Closure, 2-Factors and Cycle Coverings in Claw-Free Graphs

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Abstract

In this paper we study cycle coverings and 2-factors of a claw-free graph and those of its closure, which has been defined by the first author (On a closure concept in claw-free graphs, *J. Combinatorial Theory Ser. B* **70** (1997) 217–224). For a claw-free graph G and its closure $cl(G)$, we prove (1) $V(G)$ is covered by k cycles in G if and only if $V(cl(G))$ is covered by k cycles of $cl(G)$, and (2) G has a 2-factor with at most k components if and only if $cl(G)$ has a 2-factor with at most k components.

For graph theoretic notation not defined in this paper, we refer the reader to [2]. A vertex x of a graph G is said to be *locally connected* if the neighborhood $N_G(x)$ of x in G induces a connected graph. A locally connected vertex x is said to be *eligible* if $N_G(x)$ induces a noncomplete graph. Let x be an eligible vertex of a graph G. Consider the operation of joining every pair of nonadjacent vertices in $N_G(x)$ by an edge so that $N_G(x)$ induces a complete graph in the resulting graph. This operation is called *local completion* of G at x. For a graph G, let $G_0 = G$. For $i \geq 0$, if G_i is defined and it has an eligible vertex x_i, then apply local completion of G_i at x_i to obtain a new graph G_{i+1}. If G_i has no eligible vertex, let $cl(G) = G_i$ and call it the *closure* of G. The above operation was introduced and the following theorems were proved in [3].

Theorem A ([3]). *If G is a claw-free graph, then*

(1) *a graph obtained from G by local completion is also claw-free, and*

(2) *$cl(G)$ is uniquely determined.* □

Theorem B ([3]). *Let G be a claw-free graph. Then G is hamiltonian if and only if $cl(G)$ is hamiltonian.* □

Recently, several other properties on paths and cycles of a claw-free graph and those of its closure were studied in [1]. In particular, the following theorem was proved.

Theorem C ([1]).

(1) *A claw-free graph G is traceable if and only if $cl(G)$ is traceable.*

(2) *There exist infinitely many claw-free graphs G such that $cl(G)$ is hamiltonian-connected while G is not hamiltonian-connected.*

(3) *For any positive integer k, there exists a k-connected claw-free graph G such that $cl(G)$ is pancyclic while G is not pancyclic.* □

Let $H_1, \ldots, H_k$ be subgraphs of G. Then G is said to be covered by $H_1, \ldots, H_k$ if $V(G) = V(H_1) \cup \cdots \cup V(H_k)$.

We consider two interpretations of a hamiltonian cycle. First, a hamiltonian cycle of a graph G is a cycle which covers G. Second, it is considered as a 2-factor with one component. These interpretations may lead us to possible extensions of Theorem B to cycle coverings and 2-factors. This is the motivation of this paper.

We prove the following theorems as generalizations of Theorem B.

Theorem 1. *Let G be a claw-free graph. Then G is covered by k cycles if and only if $cl(G)$ is covered by k cycles.*

Theorem 2. *Let G be a claw-free graph. If $cl(G)$ has a 2-factor with k components, then G has a 2-factor with at most k components.*

Note that the conclusion of Theorem 2 says G has a 2-factor with "at most" k components. Under the assumption of Theorem 2, G does not always have a 2-factor with exactly k components if $k \geq 2$. Let G be a graph with $k-1$ components $H_1, \ldots, H_{k-1}$, where H_1 is the graph shown in Figure 1 and $H_2, \ldots, H_{k-1}$ are cycles of arbitrary lengths. Then G is claw-free and $cl(G) = cl(H_1) \cup H_2 \cup \cdots \cup H_{k-1}$, where $cl(H_1)$ is isomorphic to K_9. Since K_9 has a 2-factor with two components, $cl(G)$ has a 2-factor with k components. However, G has no 2-factor with k components since H_1 does not have a 2-factor with two components.

- insert figure 1

Before proving the above theorems we introduce some notation which is used in the subsequent arguments. For a graph G and $\emptyset \neq S \subset V(G)$, the subgraph induced by S is denoted by $G[S]$. When we consider a path or a cycle, we always assign an orientation. Let $P = x_0 x_1 \cdots x_m$. We call x_0 and x_m the starting vertex and the terminal vertex of P, respectively. The set of internal vertices of P is denoted by $\mathrm{int}(P)$: $\mathrm{int}(P) = \{x_1, x_2, \ldots, x_{m-1}\}$. The length of P is the number of edges in P, and is denoted by $l(P)$. We define $x_i^{+(P)} = x_{i+1}$ and $x_i^{-(P)} = x_{i-1}$. Furthermore, we define $x_i^{++(P)} = x_{i+2}$. When it is obvious which path is considered in the context, we sometimes write x_i^+ and x_i^- instead of $x_i^{+(P)}$ and $x_i^{-(P)}$, respectively. For x_i, $x_j \in V(P)$ with $i \leq j$, we denote the subpath $x_i x_{i+1} \cdots x_j$ by $x_i \overrightarrow{P} x_j$. The same path traversed in the opposite direction is denoted by $x_j \overleftarrow{P} x_i$. We use similar notations with respect to cycles with a given orientation.

We present several lemmas before proving the main theorems.

Lemma 3. *Let G be a claw-free graph and let x be a locally connected vertex. Let T_1,*

$T_2 \subset V(G)$ *with* $T_1 \cap T_2 = \{x\}$. *Suppose both* $G[T_1]$ *and* $G[T_2]$ *are hamiltonian but* $G[T_1 \cup T_2]$ *is not hamiltonian. Choose cycles* C_1 *and* C_2 *with* $V(C_1) \cup V(C_2) = T_1 \cup T_2$ *and* $V(C_1) \cap V(C_2) = \{x\}$ *and a path* P *in* $G[N_G(x)]$ *with starting vertex in* $\{x^{+(C_1)}, x^{-(C_1)}\}$ *and terminal vertex in* $\{x^{+(C_2)}, x^{-(C_2)}\}$ *so that* P *is as short as possible. Then* $2 \le l(P) \le 3$ *and* $\mathrm{int}(P) \cap (T_1 \cup T_2) = \emptyset$.

Proof. First, note that each hamiltonian cycle D_i in $G[T_i]$ ($i = 1,\ 2$) satisfies $V(D_1) \cup V(D_2) = T_1 \cup T_2$ and $V(D_1) \cap V(D_2) = \{x\}$. Furthermore, since x is a locally connected vertex of G, there exists a path in $G[N_G(x)]$ with starting vertex in $\{x^{+(D_1)}, x^{-(D_1)}\}$ and terminal vertex in $\{x^{+(D_2)}, x^{-(D_2)}\}$. Therefore, we can make a choice for (C_1, C_2, P). Let $u_1 = x^{+(C_1)}$, $v_1 = x^{-(C_1)}$, $u_2 = x^{+(C_2)}$ and $v_2 = x^{-(C_2)}$. We may assume the starting and terminal vertices of P are u_1 and u_2, respectively.

If $u_1u_2 \in E(G)$, then $C' = xv_1\overleftarrow{C}_1u_1u_2\overrightarrow{C}_2v_2x$ is a cycle in G with $V(C') = V(C_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption. Hence we have $u_1u_2 \notin E(G)$. Similarly we have u_1v_2, v_1u_2, $v_1v_2 \notin E(G)$. Since $\{u_1, v_1, u_2\} \subset N_G(x)$ and G is claw-free, we have $u_1v_1 \in E(G)$. Similarly $u_2v_2 \in E(G)$.

Let $w = u_1^{+(P)}$. We claim $w \notin V(C_1) \cup V(C_2)$. Assume $w \in V(C_1) \cup V(C_2)$. Since $w \in V(P) \subset N_G(x)$, $w \neq x$. Thus, $w \in u_1\overrightarrow{C}_1v_1 \cup u_2\overrightarrow{C}_2v_2$.

First, suppose $w \in u_1\overrightarrow{C}_1v_1$. Then by the choice of P, $w \in u_1^+\overrightarrow{C}_1v_1^-$. Since $\{x, w^+, w^-\} \subset N_G(w)$ and G is claw-free, we have $\{xw^+, xw^-, w^+w^-\} \cap E(G) \neq \emptyset$. If $w^+w^- \in E(G)$, let $C_1' = xwu_1\overrightarrow{C}_1w^-w^+\overrightarrow{C}_1v_1x$, $C_2' = C_2$ and $P' = w\overrightarrow{P}u_2$. If $w^-x \in E(G)$, then let $C_1' = xw\overrightarrow{C}_1v_1u_1\overrightarrow{C}_1w^-x$, $C_2' = C_2$ and $P' = w\overrightarrow{P}u_2$. If $w^+x \in E(G)$, then let $C_1' = xw\overleftarrow{C}_1u_1v_1\overleftarrow{C}_1w^+x$, $C_2' = C_2$ and $P' = w\overrightarrow{P}u_2$. Then in each case, since $V(C_1') = V(C_1)$, we have $V(C_1') \cup V(C_2') = V(C_1) \cup V(C_2) = T_1 \cup T_2$ and $V(C_1') \cap V(C_2') = \{x\}$. Furthermore, $w = x^{+(C_1')}$ and $l(P') < l(P)$. This contradicts the choice of (C_1, C_2, P).

Now, suppose $w \in u_2\overrightarrow{C}_2v_2$. Since $\{u_2, v_2\} \cap N_G(u_1) = \emptyset$, we have $w \in u_2^+\overrightarrow{C}_2v_2^-$. Since $\{x, w^-, w^+\} \subset N_G(w)$ and G is claw-free, $\{xw^-, xw^+, w^-w^+\} \cap E(G) \neq \emptyset$. If $xw^- \in E(G)$, let $C = xv_1\overleftarrow{C}_1u_1w\overrightarrow{C}_2v_2u_2\overrightarrow{C}_2w^-x$. If $xw^+ \in E(G)$, let $C = xw^+\overrightarrow{C}_2v_2u_2\overrightarrow{C}_2wu_1\overrightarrow{C}_1v_1x$. Then in either case C is a cycle in G with $V(C) = V(C_1) \cup V(C_2) = T_1 \cup T_2$. This contradicts the assumption. If $w^-w^+ \in E(G)$, then let $C_1' = xwu_1\overrightarrow{C}_1v_1x$, $C_2' = xu_2\overrightarrow{C}_2w^-w^+\overrightarrow{C}_2v_2x$ and $P' = w\overrightarrow{P}u_2$. Then $V(C_1') \cup V(C_2') = V(C_1) \cup V(C_2) = T_1 \cup T_2$, $V(C_1') \cap V(C_2') = \{x\}$,

$w = x^{+(C_1')}$ and $l(P') < l(P)$. This contradicts the choice of (C_1, C_2, P). Therefore, $w \notin V(C_1) \cup V(C_2)$.

Let $w' = u_2^{-(P)}$. (Possibly $w' = w$.) Then by the same arguments we have $w' \notin V(C_1) \cup V(C_2)$.

By the choice of (C_1, C_2, P), P is an induced path. Hence if $l(P) \geq 4$, then $\{u_1, u_1^{++(P)}, u_2\}$ is an independent set. Since $V(P) \subset N_G(x)$ and G is claw-free, this is a contradiction. Thus, $l(P) \leq 3$. Since $u_1u_2 \notin E(G)$, $l(P) \geq 2$. These imply $\mathrm{int}(P) \cap (T_1 \cup T_2) = \emptyset$. $\square$

By similar arguments, we have the following lemma.

Lemma 4. *Let G be a claw-free graph and let x be a locally connected vertex of G. Let $T \subset V(G)$ with $x \in T$, and let $u \in N_G(x) - T$. Suppose $G[T]$ is hamiltonian but $G[T \cup \{u\}]$ is not hamiltonian. Choose a hamiltonian cycle C in $G[T]$ and a path P in $G[N_G(x)]$ with starting vertex in $\{x^{+(C)}, x^{-(C)}\}$ and terminal vertex u so that P is as short as possible. Then $2 \leq l(P) \leq 3$ and $\mathrm{int}(P) \cap (T \cup \{u\}) = \emptyset$.* $\square$

We prove one more lemma.

Lemma 5. *Let G be a claw-free graph and let x be an eligible vertex of G. Let G' be the graph obtained from G by local completion at x. Let C' be a cycle in G' with $x \in V(C')$. Then either (1) or (2) follows.*

(1) *There exists a cycle C in G with $V(C) = V(C')$.*

(2) *There exist T_1, $T_2 \subset V(G)$ such that*

(2.1) *$T_1 \cup T_2 = V(C')$ and $T_1 \cap T_2 = \{x\}$, and*

(2.2) *$G[T_i]$ is hamiltonian or isomorphic to K_2 ($i = 1, 2$).*

Proof. Let $B = E(G') - E(G)$. Note that for each $uv \in B$, $\{u, v\} \subset N_G(x)$. Choose a cycle C in G' with $V(C) = V(C')$ so that $|E(C) \cap B|$ is as small as possible. If $E(C) \cap B = \emptyset$, then C is a cycle satisfying (1). Therefore, we may assume $E(C) \cap B \neq \emptyset$.

We claim $|E(C) \cap B| = 1$. Assume, to the contrary, $|E(C) \cap B| \geq 2$, say e_1, $e_2 \in E(C) \cap B$, $e_1 \neq e_2$. Let $e_i = x_i y_i$ ($i = 1, 2$). We may assume x_1, y_1, x_2, y_2 and x appear in this order along C. (Possibly, $y_1 = x_2$.) Then x_1, x_2 and x^- are distinct vertices in $N_G(x)$. Since G is claw-free, $\{x_1x_2, x_1x^-, x_2x^-\} \cap E(G) \neq \emptyset$. If $x_1x_2 \in E(G)$, let $C_0 = y_2 \overrightarrow{C} x_1 x_2 \overleftarrow{C} y_1 y_2$.

Then $V(C_0) = V(C)$ and $E(C_0) = (E(C) - \{x_1y_1, x_2y_2\}) \cup \{x_1x_2, y_1y_2\}$. This implies $|E(C_0) \cap B| < |E(C) \cap B|$, which contradicts the minimality of $|E(C) \cap B|$. If $x_1x^- \in E(G)$, let $C_0 = x\overrightarrow{C}x_1x^-\overleftarrow{C}y_1x$. Then $V(C_0) = V(C)$ and $E(C_0) = (E(C) - \{x_1y_1, xx^-\}) \cup \{xy_1, x_1x^-\}$. Since $xy_1 \in E(G)$, we have $|E(C_0) \cap B| < |E(C) \cap B|$, again a contradiction. We have a similar contradiction if $x^-x_2 \in E(G)$. Therefore, the claim is proved.

Let $E(C) \cap B = \{x_1y_1\}$. We may assume x, x_1 and y_1 appear in this order along C. Let $T_1 = x\overrightarrow{C}x_1$ and $T_2 = y_1\overrightarrow{C}x$. Then $T_1 \cup T_2 = V(C)$ and $T_1 \cap T_2 = \{x\}$. Since $x_1y_1 \in B$, xx_1, $xy_1 \in E(G)$. If $x_1 \neq x^+$, then $x\overrightarrow{C}x_1x$ is a hamiltonian cycle in $G[T_1]$. If $x_1 = x^+$, then $G[T_1] \simeq K_2$. Similarly, $G[T_2]$ is either hamiltonian or isomorphic to K_2. $\square$

Let G' be a graph obtained from a claw-free graph by local completion at a vertex. Using Lemmas 3, 4, 5 we prove that for each cycle in G' there exists a cycle in G which contains it. We can also impose some restriction on its length.

Theorem 6. *Let G be a claw-free graph and let x be a locally connected vertex of G. Let G' be the graph obtained from G by local completion at x. Then for each cycle C' in G' there exists a cycle C in G with $V(C') \subset V(C)$ and $l(C') \leq l(C) \leq l(C') + 3$.*

Proof. If $E(C') \cap (E(G') - E(G)) = \emptyset$, then C' is a required cycle. Hence we may assume $E(C') \cap (E(G') - E(G)) \neq \emptyset$.

If $x \in V(C')$, let $C'_1 = C'$. Suppose $x \notin V(C')$. Let $e = uu^{+(C')} \in E(C') \cap (E(G') - E(G))$. Then $\{u, u^{+(C')}\} \subset N_G(x)$. Let $C'_1 = u^{+(C')}\overrightarrow{C'}uxu^{+(C')}$. In either case, we have a cycle C'_1 with $V(C') \cup \{x\} \subset V(C'_1)$ and $l(C') \leq l(C'_1) \leq l(C') + 1$.

If there exists a cycle C in G with $V(C'_1) = V(C)$, then C is a required cycle. Therefore, we may assume G has no such cycle. Then by Lemma 5, there exist T_1, $T_2 \subset V(G)$ with $T_1 \cap T_2 = \{x\}$ and $T_1 \cup T_2 = V(C'_1)$ such that $G[T_i]$ is hamiltonian or $G[T_i] \simeq K_2$.

Suppose both $G[T_1]$ and $G[T_2]$ are hamiltonian. Then by Lemma 3 there exist cycles C_1 and C_2 in G and a path P in $G[N_G(x)]$ such that

(1) $V(C_1) \cup V(C_2) = T_1 \cup T_2 = V(C'_1)$, $V(C_1) \cap V(C_2) = \{x\}$, and

(2) P joins $\{x^{+(C_1)}, x^{-(C_1)}\}$ and $\{x^{+(C_2)}, x^{-(C_2)}\}$, $2 \leq l(P) \leq 3$ and $\mathrm{int}(P) \cap (T_1 \cup T_2) = \emptyset$.

Let $u = x^{+(C_1)}$ and $v = x^{+(C_2)}$. We may assume P joins u and v. Let $C = x\overleftarrow{C_1}u\overrightarrow{P}v\overrightarrow{C_2}x$. Then C is a cycle in G, $V(C'_1) \subset V(C)$ and $l(C'_1) \leq l(C) \leq l(C'_1) + 2$. Therefore,

$V(C') \subset V(C_1') \subset V(C)$ and $l(C') \leq l(C_1') \leq l(C) \leq l(C_1') + 2 \leq l(C') + 3$.

Using Lemma 4 instead of Lemma 3, we can, by similar arguments, deal with the case in which $G[T_1]$ or $G[T_2]$ is isomorphic to K_2. $\square$

Now Theorem 1 is a consequence of the following corollary of Theorem 6.

Corollary 7. *Let G be a claw-free graph and let x be an eligible verex of G. Let G' be the graph obtained from G by local completion at x. Then G is covered by k cycles if and only if G' is covered by k cycles.*

Proof. Since the "only if" part is trivial, we have only to prove the "if" part of the corollary. Suppose G' is covered by k cycles, say $V(G') = V(C_1') \cup \cdots \cup V(C_k')$ for cycles $C_1', \ldots, C_k'$ in G'. By Theorem 6 for each C_i' there exists a cycle C_i in G with $V(C_i') \subset V(C_i)$ $(1 \leq i \leq k)$. Then $V(G) = V(C_1) \cup \cdots \cup V(C_k)$. $\square$

Now we prove Theorem 2. Actually, we prove a stronger statement.

Theorem 8. *Let G be a claw-free graph and let x be an eligible vertex of G. Let G' be the graph obtained from G by local completion at x. Then for each set of k disjoint cycles $\{D_1, \ldots, D_k\}$ in G' there exists a set of at most k disjoint cycles $\{C_1, \ldots, C_l\}$ $(l \leq k)$ in G with $\cup_{i=1}^{k} V(D_i) \subset \cup_{i=1}^{l} V(C_i)$.*

Proof. Let $S_0 = \cup_{i=1}^{k} V(D_i)$. Assume, to the contrary, that $G[S]$ has no 2-factor with at most k components for any $S \subset V(G)$ with $S_0 \subset S$. Let $B = E(G') - E(G)$. Note $\{a, b\} \subset N_G(x)$ for each $ab \in B$. Let

$$\mathfrak{F} = \{(S, F) : S_0 \subset S \subset V(G) \text{ and } F \text{ is a 2-factor of } G'[S]\}.$$

Since $(S_0, \cup_{i=1}^{k} E(D_i)) \in \mathfrak{F}$, $\mathfrak{F} \neq \emptyset$. Let $\mathfrak{F}_0$ be the set of pairs $(S, F) \in \mathfrak{F}$ chosen so that

(a) the number of components of F is as small as possible, and

(b) $|F \cap B|$ is as small as possible, subject to (a).

Let $(S, F) \in \mathfrak{F}_0$. Suppose F consists of l components (cycles) $C_1, \ldots, C_l$: $F = E(C_1) \cup \cdots \cup E(C_l)$ (disjoint). Since $(S_0, \cup_{i=1}^{k} E(D_i)) \in \mathfrak{F}$, $l \leq k$. By the assumption $F \cap B \neq \emptyset$.

If $x \notin S$, choose i with $E(C_i) \cap B \neq \emptyset$, say $e = uv \in E(C_i) \cap B$ and $v = u^{+(C_i)}$. Let $C_i' = xv\overrightarrow{C_i}ux$ and $F' = \big(F - E(C_i)\big) \cup E(C_i')$. Then F' is a 2-factor of $G'[S \cup \{x\}]$ with l

components and $|F' \cap B| = |F \cap B| - 1$. This contradicts the choice of (S, F) given in (b). Therefore, we have $x \in S$. We may assume $x \in V(C_1)$.

We claim $B \cap (\cup_{i=2}^{l} E(C_i)) = \emptyset$. Assume $B \cap (\cup_{i=2}^{l} E(C_i)) \neq \emptyset$, say $f = u'v' \in B \cap E(C_j)$ $(j \geq 2)$. Then $\{u', v', x^{+(C_1)}\} \subset N_G(x)$ and hence $u'x^{+(C_1)} \in E(G')$. We may assume $j = 2$ and $v' = u'^{+(C_2)}$. Let $C' = xv'\overrightarrow{C_2}u'x^{+(C_1)}\overrightarrow{C_1}x$ and $F' = \big(F - (E(C_1) \cup E(C_2))\big) \cup E(C')$. Then F' is a 2-factor of $G'[S]$ with $l-1$ components. This contradicts the choice of (S, F).

Since $F \cap B \neq \emptyset$, $B \cap E(C_1) \neq \emptyset$. If there exists a cycle C_1' in G with $V(C_1') = V(C_1)$, then $\big(F - E(C_1)\big) \cup E(C_1')$ is a 2-factor of $G[S]$ with l components. This contradicts the assumption. Since $x \in V(C_1)$, by Lemma 5, there exist T_0, $T_1 \subset V(G)$ such that $T_0 \cup T_1 = V(C_1)$, $T_0 \cap T_1 = \{x\}$, and $G[T_i]$ is hamiltonian or isomorphic to K_2 $(i = 0, 1)$.

First, consider the case in which both $G[T_0]$ and $G[T_1]$ are hamiltonian. Let C_0' and C_1' be cycles in $G[T_0 \cup T_1]$ with $V(C_0') \cup V(C_1') = T_0 \cup T_1 = V(C_1)$ and $V(C_0') \cap V(C_1') = \{x\}$. Let $u_i = x^{+(C_i')}$ and $v_i = x^{-(C_i')}$ $(i = 0, 1)$. Since x is a locally connected vertex of G, $G[N_G(x)]$ has a path P with starting vertex in $\{u_0, v_0\}$ and terminal vertex in $\{u_1, v_1\}$. Since $G[S]$ has no 2-factors with l components, u_0u_1, u_0v_1, v_0u_1, $v_0v_1 \in B$. By the choice of (S, F) given in (b), $|E(C_1) \cap B| = 1$.

Now choose $(S, F) \in \mathfrak{F}_0$, C_0', C_1' and P so that

(c) P is as short as possible.

Then by Lemma 3, $2 \leq l(P) \leq 3$ and $\mathrm{int}(P) \cap V(C_1) = \emptyset$. We may assume that the starting vertex and the terminal vertex of P are v_0 and u_1, respectively.

Let $a = v_0^{+(P)}$. Then $a \notin V(C_1)$. Assume $a \notin S$. Since $V(P) \subset N_G(x)$, $ax \in E(G)$ and hence $au_1 \in E(G')$. Let $C' = xu_0\overrightarrow{C_0'}v_0au_1\overrightarrow{C_1'}v_1x$ and $F' = \big(F - E(C_1)\big) \cup E(C')$. Then F' is a 2-factor of $G'[S \cup \{a\}]$ with l components and $F' \cap B \subset \{au_1\}$. Since $|B \cap E(C_1)| = 1$, $|F' \cap B| = |F \cap B| = 1$. Furthermore, $C_0'' = xu_0\overrightarrow{C_0'}v_0ax$ and $C_1'' = C_1'$ are two cycles in G with $V(C_0'') \cup V(C_1'') = V(C')$ and $V(C_0'') \cap V(C_1'') = \{x\}$. Since $a\overrightarrow{P}u_1$ is shorter than P, this contradicts the choice of (S, F) given in (c). Therefore, we have $a \in S$. We may assume $a \in V(C_2)$. Let $a' = a^{+(C_2)}$ and $a'' = a^{-(C_2)}$.

If $a'x \in E(G)$, then $\{a', u_1\} \subset N_G(x)$ and hence $a'u_1 \in E(G')$. Let

$$C' = xu_0\overrightarrow{C_0'}v_0a\overleftarrow{C_2}a'u_1\overrightarrow{C_1'}v_1x$$

and $F' = \big(F - (E(C_1) \cup E(C_2))\big) \cup E(C')$. Then F' is a 2-factor of $G'[S]$ with $l-1$ components. This contradicts the choice of (S, F). Hence we have $a'x \notin E(G)$. By the same argument we have $a''x \notin E(G)$. Since a and $\{x, a', a''\}$ do not form a claw in G, $a'a'' \in E(G)$. If $l(C_2) \geq 4$, let $C' = xu_0\overrightarrow{C_0'}v_0au_1\overrightarrow{C_1'}v_1x$ (note $au_1 \in E(G')$), $C'' = a'\overrightarrow{C_2}a''a'$ and $F' = \big(F - (E(C_1) \cup E(C_2))\big) \cup E(C') \cup E(C'')$. Then F' is a 2-factor of $G'[S]$ with l components and $F' \cap B \subset \{au_1\}$. Since $|B \cap E(C_1)| = 1$, $|F' \cap B| = |F \cap B|$. Furthermore, $C_0'' = xu_0\overrightarrow{C_0'}v_0ax$ and $C_1'' = C_1'$ are two cycles in G with $V(C_0'') \cup V(C_1'') = V(C')$ and $V(C_0'') \cap V(C_1'') = \{x\}$. Since $a\overrightarrow{P}u_1$ is shorter than P, this contradicts the choice of (S, F) given in (c). Therefore, we have $l(C_2) = 3$, which implies $C_2 = aa'a''a$.

If $a' \in N_G(v_0)$, let $C' = xu_0\overrightarrow{C_0'}v_0a'a''au_1\overrightarrow{C_1'}v_1x$ and $F' = \big(F - (E(C_1) \cup E(C_2))\big) \cup E(C')$. Then F' is a 2-factor of $G'[S]$ with $l-1$ components. This contradicts the choice of (S, F). If $a' \in N_G(u_1)$, let $C' = xu_0\overrightarrow{C_0'}v_0aa''a'u_1\overrightarrow{C_1'}v_1x$ and $F' = F - \big((E(C_1) \cup E(C_2))\big) \cup E(C')$. Then F' is a 2-factor of $G'[S]$ with $l-1$ components, which contradicts the assumption. Therefore, $a' \notin N_G(v_0) \cup N_G(u_1)$. Similarly, $a'' \notin N_G(v_0) \cup N_G(u_1)$.

Let $b = u_1^{-(P)}$. Let $b \in V(C_i)$, $2 \leq i \leq l$, $b' = b^{+(C_i)}$ and $b'' = b^{-(C_i)}$. By symmetry, we have $\{b', b''\} \cap (N_G(x) \cup N_G(u_1) \cup N_G(v_0)) = \emptyset$ and $l(C_i) = 3$.

Suppose $l(P) = 2$. Then $b = a$ and hence $C_i = C_2$. Since $a' \notin N_G(v_0) \cup N_G(u_1)$ and $v_0u_1 \notin E(G)$, a and $\{a', v_0, u_1\}$ form a claw in G, a contradiction. Therefore, we have $l(P) = 3$. Since u_1a', $u_1a'' \notin E(G)$, $C_i \neq C_2$. We may assume $b \in V(C_3)$.

By the choice of P given in (c), bv_0, $au_1 \notin E(G)$. Since $v_0a' \notin E(G)$ and a and $\{a', b, v_0\}$ do not form a claw, $a'b \in E(G)$. Similarly, we have $a''b$, ab', $ab'' \in E(G)$. Now let $C' = aa''a'bb'b''a$ and $F' = \big(F - (E(C_2) \cup E(C_3)\big) \cup E(C')$. Then F' is a 2-factor of $G'[S]$ with $l-1$ components. This contradicts the choice of (S, F) given in (a), and the theorem follows in this case.

By replacing Lemma 3 with Lemma 4, we can follow the same arguments to obtain a contradiction if $G[T_1]$ or $G[T_2]$ is isomorphic to K_2. Therefore, the theorem is proved. □

Concluding Remarks.

Let S be a set of vertices in a claw-free graph G. Then by Theorem 6 the minimum number of cycles covering S in G is the same as the minimum number of cycles covering

S in $cl(G)$. Furthermore, by Theorem 8, the minimum number of disjoint cycles covering S in G is the same as the minimum number of disjoint cycles covering S in $cl(G)$, (if there exist such cycles). Therefore, these invariants (and hence the minimum number of cycles covering $V(G)$) are stable in the sense of [1]. Furthermore, the existence of a 2-factor is a stable property.

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