# Closure and stable hamiltonian properties in claw-free graphs


Stephan Brandt
FB Mathematik & Informatik, WE 2
Freie Universität Berlin
Arnimallee 3
14195 Berlin, Germany
e-mail brandt@math.fu-berlin.de

Odile Favaron
L.R.I., Bât. 490,
Université de Paris-Sud
91405-Orsay cedex
France
e-mail of@lri.fr

Zdeněk Ryjáček *
Katedra matematiky
Západočeská univerzita
Univerzitní 22, 306 14 Plzeň
Czech Republic
e-mail ryjacek@kma.zcu.cz


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**Abstract**

In the class of $k$-connected claw-free graphs, we study the stability of some hamiltonian properties under a closure operation introduced by the third author. We prove that

- $(i)$ the properties of pancyclicity, vertex pancyclicity and cycle extendability are not stable for any $k$ (i.e., for any of these properties there is an infinite family of graphs $G_k$ of arbitrarily high connectivity $k$ such that the closure of $G_k$ has the property while the graph $G_k$ does not),
- $(ii)$ traceability is a stable property even for $k = 1$,
- $(iii)$ homogeneous traceability is not stable for $k = 2$ (although it is stable for $k = 7$).

The paper is concluded with several open questions concerning stability of homegeneous traceability and hamiltonian connectedness.



**Keywords:** closure, claw-free graphs, stable property, hamiltonicity, pancyclicity, cycle extendability, traceability.

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# 1. Introduction

The graphs $G = (V(G), E(G))$ we consider in this paper are finite of order $|V(G)| = n$, undirected, without loops and multiple edges. For terminology and notation not defined here we refer to [3]. For any set $A \subset V(G)$ we denote by $\langle A \rangle$ the induced subgraph on $A$, $G - A$ stands for $\langle V(G) \setminus A \rangle$. The words cycle and path mean elementary cycle or path. The vertex connectivity of $G$ will be denoted by $\kappa(G)$, the circumference of $G$ (i.e., the length of a longest cycle in $G$) by $c(G)$, the girth of $G$ (i.e., the length of a shortest cycle in $G$) by $g(G)$ and the length of a longest path in $G$ by $p(G)$. By a *clique* we mean a (not necessarily maximal) complete subgraph of $G$.

We say that a graph $G$ is *hamiltonian* if $c(G) = n$, *pancyclic* if it contains cycles of every length $\ell$, $3 \le \ell \le n$, *vertex pancyclic* if it contains cycles of every length through every vertex, *cycle extendable* if for every cycle $C$ of $G$ there exists a cycle $C'$ such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. Similarly, $G$ is *traceable* if $p(G) = n$, i.e. if $G$ contains a hamiltonian path, *homogeneously traceable* if every vertex is an endvertex of some hamiltonian path in $G$, and *hamiltonian connected* if there exists a hamiltonian path between every pair of distinct vertices. For a path, the notation $P(a, b)$ means that $P$ is an $a, b$-path, i.e. its two endvertices are $a$ and $b$. We call a graph *eulerian* if it contains a closed spanning trail.

The four-vertex star $K_{1,3}$ is called the *claw*. If $G$ contains no copy of the claw as an induced subgraph, we say that $G$ is *claw-free*. The line graph $L(G)$ of a graph $G$ is always claw-free.

For a vertex $x \in V(G)$, the set $N_G(x) = \{y \in V(G) |\ xy \in E(G)\}$ is called the *neighborhood* of $x$ in $G$. If $\langle N_G(x) \rangle$ is a connected graph, we say that $x \in V(G)$ is a *locally connected vertex*. A locally connected vertex with a noncomplete neighborhood will be called an *eligible vertex*. For an eligible vertex $x \in V(G)$, the operation of joining all pairs of nonadjacent vertices in $\langle N_G(x) \rangle$ by an edge (i.e. replacing $\langle N_G(x) \rangle$ by the clique on $N_G(x)$) will be called the *local completion* of $G$ at $x$. The graph obtained from $G$ by a local completion at $x$ will be denoted by $G'_x$. Thus, $\langle N_{G'_x}(x) \rangle$ is always a clique.

In [8], Ryjáček proved that by recurrently performing the local completion operation to eligible vertices of an arbitrary claw-free graph $G$ until no such vertex remains, we get a claw-free graph which is uniquely determined by the graph $G$, i.e. which is independent of the order of the eligible vertices used during the construction. This new graph is called the *closure* of $G$ and is denoted by $\mathrm{cl}(G)$. (Note that $\mathrm{cl}(G)$ is different from the well-known closure by Bondy and Chvátal [2].) By the construction of $\mathrm{cl}(G)$, the neighborhood in $\mathrm{cl}(G)$ of every vertex is either a clique (if it is connected) or a disjoint union of two cliques (if it is disconnected). This immediately implies that, for any claw-free graph $G$, $\mathrm{cl}(G)$ is the line graph of a triangle-free graph.

It was also proved in [8] that for every eligible vertex $x \in V(G)$ and for every longest cycle $C'$ in $G'_x$ there is a cycle $C$ in $G$ such that $V(C) = V(C')$. Consequently, the circumferences of $G$ and of $\mathrm{cl}(G)$ are the same, which in particular implies that $G$ is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian. A natural question is which other properties of claw-free graphs have the same behavior with respect to the closure operation. This leads to the following definition.

**Definition.**

($i$) *Let $\mathcal{C}$ be a subclass of the class of claw-free graphs. We say that the class $\mathcal{C}$ is* stable under the closure *(or simply* stable*) if* $\mathrm{cl}(G) \in \mathcal{C}$ *for every* $G \in \mathcal{C}$.

($ii$) *Let $\mathcal{C}$ be a stable class and let $\mathcal{P}$ be a property. We say that the property $\mathcal{P}$ is* stable under the closure *(or simply* stable*)* in the class $\mathcal{C}$ *if* $G \in \mathcal{C}$ *has* $\mathcal{P}$ *if and only if* $\mathrm{cl}(G)$ *has* $\mathcal{P}$.

Whenever we speak of the stability of a given parameter (such as e.g. the circumference), we mean the property that the parameter has a specific value.

It is easy to see that, for any $k \geq 1$, the class of $k$-connected claw-free graphs is an example of a stable class.

Now the main result of [8] can be stated as follows.

**Theorem A.** *The length of a longest cycle and the property of hamiltonicity are stable properties in the class of claw-free graphs.*

In the present paper we study stability of some other hamiltonian properties of claw-free graphs such as pancyclicity, vertex-pancyclicity, cycle extendability, traceability and homogeneous traceability.

If $G$ has any of the properties $\mathcal{P}$ indicated above, then $\mathrm{cl}(G)$ has $\mathcal{P}$ as well, since the closure $\mathrm{cl}(G)$ is obtained by adding edges to $G$. Thus, $\mathcal{P}$ is stable (in a certain class), if and only if $\mathrm{cl}(G)$ has $\mathcal{P}$ implies that $G$ has $\mathcal{P}$.

# 2. Main results and problems

It is easy to see that if a property $\mathcal{P}$ is stable in a class $\mathcal{C}$, then $\mathcal{P}$ is stable also in any stable subclass $\mathcal{C}' \subset \mathcal{C}$. Thus, for every property $\mathcal{P}$, we will be interested in finding either a "large" class in which $\mathcal{P}$ is stable, or a "small" class in which $\mathcal{P}$ is not stable.

Thus, if a certain property is not stable under the minimum connectivity assumption that is necessary for the property itself, then, since the class of $k$-connected claw-free graphs is stable for any $k \geq 0$, we will be interested in the question whether the property becomes stable under the assumption of sufficiently large connectivity or whether it remains unstable for arbitrarily large connectivity.

We start with the investigation of hamiltonian properties related to paths. Our main theorem shows that the *maximum length of a path, and in consequence also traceability, are stable properties in the class of claw-free graphs.*

**Theorem 2.1.** *Let $G$ be a claw-free graph. Then*

$$p(G) = p(cl(G)).$$

**Corollary 2.2.** *Let $G$ be a claw-free graph. Then $G$ is traceable if and only if $cl(G)$ is traceable.*

The proof of Theorem 2.1 is given in Section 3. In fact, our original intention was to prove that in the class of 2-connected claw-free graphs, the maximum length of a path with a given endvertex (and, consequently, homogeneous traceability) are stable. The next proposition shows that this is not possible, since *homogeneous traceability is not stable in the class of 2-connected claw-free graphs.*

**Theorem 2.3.** *For any $n \geq 14$ there is a 2-connected claw-free graph on $n$ vertices such that $cl(G)$ is homogeneously traceable while $G$ is not homogeneously traceable.*

**Proof.** Let $G$ be the graph shown in Figure 1 (where the circle parts $K_1$, $K_2$, $K_3$ represent three cliques such that $|V(K_1)| \geq 3$, $|V(K_2)| \geq 3$, $|V(K_3)| \geq 5$ and $V(K_i) \cap V(K_j) = \{x_{i,j}\}$ for $1 \leq i < j \leq 3$). Then every hamiltonian path in $G$ has one endvertex in $V(G) \setminus (V(K_1 \cup K_2 \cup K_3))$ and thus, since both $\{x_{1,2}, x_{1,3}\}$ and $\{x_{1,2}, x_{2,3}\}$ are 2-element cutsets, there is no hamiltonian path in $G$ with $x_{1,2}$ as an endvertex. However, $\langle V(K_1) \cup V(K_2) \cup V(K_3) \rangle$ is a clique in $\mathrm{cl}(G)$ and thus every vertex in this clique is an endvertex of a hamiltonian path in $\mathrm{cl}(G)$. Therefore $\mathrm{cl}(G)$ is homogeneously traceable while $G$ is not. ■

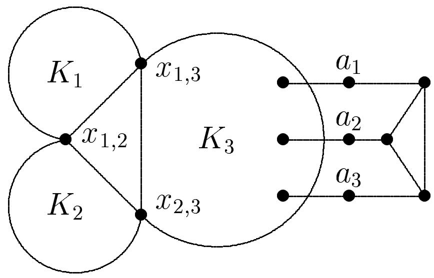


Figure 1

It is easy to check that the graph $G$ of Fig. 1 is a line graph and hence the property of homogeneous traceability is not stable even in the class of 2-connected line graphs. If $G$ is homogeneously traceable then clearly $G$ must be 2-connected. Thus Theorem 2.3 says that homogeneous traceability is not stable under the minimum connectivity assumption that is necessary for the property itself. However, it was proved in [8] that every 7-connected claw-free graph is hamiltonian, which implies that homogeneous traceability is stable in the class of 7-connected claw-free graphs. This guarantees the existence of the integer $k$ in the following question.

**Problem 1.** *Determine the smallest integer $k$ for which homogeneous traceability is stable in the class of $k$-connected claw-free graphs.*

Note that a hamiltonian connected graph must be 3-connected, since the vertices of a 2-element cutset cannot be endvertices of a hamiltonian path. However, the graph $G$ in Fig. 2 is an example of a claw-free graph with complete closure but having no hamiltonian path joining the vertices $a, b$, although $\{a, b\}$ is not a cutset of $G$. We did not succeed in constructing a similar 3-connected example and we therefore pose the following

**Problem 2.** *Is the property of hamiltonian connectedness stable in the class of 3-connected claw-free graphs?*

The answer is positive in the subclass of 3-connected, locally connected, claw-free graphs, since every such graph is hamiltonian connected as shown by Asratian [1].

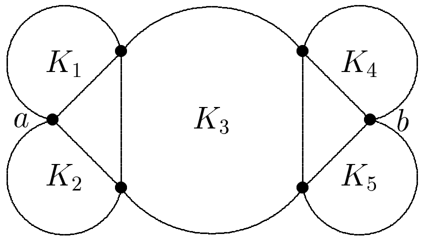


Figure 2

Zhan [9] proved that every 7-connected line graph is hamiltonian connected. Since we have seen that hamiltonian connectedness in general is not stable, the analogous problem in claw-free graphs remains open. Hence it would be interesting to consider the following weaker version of Problem 2.

**Conjecture 3.** *There is an integer $k \geq 3$ such that hamiltonian connectedness is a stable property in the class of $k$-connected claw-free graphs.*

This conjecture was recently proved by Brandt [4] for $k = 9$ by showing that every 9-connected claw-free graph is hamiltonian connected. However, this is probably not the minimal value of $k$.

Now we consider hamiltonian properties related to cycles. Ryjáček [8] proved that the length of a longest cycle and the property of hamiltonicity are stable in claw-free graphs. However, the properties investigated here will turn out to be unstable. Theorem 2.4 shows that *pancyclicity is an unstable property for arbitrarily large connectivity.*

**Theorem 2.4.** *For every $k \geq 2$ there exists a $k$-connected claw-free graph $G$ such that $G$ is not pancyclic but $cl(G)$ is pancyclic.*

By a slight (but more technical) modification of the proof, which will be performed in Section 4, we can show that the same statement holds for vertex pancyclicity as well. So the property of *vertex pancyclicity is* thus also *unstable for arbitrarily large connectivity.*

Our next theorem shows that the property of *cycle extendability is* also *not stable for arbitrarily large connectivity.*

**Theorem 2.5.** *For any $k \geq 2$ there is a $k$-connected claw-free graph $G_k$ such that $cl(G_k)$ is cycle extendable while $G_k$ is not cycle extendable.*

**Proof.** Let $k \geq 2$ and let $G_k$ be the graph obtained from two vertex disjoint cliques $K_{k-1}$ and $K_{2k-2}$ by adding $2k-2$ edges in such a way that every vertex of the $K_{k-1}$ is incident to exactly two and every vertex of the $K_{2k-2}$ is incident to exactly one of these edges.

Then $G_k$ is $k$-connected, claw-free, $cl(G_k)$ is complete (and hence cycle extendable), but the cycles of length $k-1$ in the $K_{k-1}$ are nonextendable in $G_k$. ■

# 3. Traceability is stable

The main result of this section, Proposition 3.2, is considerably stronger than Theorem 2.1. It shows that, in most cases, for any longest $a,b$-path $P'$ in $G'_x$ there is an $a,b$-path $P$ in $G$ with $V(P) = V(P')$, and gives a structural description of the only two situations when only one (but not an arbitrary one) of the endvertices of $P$ can be prescribed.

We first introduce some additional notation that will be used throughout this section.

Let $x$ be an eligible vertex of a claw-free graph $G$. The edges in $E(G)$ are said to be *black edges*, and those in $E(G'_x) \setminus E(G)$ are called *blue edges*. Hence a blue edge is not incident to $x$ but both its endvertices are adjacent to $x$. Let $a$ and $b$ be two distinct vertices of $G$ and $P'(a,b)$ an $a,b$-path in $G'_x$. We consider $P'(a,b)$ to be oriented from $a$ to $b$ and use the standard notation $w^-, w^+$ for the predecessor and successor of a vertex $w$ on $P'$ and $v_1\overrightarrow{P'}v_2$ (or $v_2\overleftarrow{P'}v_1$) for the segment of $P'$ between two vertices $v_1, v_2 \in V(P')$ with the same (opposite) orientation with respect to the orientation of $P'$. If a vertex $w_1$ precedes a vertex $w_2$ on $P'$ in this orientation, we also write $w_1 \prec w_2$. The blue edges of $P'(a,b)$ are denoted $e_1 = y_1z_1,\ e_2 = y_2z_2, \ldots, e_k = y_kz_k$, occurring on $P'$ in this order from $a$ to $b$ (i.e., $z_i = y_i^+$ and $z_i \prec y_{i+1}$ or $z_i = y_{i+1}$). Finally, whenever vertices of a claw are listed, its *center* (i.e., the only vertex of degree 3) is always the first vertex of the list.

**Lemma 3.1.** *Let $x$ be an eligible vertex of a claw-free graph $G$ and let $G'_x$ be the local completion of $G$ at $x$. Then for every pair of distinct vertices $a$ and $b$ of $G$ and for every longest $a,b$-path $R(a,b)$ in $G'_x$ there is an $a,b$-path $P'(a,b)$ in $G'_x$ such that $V(P') = V(R)$ and $P'$ contains at most one edge of $E(G'_x) \setminus E(G)$.*

**Proof.** Let $P'(a,b)$ be an $a,b$-path in $G'_x$ such that $V(P') = V(R)$ and the number $k$ of blue edges of $P'(a,b)$ is minimum.

<u>Claim 1.</u> Let $e_i,\ e_j \in E(P')$ be two blue edges with $i < j$. Then

$(i)$ $y_iy_j \notin E(G)$ and $z_iz_j \notin E(G)$,

$(ii)$ if $x \prec y_i$, then $x^+z_i \notin E(G)$,

$(iii)$ if $x \prec y_i$ and $x \neq a$, then $x^-y_i \notin E(G)$.

Similarly, if $z_i \prec x$ and $x \neq b$, then $x^+z_i \notin E(G)$.

<u>Proof of Claim 1.</u>

$(i)$ Suppose that at least one of $y_iy_j,\ z_iz_j$ is in $E(G)$. Then the path $P''(a,b) := a\overrightarrow{P'}y_iy_j\overleftarrow{P'}z_iz_j\overrightarrow{P'}b$ has the same vertex set as $R(a,b)$ and at least one blue edge less than $P'(a,b)$.

$(ii)$ If $x^+z_i \in E(G)$, then the path $P''(a,b) := a\overrightarrow{P'}xy_i\overleftarrow{P'}x^+z_i\overrightarrow{P'}b$ contradicts the choice of $P'$.

(*iii*) If $x^-y_i \in E(G)$, then the path $P''(a,b) := a\overrightarrow{P'}x^-y_i\overleftarrow{P'}xz_i\overrightarrow{P'}b$ contradicts the choice of $P'$. $\square$

Suppose now that $P'(a,b)$ contains at least two blue edges $y_1z_1$ and $y_2z_2$. If $x \prec y_1 \prec y_2$, then, by Claim 1(*i*), $z_1z_2 \notin E(G)$, by Claim 1(*ii*), $x^+z_1 \notin E(G)$ and $x^+z_2 \notin E(G)$, and hence $\langle x, x^+, z_1, z_2\rangle \simeq K_{1,3}$. Similarly we cannot have $z_1 \prec z_2 \prec x$ and hence $z_1 \prec x \prec y_2$. If $y_1x^+ \in E(G)$, then the path $P''(a,b) := a\overrightarrow{P'}y_1x^+\overrightarrow{P'}y_2x^-\overleftarrow{P'}z_1xz_2\overrightarrow{P'}b$ contains the blue edge $y_2x^-$ but avoids $y_1z_1$ and $y_2z_2$, which contradicts the choice of $P'$. Hence $y_1x^+ \notin E(G)$. Since, by Claim 1(*iii*), also $x^+z_1 \notin E(G)$, $\langle x, y_1, z_1, x^+\rangle \simeq K_{1,3}$, a contradiction. $\blacksquare$

**Proposition 3.2.** *Let $x$ be an eligible vertex of a claw-free graph $G$, $G'_x$ the local completion of $G$ at $x$, and $a$, $b$ two distinct vertices of $G$. Then for every longest $a,b$-path $P'(a,b)$ in $G'_x$ there is a path $P$ in $G$ such that $V(P) = V(P')$ and $P$ admits at least one of $a$, $b$ as an endvertex. Moreover, there is an $a,b$-path $P(a,b)$ in $G$ such that $V(P) = V(P')$ except perhaps in each of the following two situations (up to symmetry between $a$ and $b$):*

- (*i*) *There is an induced subgraph $H \subset G$ isomorphic to the graph $S$ in Fig. 3 such that both $a$ and $x$ are vertices of degree 4 in $H$. In this case $G$ contains a path $P_b$ such that $b$ is an endvertex of $P$ and $V(P_b) = V(P')$. If, moreover, $b \in V(H)$, then $G$ contains also a path $P_a$ with endvertex $a$ and with $V(P_a) = V(P')$.*
- (*ii*) *$x = a$ and $ab \in E(G)$. In this case there is always both a path $P_a$ in $G$ with endvertex $a$ and with $V(P_a) = V(P')$ and a path $P_b$ in $G$ with endvertex $b$ and with $V(P_b) = V(P')$.*

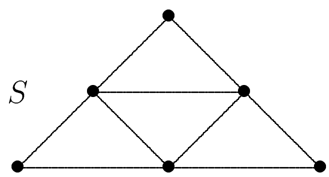

Figure 3

**Proof.** Let, to the contrary, $P'(a,b)$ be a longest $a,b$-path in $G'_x$ such that there is no $a,b$-path $P(a,b)$ in $G$ with $V(P) = V(P')$. We show that then one of the cases (*i*), (*ii*) occurs and, in each of these cases, there are paths $P_a$ and $P_b$ in $G$ with the required properties.

If $k = 0$, then $P'$ is a black path; hence $k \geq 1$. By the maximality of $P'$ and since $\langle N_{G'}(x)\rangle$ is a clique, we have $x \in V(P')$ and $N_G(x) = N_{G'}(x) \subset V(P')$. By Lemma 3.1, we can suppose that $P'$ contains exactly one blue edge $e = yz$ and, without loss of generality, $x \prec y$. We denote by $B_a := a\overrightarrow{P'}x^-$, $B := x^+\overrightarrow{P'}y$ and $B_b := z\overrightarrow{P'}b$ the three subpaths of $P'$ obtained by removing the vertex $x$ and the blue edge $yz$ (note that we do not exclude the possible special cases $y = x^+$, i.e. $V(B) = \{x^+\}$, and $x = a$, i.e. $V(B_a) = \emptyset$).

By Claim 1$(ii)$, we have $x^+z \notin E(G)$. If $y \neq x^+$, then $x^+y \in E(G)$ (otherwise $\langle x, x^+, y, z\rangle$ is a claw). If $x \neq a$, then $x^-x^+ \notin E(G)$ (otherwise $P = a\overrightarrow{P'}x^-x^+\overrightarrow{P'}yxz\overrightarrow{P'}b$ is a black $a,b$-path), $x^-y \notin E(G)$ (by Claim 1$(iii)$) and $x^-z \in E(G)$ (otherwise $\langle x, x^-x^+, z\rangle$ is a claw).

Since $\langle N_G(x)\rangle$ is connected, there is a path $Q$ in $\langle N_G(x)\rangle$ joining one of the vertices $x^+, y$ to one of the vertices $x^-, z$ (or, if $V(B_a) = \emptyset$, to $z$). Suppose that $P'(a,b)$ is chosen such that, among all $a,b$-paths in $G'_x$ with vertex set $V(P')$, containing exactly one blue edge, the path $Q$ is shortest possible. We may assume that $Q$ starts at $x^+$ since otherwise we can replace $P'$ by the path $P'' = a\overrightarrow{P'}xy\overleftarrow{P'}x^+z\overrightarrow{P'}b$ with the desired properties. Denote the vertices of $Q$ by $x_0, x_1, \ldots, x_\ell$, where $x_0 = x^+$ and $x_\ell \in \{x^-, z\}$. Since $N_G(x) \subset V(P')$, $\{x_0, x_1, \ldots, x_\ell\} \subset V(P')$. Since $x^+$ is adjacent neither to $x^-$ nor to $z$, $\ell \geq 2$. On the other hand, since $Q$ is induced in $\langle N_G(x)\rangle$ and $\langle x, x_0, x_2, x_4\rangle$ cannot be a claw, $\ell \leq 3$. Note that if $\ell = 3$ (i.e. if $|V(Q)| = 4$), then $yx_1 \in E(G)$, for otherwise $\langle x, x_1, x_\ell, y\rangle$ is a claw. We can suppose that $x_1$ (i.e. the second vertex of $Q$) belongs to $B$ or to $B_b$ (otherwise, i.e. if $x_1 \in V(B_a)$, we can replace $P'$ by the path $P'' = b\overleftarrow{P'}zxx^+\overrightarrow{P'}yx^-\overleftarrow{P'}a$ in $G'_x$, interchange the notation of $a$ and $b$, and using the fact that the statement of Proposition 3.2 is symmetric with respect to $a$ and $b$, transform this case to the case $x_1 \in V(B_b)$). Recall that $x_1 \in N(x) \setminus \{x^+, y, z, x^-\}$.

<u>Case 1:</u> $x_1 \in V(B) \setminus \{x^+, y\}$. We consider $\langle x_1, x_1^-, x, x_1^+\rangle$.

If $xx_1^- \in E(G)$, then $P''(a,b) := a\overrightarrow{P'}x^-xx_1^-\overleftarrow{P'}x^+y\overleftarrow{P'}x_1z\overrightarrow{P'}b$ is an $a,b$-path in $G'_x$ (even if $x = a$), with $V(P'') = V(P')$, and $P''$ is either a black path or it contains one blue edge $x_1z$. In the last case, the deletion of the vertex $x$ and the blue edge $x_1z$ from $P''$ yields the subpaths $B''_a = B_a := a\overrightarrow{P'}x^-$, $B'' = x_1^-\overleftarrow{P'}x^+y\overleftarrow{P'}x_1$ and $B''_b = B_b = z\overrightarrow{P'}b$, but the portion of $Q$ between $x_1$ (as an endvertex of $B''$) and $\{x^-, z\}$ (as the two endvertices of $B''_a$ and $B''_b$) is shorter than $Q$ in contradiction to the choice of $P'$. Hence $xx_1^- \notin E(G)$, which implies in particular $x_1 \neq x^{++}$.

If $xx_1^+ \in E(G)$, then $P''(a,b) := a\overrightarrow{P'}x^-xx_1^+\overrightarrow{P'}yx^+\overrightarrow{P'}x_1z\overrightarrow{P'}b$ is an $a,b$-path in $G'_x$ that is either black (if $x_1z \in E(G)$) or contradicts the minimality of $Q$. Hence $xx_1^+ \notin E(G)$.

If $x_1^-x_1^+ \in E(G)$, then similarly the $a,b$-path $P''(a,b) := a\overrightarrow{P'}x^-xx_1x^+\overrightarrow{P'}x_1^-x_1^+\overrightarrow{P'}yz\overrightarrow{P'}b$ in $G'_x$ contains one blue edge $yz$ and contradicts the minimality of $Q$. Hence $x_1^-x_1^+ \notin E(G)$.

Therefore $\langle x_1, x_1^-, x, x_1^+\rangle$ is a claw, a contradiction.

<u>Case 2:</u> $x_1 \in V(B_b) \setminus \{b, z\}$. We consider $\langle x_1, x_1^-, x^+, x_1^+\rangle$.

If $x_1^-x_1^+ \in E(G)$, then $P''(a,b) := a\overrightarrow{P'}xx_1x^+\overrightarrow{P'}yz\overrightarrow{P'}x_1^-x_1^+\overrightarrow{P'}b$ is an $a,b$-path in $G'_x$ (even if $x = a$), containing exactly one blue edge $yz$ and contradicting the minimality of $Q$. Therefore $x_1^-x_1^+ \notin E(G)$.

Suppose that $x^+x_1^- \in E(G)$. If $|V(Q)| = 4$, then we already know that $yx_1 \in E(G)$ and thus $P(a,b) := a\overrightarrow{P'}xz\overrightarrow{P'}x_1^-x^+\overrightarrow{P'}yx_1\overrightarrow{P'}b$ is a black $a,b$-path, a contradiction. Hence $|V(Q)| = 3$, i.e., $x_1$ is adjacent to at least one of $\{x^-, z\}$. If $x \neq a$, then $P(a,b) := a\overrightarrow{P'}x^-z\overrightarrow{P'}x_1^-x^+\overrightarrow{P'}yxx_1\overrightarrow{P'}b$ is a black $a,b$-path. If $x = a$, then $x^-$ does not exist, $x_1$ is adjacent to $z$, and $P(a = x, b) := xy\overleftarrow{P'}x^+x_1^-\overleftarrow{P'}zx_1\overrightarrow{P'}b$ is a black $a,b$-path. Therefore $x^+x_1^- \notin E(G)$.

Now, since $\langle x_1, x_1^-, x^+, x_1^+\rangle$ cannot be a claw, we have $x^+x_1^+ \in E(G)$. If $|V(Q)| = 4$, then again $yx_1 \in E(G)$ and $P(a,b) := a\overrightarrow{P'}xz\overrightarrow{P'}x_1y\overleftarrow{P'}x^+x_1^+\overrightarrow{P'}b$ is a black $a,b$-path. Hence $|V(Q)| = 3$ and thus $x_1$ is adjacent to one of $\{x^-, z\}$. If $x \neq a$, then $P(a,b) := a\overrightarrow{P'}x^-z\overrightarrow{P'}x_1xy\overleftarrow{P'}x^+x_1^+\overrightarrow{P'}b$ is a black $a,b$-path. Hence $x = a$, $x^-$ does not exist and $x_1$ is adjacent to $z$. The vertices $y$ and $x^+$ are distinct for otherwise the path $P(a = x, b) := xz\overrightarrow{P'}x_1x^+x_1^+\overrightarrow{P'}b$ is black. Now, $zx_1^+ \notin E(G)$, for otherwise the path $P(a = x, b) := xy\overleftarrow{P'}x^+x_1\overleftarrow{P'}zx_1^+\overrightarrow{P'}b$ is black. Similarly, $yx_1^+ \notin E(G)$, for otherwise the path $P(a = x, b) := xz\overrightarrow{P'}x_1x^+\overrightarrow{P'}yx_1^+\overrightarrow{P'}b$ is black. Hence, since $\langle x, x_1^+, y, z\rangle$ is not a claw, $xx_1^+ \notin E(G)$ and thus, since $\langle x_1, x_1^-, x_1^+, x\rangle$ is not a claw, $xx_1^- \in E(G)$. Finally, $yx_1^- \notin E(G)$ because of the black path $P(a = x, b) := xz\overrightarrow{P'}x_1^-y\overleftarrow{P'}x^+x_1\overrightarrow{P'}b$, and $yx_1 \notin E(G)$ because of the black path $P(a = x, b) := xz\overrightarrow{P'}x_1y\overleftarrow{P'}x^+x_1^+\overrightarrow{P'}b$. Hence $H = \langle x, x_1^-, x_1, x_1^+, x^+, y\rangle \simeq S$ and the vertex $a = x$ has degree 4 in $H$. We are thus in Situation $(i)$. In this case we can however construct a path $P_b$ in $G$ (e.g. $P_b := b\overleftarrow{P'}zxy\overleftarrow{P'}x^+$) such that $V(P_b) = V(P')$ and $b$ is an endvertex of $P_b$; if moreover $b = x_1^+ \in V(H)$, then $P_a := xy\overleftarrow{P'}x^+x_1^+\overleftarrow{P'}z$ is a path in $G$ with endvertex $a$ and with $V(P_a) = V(P')$.

<u>Case 3:</u> $x_1 = b$. Suppose first that $a = x$. Then, since $b = x_1 \in N_G(x)$, $ab \in E(G)$. We are thus in Situation $(ii)$ and the paths $P_a := ay\overleftarrow{P'}x^+b\overleftarrow{P'}z$ and $P_b := b\overleftarrow{P'}zxy\overleftarrow{P'}x^+$ have the required properties.

Hence we can suppose that $a \neq x$, i.e., the vertex $x^-$ exists. If $b^- = z$, then $P(a,b) := a\overrightarrow{P'}x^-zxy\overleftarrow{P'}x^+b$ is a black path, hence $b^- \neq z$. Further, $b^-x^- \notin E(G)$, since otherwise $P(a,b) := a\overrightarrow{P'}x^-b^-\overleftarrow{P'}zxy\overleftarrow{P'}x^+b$ is a black path, $b^-x \notin E(G)$ because of the black path $P(a,b) := a\overrightarrow{P'}x^-z\overrightarrow{P'}b^-xy\overleftarrow{P'}x^+b$ and $b^-x^+ \notin E(G)$ because of the black path $P(a,b) := a\overrightarrow{P'}x^-z\overrightarrow{P'}b^-x^+\overrightarrow{P'}yxb$. If $bx^- \in E(G)$, then, since also $x^-x^+ \notin E(G)$, $\langle b, b^-, x^-, x^+\rangle \simeq K_{1,3}$. Hence $bx^- \notin E(G)$.

Suppose now that $\ell = 2$. Then, since $b = x_1$ and $bx^- \notin E(G)$, $bz \in E(G)$. Since $\langle b, b^-, z, x^+\rangle$ cannot be a claw and obviously $zx^+ \notin E(G)$, we have $b^-z \in E(G)$. This implies that $H = \langle b, z, x, b^-, x^-, x^+\rangle \simeq S$ and $x$ and $b$ have degree 4 in $H$, i.e., we are (up to symmetry between $a$ and $b$) in Situation $(i)$. The black paths $P_a := a\overrightarrow{P'}x^-xy\overleftarrow{P'}x^+b\overleftarrow{P'}z$ and, if $a \in V(H)$ (i.e., $a = x^-$), also $P_b := bx^+\overrightarrow{P'}yxx^-z\overrightarrow{P'}b^-$, have the required properties.

We can thus suppose that $\ell = 3$, i.e., $V(Q) = \{x_0, x_1, x_2, x_3\}$ with $x_0 = x^+$ and $x_1 = b$. Recall that we already know that in this case $b = x_1$ is adjacent to both $x^+$ and $y$ and, by the same argument (i.e., by a claw centered at $x$), $x_2$ is adjacent to both $x^-$ and $z$. This implies that $x_2 \neq b^-$ (since $b^-x \notin E(G)$). Since $\langle b, b^-, x^+, x_2\rangle$ cannot be a claw, $b^-x^+ \notin E(G)$ (as we already know) and $x_2x^+ \notin E(G)$ (since $Q$ is an induced path), we have $x_2b^- \in E(G)$. This implies that $H = \langle b, x_2, x, b^-, x^-, x^+\rangle \simeq S$ and $b = x_1$ and $x$ are vertices of degree 4 in $H$, i.e., we are again (up to symmetry between $a$ and $b$) in Situation $(i)$. The black paths $P_a$ and $P_b$ are the same as in the case $\ell = 2$. ■

**Remarks. 1.** The graph in Figure 2 and the graph formed by two cliques sharing an edge $xb$ show that the presence of the situations $(i)$ and $(ii)$ can effectively prevent the existence of a path $P(a, b)$ in $G$ with $V(P) = V(P')$.

**2.** By Proposition 3.2, the length of a longest path in $G$ and in $G'_x$ are the same. This immediately implies Theorem 2.1.

We say that a vertex $a \in V(G)$ is a *simplicial vertex* if $\langle N_G(a)\rangle$ is complete.

**Corollary 3.3.** *Let $a$ be a simplicial vertex of a claw-free graph $G$. Then the maximum length of a path in $G$ with endvertex $a$ is the same as in* $\mathrm{cl}(G)$.

**Proof.** If $a$ is simplicial, then $a$ is not eligible and hence $x \neq a$. If there is a longest path $P'$ in $G'_x$ with endvertex $a$ such that there is no path $P$ in $G$ with $V(P) = V(P')$ having $a$ as an endvertex, then, by Proposition 3.2, $a$ has degree 4 in some induced subgraph $H \subset G$ isomorphic to $S$, which contradicts the simpliciality of $a$. ■

# 4. Pancyclicity is not stable

**Lemma 4.1.** *If a graph $G$ has diameter $r$ and contains a cycle of length $s > 2r+1$, then $G$ contains a cycle of length $\ell$ for some $s/2 \leq \ell < s$.*

**Proof.** Consider a shortest path $P$ in $G$ joining two antipodal vertices $u, v$ of the $s$-cycle $C$. Then some subpath of $P$ is a shortcut in $C$. This can be seen as follows:

Mark the vertices of $P$ which intersect $C$. Now start from one endpoint, say $u$, of $P$ and for any two consecutive marked vertices $x, y$ of $P$ mark the edges of a shortest path of $C$ joining $x$ to $y$. It is easily seen the the marked edges span a connected subgraph of $C$, containing $u$ and $v$. Since there are more marked edges in $C$ than edges in $P$, there must be two consecutive marked vertices $x_0, y_0$ of $P$, such that a shortest $(x_0, y_0)$-path in $C$ joining them has more edges than the $(x_0, y_0)$-path joining them in $P$. Replacing the path of $C$ by the one of $P$ gives the required shorter cycle. ■

**Proof of Theorem 2.4.** Consider the Ramanujan graphs created by Lubotzky, Phillips and Sarnak [6]. It was shown that for infinitely many $d$ there exist infinitely many $n$ such that there is a connected, vertex-transitive, $d$-regular graph on $n$ vertices of arbitrarily large girth $g$ and diameter $r \leq 3g + 1$. By a result of Mader [7], $d$-regular, connected, vertex-transitive graphs are $d$-edge-connected.

Choose an integer $d > \max\{3, k\}$ for which an infinite sequence of such Ramanujan graphs exists, and let $G$ be such a graph with $g(G) > d+2\binom{d}{2}$. Since $G$ is $d$-edge-connected, its line graph $L(G)$ is $d$-connected. Now modify $L(G)$ in the following way:

For every vertex $v \in V(G)$, subdivide twice in $L(G)$ every edge of the $K_d$ corresponding to the edges of $G$ incident with $v$, add $3g(G) - 2 - d - 2\binom{d}{2}$ additional vertices and add all possible edges between any two of the $3g(G) - 2 - d$ newly generated vertices.

The resulting graph $M(G)$ is claw-free and not pancyclic, since it cannot have a $(3g(G)-1)$-cycle (such a cycle cannot stay within any modified $K_d$, since it has $3g(G)-2$ vertices, so it must go around a cycle $C$ of $G$, which is impossible, since for every $v \in V(C)$ it picks up at least three edges in every modified $K_d$). On the other hand, for any collection of edge disjoint $u, v$-paths in $G$ there is a corresponding collection of vertex disjoint $u', v'$-paths in $M(G)$ where $u'$ is any vertex in the modified $K_d(u)$ and $v'$ any vertex different from $u'$ in the modified $K_d(v)$. Moreover, each modified $K_d(u)$ contains at least $d - 1 \geq k$ vertex disjoint paths between any pair $u'_1$, $u'_2$ of its vertices. Therefore $M(G)$ is $k$-connected.

Set $H = \mathrm{cl}(M(G))$. Then the modified $K_d$'s turn in $H$ into cliques of cardinality $3g(G) - 2$. It remains to prove that $H$ is pancyclic.

Let $S$ be an eulerian subgraph of $G$ and put $e(S) = |E(S)|$. Then $H$ contains cycles of all lengths in the interval $[e(S), e(S)+(3g(G)-d-2)|V(S)|]$. The lower bound follows from the fact that we have a cycle which picks up exactly the vertices from the line graph corresponding to the edges of $S$. Now we can start with such a cycle and include any number of added vertices in every modified $K_d$ belonging to a vertex of $S$ in the cycle. It is thus sufficient to show that $G$ contains a sequence of eulerian subgraphs of slowly decreasing orders.

First note that by a result of Jaeger [5], $G$ has a spanning eulerian subgraph $S_1$, since $G$ is 4-edge-connected. This, in particular, gives rise to a hamiltonian cycle in $H$. Starting with $S_1$, we will construct a sequence of eulerian subgraphs $S_1, S_2, \ldots, S_t$ of $G$ satisfying $|V(S_i)| \geq |V(S_{i+1})| \geq |V(S_i)|/2$ and $e(S_i) > e(S_{i+1})$, until we end with a cycle $S_t$ of length at most $6g(G)+3$. Indeed, if $S_i$ is not a cycle then fix an eulerian trail $T$ of $S_i$, i.e. a closed walk visiting every edge exactly once. Take a shortest closed subwalk $T'$ of $T$. Deleting the edges of $T'$, the graph $S_{i+1}$ spanned by the remaining edges is eulerian and has more than $|V(S_i)|/2$ vertices. So suppose that $S_i$ is a cycle. If $|V(S_i)| > 6g(G)+3$ then we can apply Lemma 4.1 to obtain a cycle $S_{i+1}$ of length $|V(S_i)| > |V(S_{i+1})| \geq |V(S_i)|/2$.

It is now straightforward to check that these eulerian subgraphs generate cycles of all lengths in the interval $[6g(G)+3, |V(H)|]$, since if $|V(S_{i+1})| \geq |V(S_i)|/2$, then $e(S_{i+1}) + (3g(G) - d - 2)|V(S_{i+1})| \geq (3g(G) - d - 1)|V(S_{i+1})| \geq (3g(G) - d - 1)|V(S_i)|/2 \geq (3g(G) - d - 1)e(S_i)/d \geq e(S_i)$. The remaining short cycles of lengths between 3 and $g(G)$ are obtained locally in a modified $K_d$ and those between $g(G)$ and $6g(G)+3$ from a cycle of length $g(G)$ in $G$. ■

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