_1^1z_1u_1^2$ is a path in $\langle N_G(z_2)\rangle_G$ and the fact that G is closed implies $u_1^1u_1^2 \in E(G)$, contradicting Claim 4; if $u_2^2 \in V(K_2^*)$, $u_2^2 \neq u_1^2$, then we get $\langle\{a_2, u_1^2, b_2, a_1, z_1, z_2, b_3, c_3\}\rangle_G \simeq N_{1,2,2}$. Hence we have $u_2^2 \in V(K_3^*)$.

Put $H^{**} = \langle V(H^*) \cup \{z_1, z_2\}\rangle_G$. We want to show that $V(G) = V(H^{**})$. Suppose, to the contrary, that there is another vertex $z_3 \in V(G) \setminus V(H^{**})$ having a neighbor in H^{**}. If z_3 has no neighbor in H^*, then $z_3z_1 \in E(G)$ or $z_3z_2 \in E(G)$, but then $z_3z_1 \in E(G)$ and $\langle\{z_1, z_3, z_2, u_1^2\}\rangle_G \not\simeq C$ implies $z_3z_2 \in E(G)$ and, symmetrically, $z_3z_2 \in E(G)$ and $\langle\{z_2, z_3, z_1, u_2^2\}\rangle_G \not\simeq C$ implies $z_3z_1 \in E(G)$. Hence both $z_3z_1 \in E(G)$ and $z_3z_2 \in E(G)$, i.e., $u_1^1z_1z_3$ is a path in $\langle N_G(z_2)\rangle_G$. Since G is closed, this implies $z_3u_1^1 \in E(G)$, contradicting the assumption that z_3 has no neighbor in H^*.

Let thus u_3^1, u_3^2 be the neighbors of z_3 in H^*. Repeating the proof that $z_1z_2 \in E(G)$ for the pairs z_3, z_1 and z_3, z_2, we get $z_3z_1 \in E(G)$ and $z_3z_2 \in E(G)$. But then again $u_1^1z_1z_3$ is a path in $\langle N_G(Z_2)\rangle_G$, implying (since G is closed) that $u_3^1 = u_1^1 (= u_2^1)$. By symmetry, we can suppose that $u_3^2 \in V(K_2^*)$; since G is closed, we have $u_3^2 \neq u_1^2$. But then, since obviously $z_1u_3^2 \notin E(G)$, we get $\langle\{a_2, u_3^2, b_2, a_1, z_3, z_1, b_3, u_2^2\}\rangle_G \simeq N_{1,2,2}$. Hence $V(G) = V(H^{**})$.

It is straightforward to check that H^{**} (and hence also G) is hamiltonian. This contradiction proves Claim 7. $\square$

<u>Claim 8.</u> $G = H^*$.

<u>Proof of Claim 8.</u> We first show that $V(H^*) = V(G)$. Let, to the contrary, $V(G) \setminus V(H^*) \neq \emptyset$. By Claim 7 and by the connectedness of G, there is exactly one vertex $z \in V(G) \setminus V(H^*)$ with $N_G(z) \cap V(H^*) \neq \emptyset$. By Claims 2 and 6 and by the symmetry, we can suppose that $y_i \in N_G(z) \cap (V(K_i^*) \setminus \{a_i, b_i\})$ for $i = 1, 2$ and $N_G(z) \cap (V(K_3^*) = \emptyset$. By Claim 5, $a_1b_1 \in E(G)$ and $a_2b_2 \in E(G)$. Since G is closed, z has no other neighbors in H^*.

If $V(G) \setminus (V(H^*) \cup \{z\}) \neq \emptyset$, then, by the connectedness of G, z has a neighbor w outside H^*, but then from $\langle\{z, w, y_1, y_2\}\rangle_G \not\simeq C$ we get $wy_1 \in E(G)$ or $wy_2 \in E(G)$, contradicting Claim 7. Hence $V(G) \setminus (V(H^*) \cup \{z\}) = \emptyset$, implying that G is hamiltonian. This contradiction proves that $V(G) = V(H^*)$.

Now it is straightforward to check that adding any edge to H^* contradicts Lemma 12(v), Lemma 12(vi), Claim 4 or (since G is closed) the fact that H is an induced subgraph of G. Hence $G = H^*$. $\square$

It remains to prove that $H^* \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \{P_{3,3,3}\}$.

- If $H = P_{T,T,T}$, then evidently $H^* \in \mathcal{F}_3$.

- If $H = P_{3,T,T}$, then, by Claim 1, $|V(K_a)| = |V(K_b)| = 3$. By symmetry, suppose that $a_1b_1 \notin E(G)$. If both $|V(K_1^1)| \geq 3$ and $|V(K_1^2)| \geq 3$, then, for some $d^i \in V(K_1^i) \setminus \{a_1, b_1, c_1\}$, $i = 1, 2,$, we get $\langle \{a_1, a_2, a_3, d^1, c_2, b_3, b_1, d^2\} \rangle_G \simeq N_{1,1,3}$. Hence either $|V(K_1^1)| = 2$ or $|V(K_1^2)| = 2$ and thus $H^* \in \mathcal{F}_4$.
- If $H = P_{3,3,T}$ or $H = P_{3,3,3}$, then again $|V(K_a)| = |V(K_b)| = 3$ and, by symmetry, we can suppose that $a_1b_1 \notin E(G)$ and $a_2b_2 \notin E(G)$. If e.g. $|V(K_1^1)| \geq 3$, then, for a $d \in V(K_1^1) \setminus \{a_1, c_1\}$, $\langle \{a_1, a_3, a_2, d, c_3, c_2, b_2, b_1\} \rangle_G \simeq N_{1,1,3}$. By symmetry, this proves that if $H = P_{3,3,T}$, then $H^* \in \mathcal{F}_5$, and if $H = P_{3,3,3}$, then $H^* = H$.

■

Proof of Corollary 10. Let G be a 2-connected nonhamiltonian CE-free graph. Since every CE-free graph is $CN_{1,2,2}N_{1,1,3}$-free, by Theorem 9 we have $G \simeq P_{3,3,3}$, or $\mathrm{cl}(G) \in \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5$. By Corollary 5, $\mathrm{cl}(G)$ is also CE-free and it is straightforward to check that neither $P_{3,3,3}$ nor any graph in $\mathcal{F}_4 \cup \mathcal{F}_5$ is E-free. Hence $\mathrm{cl}(G) \in \mathcal{F}_3$. Moreover, if $\mathrm{cl}(G) \in \mathcal{F}_3 \setminus \mathcal{F}_6$, then e.g. for a vertex $z \in V(K_a) \setminus \{a_1, a_2, a_3\}$ we have $\langle \{b_1, b_2, b_3, c_1, c_2, a_3, z\} \rangle_G \simeq E$. Hence $|V(K_a)| = 3$ and, by symmetry, $|V(K_b)| = 3$, i.e. $\mathrm{cl}(G) \in \mathcal{F}_6$. It remains to show that also $G \in \mathcal{F}_6$. By symmetry, it is sufficient to show that $\langle V(K_1) \rangle_G$ is a clique. Suppose thus that there is a vertex $z \in V(K_1)$ at distance 2 (in G) from b_1 and let $u \in N_G(z) \cap N_G(b_1)$. Then $\langle \{b_2, b_3, b_1, c_2, c_3, u, z\} \rangle_G \simeq E$ – a contradiction. Hence all vertices in $V(K_1)$ are adjacent to b_1 (and, by symmetry, also to a_1). Since G is claw-free, $\langle V(K_1) \rangle_G$ is a clique. ■

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