Induced $S(K_{1,3})$ and hamiltonian square

Mohamed El Kadi Abderrezzak
Evelyne Flandrin

L.R.I., URA 410 C.N.R.S.
Bât. 490, Université de Paris-sud
91405-Orsay cedex, FRANCE *

Zdeněk Ryjáček †

Department of Mathematics
University of West Bohemia
306 14 Pilsen
Czech Republic

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Abstract

We prove that the square of a connected graph such that every induced $S(K_{1,3})$ has at least three edges in a block of degree at most 2 is hamiltonian. We also show that the insertion, and, under certain conditions also deletion, of a block of degree 2 into (from) a connected graph does not change the hamiltonicity of its square.

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1 Introduction and notation

The graphs considered in this paper are undirected and simple. All concepts not defined in this paper can be found in [1].

If G is a graph, we denote by $V(G)$ the vertex set of G, by $E(G)$ the edge set of G. The neighborhood in G of a vertex u is denoted by $N(u)$. We denote the set $N(u) \cup \{u\}$ by $N[u]$. For $A \subseteq V(G)$, $< A >$ represents the subgraph of G induced by A.

The square of G, denoted G^2, is the graph with vertex set $V(G)$ in which two vertices are adjacent if their distance in G is one or two. The graph $S(K_{1,3})$ is the graph $K_{1,3}$ in which each edge is subdivided once.

A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property. The degree of a block B of a graph G, denoted by $d(B)$, is the number of cut vertices of G belonging to $V(B)$. A block of degree 1 is called an endblock of G.

The length of a path in G is the number of its edges. We will use the notation $P_3(u)$ (where $u \in V(G)$) for a path of length 2 in G having u as endvertex. For a connected subgraph H of G, and for any two vertices u and v in H, denote by uP_Hv a (n arbitrary) path connecting u and v with the internal vertices in H.

The notation $G = F_1xF_2$ means that x is a cut vertex of G and F_1, F_2 are two connected subgraphs of G such that $V(F_1) \cap V(F_2) = \{x\}$ and $V(F_1) \cup V(F_2) = V(G)$.

This work is motivated by the following result due to G. Hendry and W. Vogler:

Theorem 1 [2]. Every 1-connected $S(K_{1,3})-$free graph has a hamiltonian square.

We looked for weaker conditions still implying that the square of a 1-connected graph is hamiltonian. More precisely, instead of forbidding the existence of an induced $S(K_{1,3})$, we put on every induced $S(K_{1,3})$ certain conditions under which the square of the graph remains hamiltonian.

Theorem 2. If G is a connected graph such that every induced $S(K_{1,3})$ has at least three edges in a block of degree at most 2, then G^2 is hamiltonian.

The following result by Thomassen [4] is an immediate corollary of Theorem 2.

Theorem 3 [4]. If the block graph of G is a path, then G^2 is hamiltonian.

The following theorem shows that, under certain conditions, insertion or deletion of a part of G does not change the hamiltonicity (or nonhamiltonicity) of G^2.

Theorem 4. Let G_1 and G_2 be connected graphs with $|V(G_i)| \geq 2$, $c_i \in V(G_i)$, $i = 1, 2$ and let B be a block with $|V(B)| \geq 3$, b_1 and $b_2 \in V(B)$. Let $G = G_1(c_1 = c_2)G_2$ and $G' = G_1(c_1 = b_1)B(b_2 = c_2)G_2$.
(i) If G^2 is hamiltonian, then $(G')^2$ is hamiltonian.
(ii) If moreover, c_i is not a cutvertex of G_i and is contained in an endblock of G_i, $i = 1, 2$, then the converse is also true.

2 Proof of Theorem 2

Let us first mention the following result by H. Fleischner that we will use many times in the proofs.

<u>**Theorem 5 [3].**</u> Let y and z be arbitrarily chosen vertices of a 2−connected graph G. Then G^2 contains a hamiltonian cycle C such that the edges of C in y are in G and at least one of the edges of C in z is in G. If y and z are adjacent in G, then these are three different edges.

In the rest of this section, G is always a graph of connectivity one.

First we give some additionnal definitions.

Let x be a cut vertex of G, and H' be a component of $< G - x >$. Then the subgraph $H =< H' \cup \{x\} >$ is called a branch of G at x.

Let F be a connected subgraph of G and x some vertex of F. F is said to be nontrivial at x if it contains a $P_3(x)$ as a proper induced subgraph (i.e., F is trivial at x if $F = P_3(x)$ or $V(F) \subseteq N[x]$).

Now suppose that Theorem 2 is not true and choose a graph G having the following properties:

(i) G is connected and every induced $S(K_{1,3})$ in G has at least 3 edges in a block of degree at most 2,

(ii) G^2 is not hamiltonian,

(iii) $|V(G)|$ is minimal with respect to (i) and (ii).

<u>**Claim 1:**</u> Let F be a connected graph, $x \in V(F)$ and xyz a $P_3(x)$ such that y and z are not in $V(F)$. If $(Fx(yz))^2$ is hamiltonian then F^2 contains a hamiltonian path connecting x and some vertex $x' \in N(x)$.

<u>**Proof:**</u> Let $G = Fx(yz)$ and let C be a hamiltonian cycle of F^2. Since the only adjacencies of z in F^2 are x and y and $N_F(y) = \{x\}$, there exists necessarily some vertex $x' \in V(F) - \{x\}$ such that $C = xP_Fx'yzx$ where xP_Fx' is a hamiltonian path of F^2 between x and x' and consequently $x' \in N(x)$.

<u>**Claim 2:**</u> If an induced $H \simeq S(K_{1,3}) \subset G$ has at least three edges in a block B of degree at most two, then some three edges of H in B induce a path P_4.

<u>**Proof:**</u> immediate.

<u>**Claim 3:**</u> Let x be a cutvertex of G and F_1, F_2 two connected subgraphs of G such that $V(F_1) \cap V(F_2) = \{x\}$. Assume that F_2 is not trivial at x, i.e., F_2 contains an induced $P_3(x) = xyz$ as a proper induced sugraph. Then the graph $G' = F_1xyz$ also satisfies all the hypothesis of Theorem 2.

<u>**Proof:**</u> If not, there exists in G' some $S(K_{1,3})$ that has no connected part of order at least

4 in a block of degree at most 2. But if so, it was the same in G, since we neither created any new $S(K_{1,3})$ nor increased the degree of any block.

Proof of Theorem 2. By the assumptions, G^2 is not hamiltonian. Thus, by Theorem 1, G contains some $S(K_{1,3})$ as an induced subgraph. By (i), the $S(K_{1,3})$ has at least 3 edges in some block H of G of degree at most 2. Notice that $|V(H)| \geq 4$.

Case 1: $d(H) = 1$.

Let c be the cutvertex of G, belonging to H and let R be the union of all branches of G at c which intersect H only at c.

If H is trivial at c, then, by Claim 2, $V(H) - \{c\} = \{b_1, b_2, \cdots, b_h\} \subseteq N(c)$. The graph $G' = Rcb_1$ satisfies the property (i). So by minimality of G, the graph G'^2 is hamiltonian and, using similar arguments as in the proof of Claim 1, R^2 contains a hamiltonian path $c'P_Rc''$ between some $c' \in N[c]$ and some $c'' \in N(c)$. Let

$$C = c'P_Rc''b_1 \cdots b_hc'$$

It is easy to see that C is a hamiltonian cycle in G^2, a contradiction.

Hence H is not trivial at c, i.e., it contains a proper induced path $P_3(c) = cb_1b_2$. The graph $G'' = Rcb_1b_2$ is connected and, by Claim 3, G'' satisfies the condition (i). Since $|V(G'')| < |V(G)|$, $(G'')^2$ is hamiltonian and, by Claim 1, the graph R^2 contains a hamiltonian path cP_Rc'' connecting c and some $c'' \in N(c)$. On the other hand, by Theorem 5, H^2 contains a hamiltonian path b_1P_Hc connecting b_1 and c.

Hence the cycle $C = cP_Rc''b_1P_Hc$ is a hamiltonian cycle in G^2, a contradiction with the condition (ii) on G.

Case 2: $d(H) = 2$.

Let c_1 and c_2 be the two cutvertices of G belonging to H and let B_i, $i = 1, 2$, be the union of all branches at c_i not containing H. This means that $G = B_1c_1Hc_2B_2$. We distinguish, up to symmetry, the following two subcases.

Subcase 2.1: B_1 is trivial at c_1 and B_2 is trivial at c_2.

The subgraph H is a block and thus, by Theorem 5, $V(H)$ can be partitioned into two subpaths $a_1P_H^1a_2$ and $c_2P_H^2c_1$, where $a_1 \in N(c_1)$ and $a_2 \in N(c_2)$.

If $V(B_1) = \{b_1, b_2, \cdots, b_k, c_1\} \subseteq N[c_1]$, $k \geq 1$, and $B_2 = P_3(c_2) = c_2d_1d_2$ then the cycle $C = c_1b_1b_2 \cdots b_ka_1P_H^1a_2d_1d_2c_2P_H^2c_1$ is a hamiltonian cycle in G^2 and contradicts (ii).

The proof is similar if $B_1 = P_3(c_1)$ and $V(B_2) \subseteq N[c_2]$.

If $V(B_1) = \{b_1, b_2, \cdots, b_k, c_1\} \subseteq N[c_1]$ and $V(B_2) = \{d_1, d_2, \cdots, d_l, c_2\} \subseteq N[c_2]$, then the cycle $C = c_1b_1b_2 \cdots b_ka_1P_H^1a_2d_1d_2 \cdots d_lc_2P_H^2c_1$ is a hamiltonian cycle in G^2, contradicting (ii).

Finally, if $B_1 = P_3(c_1) = c_1b_1b_2$ and $B_2 = P_3(c_2) = c_2d_1d_2$, then again the cycle $C = c_1b_2b_1a_1P_H^1a_2d_1d_2c_2P_H^2c_1$ gives a similar contradiction.

Subcase 2.2: B_1 is not trivial at c_1.

Then B_1 contains a path $P_3(c_1) = c_1b_1b_2$ as a proper induced subgraph. On the other hand, since $|V(H) \cup V(B_2)| > 3$ and there exists some vertex in $V(H) \cup V(B_2)$ (for example, each vertex in $V(B_2) - \{c_2\}$) nonadjacent to c_1, the subgraph $G' = Hc_2B_2$ is not trivial. Then G' contains a path $P_3(c_1) = c_1d_1d_2$ as a proper induced subgraph. Now let $G_1 = B_1c_1d_1d_2$ and $G_2 = b_2b_1c_1G'$. By Claim 3, both G_1 and G_2 satisfy the condition (i). By the minimality of G, the graphs G_1^2 and G_2^2 are hamiltonian and thus, by Claim 1, B_1^2 and G'^2 contains hamiltonian paths $a_1P_{B_1}c_1$ and $c_1P_{G'}a_2$ respectively, where the vertices a_1 and a_2 are in $N(c_1)$. But then the cycle $C = a_1P_{B_1}c_1P_{G'}a_2a_1$ is clearly a hamiltonian cycle in G^2, contradicting the hypothesis (ii).

3 Proof of Theorem 4

Before proving Theorem 4, let us give the following lemma.

Lemma. Let $G = G_1xG_2$, where G_1 and G_2 are two connected graphs with $|V(G_i)| \geq 2$, $i = 1, 2$.
(*i*) If G^2 is hamiltonian, then each of the graphs G_i, $i = 1, 2$, has at least one of the following three properties:
(1) $< G_i - x >^2$ contains a hamiltonian path $x_iP_{G_i-x}y_i$ where $x_i, y_i \in N(x)$,
(2) G_i^2 contains a hamiltonian path $x_iP_{G_i}y_i$ where $x_i, y_i \in N(x)$ (and thus x is an interior vertex of $x_iP_{G_i}y_i$),
(3) G_i^2 contains a hamiltonian path $xP_{G_i}x_i$, where $x_i \in N(x)$.
(*ii*) If both G_1 and G_2 have some of the properties in (*i*), then G^2 is hamiltonian except possibly if G_1 and G_2 satisfy (2) or G_1 satisfies (2) and G_2 satisfies (3) (and symmetrically).

Proof of Lemma: (*i*) Let C be a hamiltonian cycle in G^2. Then clearly, for each $i = 1, 2$, $E(C) \cap E(G_i^2)$ is a system of paths $x_i^jP_i^jy_i^j$, $j = 1, \cdots, k_i$, satisfying one of the following:
(a) $x_i^j, y_i^j \in N(x)$ and $x \notin \bigcup_{j=1}^{k_i} V(x_i^jP_i^jy_i^j)$,
(b) $x_i^j, y_i^j \in N(x)$ and x is an interior vertex of some path $P_i^{j_0}$,
(c) $x_i^j, y_i^j \in N[x]$ and x is an endvertex of some path $P_i^{j_0}$.
If the system of paths satisfies (a), then $x_i^1P_i^1y_i^1x_i^2P_i^2y_i^2 \cdots x_i^{k_i}P_i^{k_i}y_i^{k_i}$ is a hamiltonian path in $< G_i - x >^2$.
If the system of paths satisfies (b), then $x_i^1P_i^1y_i^1x_i^2P_i^2y_i^2 \cdots x_i^{k_i}P_i^{k_i}y_i^{k_i}$ is a hamiltonian path in G_i^2 and x is an interior vertex.
If the system of paths satisfies (c) and if we put (without loss of generality) $x = x_i^1$, then $xP_i^1y_i^1x_i^2P_i^2y_i^2 \cdots x_i^{k_i}P_i^{k_i}y_i^{k_i}$ is a hamiltonian path in G_i^2.
(*ii*) If G_1 satisfies (1) and G_2 satisfies (1), then

$$C = xx_1P_{G_1}y_1x_2P_{G_2}y_2x$$

is a hamiltonian cycle in G^2.
If G_1 satisfies (1) and G_2 satisfies (2), then

$$C = x_1P_{G_1}y_1x_2P_{G_2}y_2x_1$$

is a hamiltonian cycle in G^2.

If G_1 satisfies (1) and G_2 satisfies (3), then

$$C = xP_{G_2}x_2x_1P_{G_1}y_1x$$

is a hamiltonian cycle in G^2.

If G_1 satisfies (3) and G_2 satisfies (3), then

$$C = xP_{G_1}x_1x_2P_{G_2}x$$

is a hamiltonian cycle in G^2.

Proof of Theorem 4: (i) First of all, by Theorem 5, B^2 contains a hamiltonian cycle $b_1a_1P_H^1a_2b_2a_2'P_H^2b_1$, where $a_1 \in N(b_1)$ and $a_2, a_2' \in N(b_2)$.

On the other hand, by the hypothesis, the graph G^2 is hamiltonian. Then G_1 satisfies one of the three conditions in the part (i) of the Lemma, with $x = c_1 = c_2$.

We thus consider the following three different cases.

Case 1: G_1 satisfies (1).

Then necessarily G_2 satisfies (2). Let

$$C' = x_1P_{G_1}y_1a_1P_B^1a_2x_2P_{G_2}y_2a_2'P_B^2(b_1 = c_1)x_1$$

It is easy to see that C' is a hamiltonian cycle in $(G')^2$.

Case 2: G_1 satisfies (2).

Then G_2 satisfies (1) and this case is similar to case 1.

Case 3: G_1 satisfies (3).

Then G_2 satisfies (3). Let

$$C' = c_1P_{G_1}x_1a_1P_B^1a_2(c_2 = b_2)P_{G_2}x_2a_2'P_B^2(b_1 = c_1).$$

Then C' is a hamiltonian cycle in $(G')^2$.

(ii) For $i = 1, 2$, suppose that c_i is contained in an endblock H_i of G_i. Let d_i be the cutvertex of G_i belonging $V(H_i)$ and R_i the connected graph such that $G_i = R_id_iH_i$. Without loss of generality one of the following cases occurs.

Case 1: $|V(H_i)| \geq 3$, $i = 1, 2$.

Then, by Theorem 5, H_1^2 contains a hamiltonian cycle $d_1u_1P_{H_1}^1v_1c_1P_{H_1}^2u_1'd_1$ where $u_1, u_1' \in N(d_1)$ and $v_1 \in N(c_1)$. This implies that H_1 has property (1) with $x = d_1$.

On the other hand, $(G')^2$ is hamiltonian. Then, using the part (i) of the Lemma, the graph R_1 satisfies one of the three properties (1), (2) or (3) with $x = d_1$.

Thus, by the part (ii) of the Lemma, the graph G_1^2 admits a hamiltonian cycle that contains the edge $c_1v_1 \in E(G_1)$. Then G_1 satisfies (3), with $x = c_1$.

Using similar arguments, we show that the graph G_2 also satisfies (3) and, applying the part (ii) of the Lemma, we obtain that the graph $G^2 = (G_1(c_1 = c_2)G_2)^2$ is hamiltonian.

Case 2: $|V(H_1)| \geq 3$ and $|V(H_2)| = 2$.

Using the same arguments as in Case 1, the graph G_1 satisfies (3), with $x = c_1$. Since $(G')^2$ is hamiltonian and $V(H_2) = \{c_2 d_2\}$, the graph G_2 satisfies (3) with $x = c_2$.

Thus, applying the part (ii) of the Lemma, we obtain that the graph G^2 is hamiltonian.

Case 3: $|V(H_1)| = 2$ and $|V(H_2)| = 2$.

It is easy to see that in this case again both G_1 and G_2 satisfy (3) with $x = c_1$ and $x = c_2$, respectively, and thus the graph G^2 is hamiltonian.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan Press (1976).

[2] G. Hendry, W. Vogler, The square of a $S(K_{1,3})-$free graph is vertex pancyclic, Journal of Graph Theory, Vol. 9 (1985) 535-537.

[3] H. Fleischner, In the square of graphs, hamiltonicity and pancyclicity, hamiltonian connectedness and panconnectedness are equivalent concepts, Monatshefte Für Mathematik 82 (1976) 125-149.

[4] C. Thomassen, The square of a graph is hamiltonian provided its block graph is a path (preprint, unpublished).