Claw-free graphs with complete closure

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Abstract

We study some properties of the closure concept in claw-free graphs that was introduced by the first author. It is known that G is hamiltonian if and only if its closure is hamiltonian, but, on the other hand, there are infinite classes of non-pancyclic graphs with pancyclic closure. We show several structural properties of claw-free graphs with complete closure and their clique cutsets and, using these results, we prove that every claw-free graph on n vertices with complete closure contains a cycle of length $n-1$.

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1 Introduction

We refer to [1] for terminology and notation not defined here and consider only finite undirected graphs $G = (V(G), E(G))$ without loops and multiple edges.

If G is a graph and $M \subset V(G)$, then the induced subgraph of G on M will be denoted by $\langle M \rangle_G$. We will simply write $G - M$ for $\langle V(G) \setminus M \rangle_G$ and $G - x$ for $G - \{x\}$ (where $x \in V(G)$). We will denote by $n_G = |V(G)|$ the order of G and by $c(G)$ the circumference of G (i.e. the length of a longest cycle in G). A graph G is *hamiltonian* if $c(G) = n_G$ and G is *pancyclic* if G contains a cycle of any length ℓ, $3 \leq \ell \leq n_G$. By a *clique* we mean a (not necessarily maximal) complete subgraph of G. If $S \subset V(G)$ is a cutset of a connected graph G (i.e. $G - S$ is disconnected) such that $\langle S \rangle_G$ is a clique, we say that S is a *clique cutset* of G.

A graph G is *claw-free* if G does not contain a copy of *the claw* $K_{1,3}$ as an induced subgraph. Whenever we list vertices of an induced claw, its *center* (i.e. the only vertex of degree 3) is always the first vertex in the list.

If C is a cycle in G with a fixed orientation and $u, v \in V(C)$, then by $u \overrightarrow{C} v$ ($v \overleftarrow{C} u$) we denote the consecutive vertices on C from u to v in the same (opposite) orientation with respect to the given orientation of C. The predecessor and successor of a vertex v on C will be denoted by v^- and v^+, respectively.

For any $x \in V(G)$, the set $N_G(x) = \{y \in V(G) |\ xy \in E(G)\}$ is called the *neighborhood of x in G*. For a set $M \subset V(G)$ we let $N_G(M) = \cup_{x \in M} N_G(x)$. We say that a vertex $x \in V(G)$ is *locally connected* if $\langle N_G(x) \rangle_G$ is a connected graph; otherwise x is said to be *locally disconnected*. A locally connected vertex x is said to be *eligible* if $\langle N_G(x) \rangle_G$ is not a clique; otherwise we say that x is *simplicial*. The set of all locally connected (eligible, simplicial, locally disconnected) vertices of G will be denoted by $V_{LC}(G)$ ($V_{EL}(G), V_{SI}(G), V_{LD}(G)$), respectively. Thus, the sets $V_{EL}(G), V_{SI}(G), V_{LD}(G)$ are pairwise disjoint, $V_{EL}(G) \cup V_{SI}(G) = V_{LC}(G)$ and $V_{LC}(G) \cup V_{LD}(G) = V(G)$. If $V_{LC}(G) = V(G)$, we say that the graph G is locally connected.

Let $x \in V_{EL}(G)$ be an eligible vertex and let $B_x = \{uv |\ u, v \in N_G(x), uv \notin E(G)\}$. Denote by G'_x the graph $G'_x = (V(G), E(G) \cup B_x)$ (i.e., G'_x is obtained from G by adding to $\langle N_G(x) \rangle_G$ all missing edges). The graph G'_x is called the *local completion of G at x*. The following proposition shows that the local completion operation preserves the claw-freeness and the value of circumference of G

Proposition A [3]. *Let G be a claw-free graph and let $x \in V_{EL}(G)$ be an eligible vertex of G. Then*

(*i*) *the graph G'_x is claw-free,*

(*ii*) $c(G'_x) = c(G)$.

Apparently, if $x \in V_{EL}(G)$, then x becomes simplicial in G'_x and, if $V_{EL}(G'_x) \neq \emptyset$, the local completion operation can be applied repeatedly to another vertex. We thus obtain the following concept (introduced in [3]).

Let G be a claw-free graph. We say that a graph H is a closure of G, *denoted $H = \mathrm{cl}(G)$, if*

(i) *there is a sequence of graphs $G_1, \dots, G_t$ and vertices $x_1, \dots, x_{t-1}$ such that $G_1 = G$, $G_t = H$, $x_i \in V_{EL}(G_i)$ and $G_{i+1} = (G_i)'_{x_i}$, $i = 1, \dots, t-1$,*

(ii) *$V_{EL}(H) = \emptyset$.*

The following result summarizes basic properties of the closure operation.

Theorem B [3]. *Let G be a claw-free graph. Then*

(i) *the closure $\mathrm{cl}(G)$ is well-defined,*

(ii) *there is a triangle-free graph H such that $\mathrm{cl}(G)$ is the line graph of H,*

(iii) *$c(G) = c(\mathrm{cl}(G))$.*

Remarks. 1. Part (i) of Theorem B says that $\mathrm{cl}(G)$ is uniquely determined, i.e., does not depend on the order of eligible vertices used during the construction.

2. It is easy to see that $\mathrm{cl}(G)$ can be equivalently characterized as the minimum graph containing G, which does not contain an induced subgraph isomorphic to the diamond $(K_4 - e)$.

Specifically, by part (iii) of Theorem B, a claw-free graph G is hamiltonian if and only if $\mathrm{cl}(G)$ is hamiltonian. On the other hand, the following theorem shows that this is not the case with the property of pancyclicity.

Theorem C [2]. *For every $k \geq 2$ there is a k-connected claw-free graph G such that G is not pancyclic but $\mathrm{cl}(G)$ is pancyclic.*

An example of an infinite family of such graphs for $k = 2$ is shown in Figure 1. The graph in Figure 1 is, moreover, an example of a nonpancyclic graph having a complete (and hence pancyclic) closure. This situation gives rise to the following question.

Problem. *Determine the maximum number $c_m(n)$ of cycle lengths that can be missing in a claw-free graph on n vertices with complete closure.*

Let $k \geq 1$ and let G be the graph in Figure 1 of order $n_G = 6k + 3$. Then G is claw-free, $\mathrm{cl}(G)$ is complete and G contains no cycle of length ℓ for $2k + 3 \leq \ell \leq 3k + 2$, i.e. G misses $k = (n_G - 3)/6$ cycle lengths. This example shows that $c_m(n) \geq (n - 3)/6$.

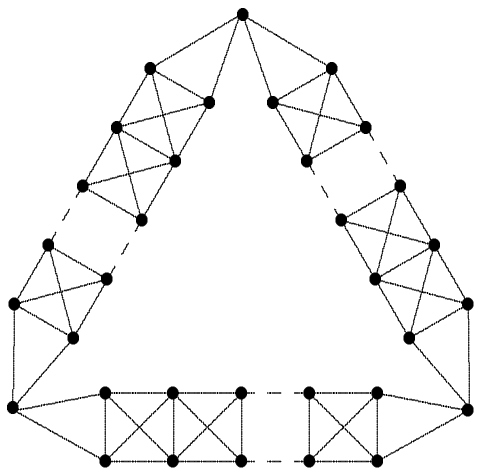

Figure 1

On the other hand, it is easy to see that a claw-free graph with complete closure on at least 4 vertices can miss neither a C_3 nor a C_4. Also, the main result of Section 3 shows that such a graph G cannot be missing a cycle of length $n_G - 1$.

More is likely to be true. No example is known when G has complete closure and large order but fails to contain one of all possible "short length" and "long length" cycles. We state this precisely as the following conjecture.

Conjecture. *Let c_1, c_2 be fixed constants. Then for large n, any claw-free graph G of order n whose closure is complete contains cycles C_i for all i, where $3 \leq i \leq c_1$ and $n - c_2 \leq i \leq n$.*

In Section 2 we prove several structural results about graphs with a clique cutset and their closures. In Section 3 we use these results to prove that every claw-free graph G with complete closure has a cycle of length $n_G - 1$.

2 Closure and clique cutsets

We begin with several simple observations.

Proposition 1. *Let G be a claw-free graph. Then $V_{SI}(G) \subset V_{SI}(\mathrm{cl}(G))$.*

Proof. It is sufficient to show that, for any $x \in V_{EL}(G)$, $V_{SI}(G) \subset V_{SI}(G'_x)$. Let $y \in V_{SI}(G)$. If $xy \notin E(G)$, then no edge in B_x contains y and hence $N_{G'_x}(y) = N_G(y)$. If

$xy \in E(G)$, then, since $\langle N_G(y)\rangle_G$ is a clique, $N_G(y) \subset N_G(x) \cup \{x\}$ and hence $\langle N_{G'_x}(y) \cup \{y\}\rangle_{G'_x} = \langle N_{G'_x}(x) \cup \{x\}\rangle_{G'_x}$. In both cases, $y \in V_{SI}(G'_x)$. ■

Corollary 2. *For any claw-free graph G, the closure $\mathrm{cl}(G)$ is constructed in at most $n_G = |V(G)|$ local completions.* ■

Proposition 3. *Let G be a claw-free graph and let H be an induced subgraph of G. Then $V_{EL}(H) \subset V_{EL}(G)$.*

Proof. Let $x \in V_{EL}(H)$ and let $z_1, z_2 \in N_H(x)$ be nonadjacent in $\langle N_H(x)\rangle_H$. If $x \in V_{SI}(G)$, then $z_1z_2 \in E(G)$, implying $z_1z_2 \in E(H)$, a contradiction. If $x \in V_{LD}(G)$, then, since x is eligible in H, the vertices z_1, z_2 are in the same component of $\langle N_G(x)\rangle_G$ and $z_1z_2 \notin E(G)$, but then, for any vertex z lying in the second component of $\langle N_G(x)\rangle_G$, $\langle\{x, z, z_1, z_2\}\rangle_G$ is a claw in G, which is again a contradiction. Hence $x \in V_{EL}(G)$. ■

Corollary 4. *Let H be an induced subgraph of a claw-free graph G. Then $\mathrm{cl}(H) \subset \langle V(H)\rangle_{\mathrm{cl}(G)}$.*

Proof. Let $H_1, \ldots, H_s$ and $x_1, \ldots, x_{s-1}$ be the sequences of graphs and corresponding eligible vertices that yield $\mathrm{cl}(H)$ (i.e., $H_1 = H$, $H_s = \mathrm{cl}(H)$, $x_j \in V_{EL}(H_j)$ and $H_{j+1} = (H_j)'_{x_j}$, $j = 1, \ldots, s-1$). By Proposition 3, $x_1 \in V_{EL}(G)$ and we can let $G_2 = G'_{x_1}$. Note that H_2 is an induced subgraph of G_2. By induction (and by Proposition 3), $x_j \in V_{EL}(G_j)$ and we can let $G_{j+1} = (G_j)'_{x_j}$, $j = 2, \ldots, s-1$. Then $\mathrm{cl}(H) = \langle V(H)\rangle_{G_s}$. Since $\mathrm{cl}(G)$ is independent of the order of eligible vertices used during the construction, there are vertices $x_{s+1}, \ldots, x_t \in V(G)$ such that the sequence of local completions of G at $x_1, \ldots, x_s, x_{s+1}, \ldots, x_t$ yields $\mathrm{cl}(G)$. Hence we have $\mathrm{cl}(H) = \langle V(H)\rangle_{G_s} \subset \langle V(H)\rangle_{G_t} = \langle V(H)\rangle_{\mathrm{cl}(G)}$. ■

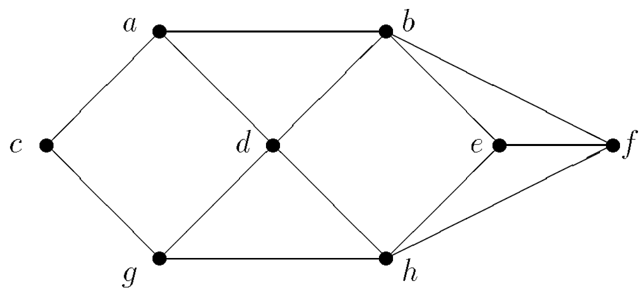

Figure 2

Example. Let G be the graph in Figure 2 and let $H = \langle\{a, c, d, g\}\rangle_G \subset G$. Then $\mathrm{cl}(H) \simeq C_4$, while $\langle V(H)\rangle_{\mathrm{cl}(G)} \simeq K_4$. Thus, it is possible that $\mathrm{cl}(H)$ is a proper subgraph of $\langle V(H)\rangle_{\mathrm{cl}(G)}$.

The following theorem is the main result of this section, giving structural information of the closure of the whole graph G in terms of the closures of its corresponding parts. Its corollaries will be useful in the next section for decomposition of $\mathrm{cl}(G)$ by means of clique cutsets.

Theorem 5. *Let $S \subset V(G)$ be a clique cutset of a claw-free graph G and let H_i, $i = 1, \ldots, k$, be the components of $G - S$. For $i = 1, \ldots, k$ let $S_i = N_G(V(H_i)) \cap S$ and $G_i = \langle V(H_i) \cup S_i \rangle_G$. Let $I_0 = \{i | \, |S_i| = 1\}$ and $S_0 = \cup_{i \in I_0} S_i$. Then*
(i) $V_{LD}(\mathrm{cl}(G)) = (\cup_{i=1}^k V_{LD}(\mathrm{cl}(G_i))) \cup S_0$,
(ii) $\mathrm{cl}(G_i) = \langle V(G_i) \rangle_{\mathrm{cl}(G)}$.

Proof. Let K^i be the largest clique in $\mathrm{cl}(G_i)$ containing the clique $\langle S_i \rangle_G$, $i = 1, \ldots, k$. Then, for every i and every $x \in V(K^i)$, either $\langle N_{\mathrm{cl}(G_i)}(x) \rangle_{\mathrm{cl}(G_i)} = K^i - x$ (and $x \in V_{SI}(\mathrm{cl}(G_i))$), or $\langle N_{\mathrm{cl}(G_i)}(x) \rangle_{\mathrm{cl}(G_i)}$ consists of two disjoint cliques, one of them being $K^i - x$ (and then $x \in V_{LD}(\mathrm{cl}(G_i))$). Let $\tilde{G}$ be the graph obtained by taking a copy of each $\mathrm{cl}(G_i)$ and a copy of $\langle S \rangle_G$ and by identifying the vertices of every S_i with the corresponding vertices of S, $i = 1, \ldots, k$. By Corollary 4, $\tilde{G} \subset \mathrm{cl}(G)$. Note that $\tilde{G}$ can contain induced claws centered at vertices of S (for example, if $S_1 = \{a_1, a_2, a_3\}$, $\{b_1, b_2\} \subset V(H_1)$, $N_S(b_1) = \{a_1\}$ and $N_S(b_2) = \{a_2, a_3\}$, then we get $a_1 b_2 \in E(\mathrm{cl}(G))$ and, if $b_1 b_2 \notin E(\mathrm{cl}(G))$, then $\langle \{a_1, b_1, b_2, x\} \rangle_{\tilde{G}}$ is a claw for any $x \in S \setminus S_1$). It is straightforward to check that if $|S_{i_0}| = 1$ for some $i_0 \in I_0$, then $S_{i_0} \subset V_{LD}(\mathrm{cl}(G))$ and $V_{LD}(\mathrm{cl}(G_{i_0})) \cup S_{i_0} = V_{LD}(\mathrm{cl}(G)) \cap V(G_{i_0})$, and hence it is sufficient to verify the theorem in $G - V(H_{i_0})$. Hence we can suppose that $|S_i| \geq 2$ for every $i = 1, \ldots, k$. Then the subgraph $\langle S \cup (\cup_{i=1}^k V(K^i)) \rangle_{\tilde{G}}$ is locally connected. Let $\hat{G}$ be the graph obtained from $\tilde{G}$ by adding to $\langle S \cup (\cup_{i=1}^k V(K^i)) \rangle_{\tilde{G}}$ all missing edges (i.e., the subgraph $K = \langle S \cup (\cup_{i=1}^k V(K^i)) \rangle_{\hat{G}}$ is a clique). Since $\tilde{G} \subset \mathrm{cl}(G)$ and $\langle S \cup (\cup_{i=1}^k V(K^i)) \rangle_{\tilde{G}}$ is locally connected, $\hat{G} \subset \mathrm{cl}(G)$. By the construction, it is now straightforward to verify the following facts:

(*a*) $\hat{G}$ is claw-free,

(*b*) if $x \in V(G_i) \setminus V(K)$, then $\langle N_{\mathrm{cl}(G_i)}(x) \rangle_{\mathrm{cl}(G_i)} = \langle N_{\hat{G}}(x) \rangle_{\hat{G}}$,

(*c*) if $x \in V(K^i) \setminus S$ for some $i = 1, \ldots, k$, then

- (α) if $x \in V_{SI}(\mathrm{cl}(G_i))$, then $\langle N_{\hat{G}}(x) \rangle_{\hat{G}} = K - x$ and hence $x \in V_{SI}(\hat{G})$, and
- (β) if $x \in V_{LD}(\mathrm{cl}(G_i))$, then one component of $\langle N_{\hat{G}}(x) \rangle_{\hat{G}}$ is $K - x$ and the other component is the same in $\mathrm{cl}(G_i)$ and in $\hat{G}$, and hence $x \in V_{LD}(\hat{G})$,

(*d*) if $x \in S$, then $x \in V_{LD}(\mathrm{cl}(G_i))$ for at most one i, $1 \leq i \leq k$, since if $x \in V_{LD}(\mathrm{cl}(G_{i_1})) \cap V_{LD}(\mathrm{cl}(G_{i_2}))$ for some i_1, i_2 with $1 \leq i_1 < i_2 \leq k$, then x centers a claw in $\hat{G}$, contradicting (a), and

(α) if $x \in V_{SI}(\mathrm{cl}(G_i))$ for all $i = 1, \ldots, k$, for which $x \in V(G_i)$, then $x \in V_{SI}(\hat{G})$,

(β) if there is an i_0, $1 \leq i_0 \leq k$, such that $x \in V_{LD}(\mathrm{cl}(G_{i_0}))$, then $x \in V_{LD}(\hat{G})$.

(Note that ($d\alpha$) includes the case when $x \notin \cup_{i=1}^{k} V(G_i)$). This immediately implies that $V(\hat{G}) = V_{SI}(\hat{G}) \cup V_{LD}(\hat{G})$, i.e., $V_{EL}(\hat{G}) = \emptyset$. Since $\hat{G} \subset \mathrm{cl}(G)$, we have $\hat{G} = \mathrm{cl}(G)$, and by (b), ($c\beta$) and ($d\beta$), $V_{LD}(\hat{G}) = \cup_{i=1}^{k} V_{LD}(\mathrm{cl}(G_i))$.

Proof of part (ii) follows immediately from the construction of $\hat{G} = \mathrm{cl}(G)$. ■

Example. Let G be the graph in Figure 2 and put $S = \{b, h\}$, $G_1 = \langle\{a, b, c, d, g, h\}\rangle_G$, $G_2 = \langle\{b, e, f, h\}\rangle_G$. Then $V_{LD}(\mathrm{cl}(G_1)) = \{a, c, d, g\}$, but $V_{LD}(\mathrm{cl}(G)) = \emptyset$. This example shows that Theorem 5 fails if $\langle S\rangle_G$ is not a clique.

Corollary 6. *Let G be a claw-free graph and let $S \subset V(G)$ be a clique cutset of G. Denote by $H_1, \ldots, H_k$ the components of $G - S$, let $S_i = N_G(V(H_i)) \cap S$ and let $G_i = \langle V(H_i) \cup S_i\rangle_G$. Suppose that $|S_i| \geq 2$, $i = 1, \ldots, k$. Then $\mathrm{cl}(G)$ is complete if and only if $\mathrm{cl}(G_i)$ is complete for every $i = 1, \ldots, k$.*

Proof. If $\mathrm{cl}(G)$ is complete, then all $\mathrm{cl}(G_i)$ are complete by part (ii) of Theorem 5. Conversely, suppose that all $\mathrm{cl}(G_i)$ are complete and let K^i, K, $\tilde{G}$ and $\hat{G}$ be the same as in the proof of Theorem 5. Then $K^i = G_i$, $\tilde{G}$ is locally connected and $\hat{G} = \mathrm{cl}(G) = K$. ■

Corollary 7. *Let G be a claw-free graph and let $x \in V_{SI}(G)$. Then $\mathrm{cl}(G)$ is complete if and only if $\mathrm{cl}(G - x)$ is complete.*

Proof. If $x \in V_{SI}(G)$, then $\langle N_G(x)\rangle_G$ is a clique cutset. The rest of the proof follows immediately from Corollary 6 by setting $S = N_G(x)$. ■

3 Cycle of length $n_G - 1$

In the main result of this section, Theorem 12, we prove that every claw-free graph G with complete closure contains a cycle of length $n_G - 1$. Before we present this result, we first prove several auxiliary statements. The first of them is of importance in its own right.

We say that a set $S \subset V(G)$ is *cyclable in G* if there is a cycle $C \subset G$ such that $V(C) = S$.

Theorem 8. *Let G be a claw-free graph and let $G_0, G_1, \ldots, G_t$, $t \geq 1$, be a sequence of graphs such that $G_0 = G$ and $G_i = (G_{i-1})'_{x_{i-1}}$ for some $x_{i-1} \in V_{EL}(G_{i-1})$, $i = 1, \ldots, t$. Let $B_i = E(G_i) \setminus E(G_{i-1})$ $(i = 1, \ldots, t)$ and $B_0 = E(G_0)$. For every cycle $C \subset G_t$ set $b_i(C) = |E(C) \cap B_i|$, $i = 0, 1, \ldots, t$. Then for every cyclable set S in G_t there is a cycle C in G_t with $V(C) = S$ such that*

(i) $b_i(C) \leq 2$ for every $i = 1, \ldots, t$,

(ii) if $x_{i-1}x_i \in E(G_{i-1})$ and $b_{i+1}(C) \geq 1$, then $b_i(C) \leq 1$ $(1 \leq i \leq t-1)$.

Proof. Since every edge $e \in E(G_t)$ is in exactly one B_k $(0 \leq k \leq t)$, we can define a weight function $w(e)$ on $E(G_t)$ by $w(e) = k$ if $e \in B_k$. For any cycle $C \subset G_t$ we define the weight of C by $w(C) = \sum_{e \in E(C)} w(e)$. Let $S \subset V(G)$ be cyclable in G_t and let C be a cycle in G_t such that $V(C) = S$ and $w(C)$ is as small as possible.

(i) Let, to the contrary, $b_i(C) \geq 3$ for some i, $1 \leq i \leq t$, and let e_1, e_2, e_3 be distinct edges in $E(C) \cap B_i$. Let $e_j = u_jv_j$ $(1 \leq j \leq 3)$, and assume the notation is chosen such that u_1, v_1, u_2, v_2, u_3 and v_3 appear in this order along C. Then u_1, u_2, u_3 are distinct vertices in $N_{G_{i-1}}(x_{i-1})$. Since $\langle \{x_{i-1}, u_1, u_2, u_3\} \rangle_{G_{i-1}}$ cannot be an induced claw, $\{u_1u_2, u_1u_3, u_2u_3\} \cap E(G_{i-1}) \neq \emptyset$. By symmetry, we can suppose that $u_1u_2 \in E(G_{i-1})$. Let $C' = v_2 \overrightarrow{C} u_1u_2 \overleftarrow{C} v_1v_2$. Then C' is a cycle in G_i with $V(C') = V(C) = S$ (recall that $v_1v_2 \in V(G_i)$ since $v_1, v_2 \in N_{G_{i-1}}(x_{i-1})$), and $E(C') = E(C) \setminus \{u_1v_1, u_2v_2\} \cup \{u_1u_2, v_1v_2\}$. By the assumption, $w(u_1v_1) = w(u_2v_2) = i$. On the other hand, since $u_1u_2 \in E(G_{i-1})$ and $v_1v_2 \in E(G_i)$, $w(u_1u_2) \leq i-1$ and $w(v_1v_2) \leq i$. Therefore, we have $w(C') \leq w(C) - (i+i) + (i-1+i) = w(C) - 1$, contradicting the minimality of C.

(ii) Assume that $b_i(C) \geq 2$ and $b_{i+1}(C) \geq 1$. Let $e_1, e_2 \in E(C) \cap B_i$, $e_1 \neq e_2$, setting $e_j = u_jv_j$ $(j = 1, 2)$ and let $e = uv \in E(C) \cap B_{i+1}$. Suppose that the notation is chosen such that u, v, u_1, v_1, u_2 and v_2 appear in this order along C. By the definition, $\{u_1, v_1, u_2, v_2\} \subset N_{G_{i-1}}(x_{i-1})$ and $\{u, v\} \subset N_{G_i}(x_i)$. Apparently, $u_1 \neq u_2$. If $u_1u_2 \in E(G_{i-1})$, then let $C' = v_2 \overrightarrow{C} u_1u_2 \overleftarrow{C} v_1v_2$. Then C' is a cycle in G_t with $V(C') = V(C) = S$ and $E(C') = E(C) \setminus \{u_1v_1, u_2v_2\} \cup \{u_1u_2, v_1v_2\}$. Since $w(u_1v_1) = w(u_2v_2) = i$, $w(u_1u_2) \leq i-1$ and $w(v_1v_2) \leq i$, we have $w(C') \leq w(C) - 2i + 2i - 1 = w(C) - 1$, a contradiction. Therefore, $u_1u_2 \notin E(G_{i-1})$. Similarly, $v_1v_2 \notin E(G_{i-1})$.

Next consider u and u_1. Apparently $u \neq u_1$, and we show that $uu_1 \notin E(G_{i-1})$. Let $uu_1 \in E(G_{i-1})$ and set $C' = v_1 \overrightarrow{C} uu_1 \overleftarrow{C} vv_1$. First suppose $v_1 \neq x_i$. Then, since $v_1, x_i \in N_{G_{i-1}}(x_{i-1})$, we have $v_1x_i \in E(G_i)$. Since $v \neq v_1$, this implies $vv_1 \in E(G_{i+1})$. Hence C' is a cycle in $G_{i+1} \subset G_t$ with $V(C') = V(C) = S$ and with $E(C') = E(C) \setminus \{uv, u_1v_1\} \cup \{uu_1, vv_1\}$. Since $w(uv) = i+1$, $w(u_1v_1) = i$, $w(uu_1) \leq i-1$ and $w(vv_1) \leq i+1$, we have $w(C') \leq w(C) - i - (i+1) + (i-1) + (i+1) = w(C) - 1$, a contradiction. Let thus $v_1 = x_i$. Then $vv_1 = vx_i \in E(G_i)$, and since again $E(C') = E(C) \setminus \{uv, u_1v_1\} \cup \{uu_1, vv_1\}$ and $w(uv) = i+1$, $w(u_1v_1) = i$, $w(uu_1) \leq i-1$ and

$w(vv_1) \le i$, we obtain $w(C') \le w(C) - i - (i+1) + (i-1) + i = w(C) - 2$, which is again a contradiction. Hence $uu_1 \notin E(G_{i-1})$. Similarly, $uu_2 \notin E(G_{i-1})$, $vv_1 \notin E(G_{i-1})$ and $vv_2 \notin E(G_{i-1})$. Hence $\{u, u_1, u_2\}$ and $\{v, v_1, v_2\}$ are independent sets in G_{i-1}. This implies that $x_{i-1}u \notin E(G_{i-1})$ (since otherwise $\langle\{x_{i-1}, u, u_1, u_2\}\rangle_{G_{i-1}}$ is a claw) and hence $x_i u \notin B_i$, which implies $x_i u \in E(G_{i-1})$. Similarly we have $x_{i-1}v \notin E(G_{i-1})$ and $x_i v \in E(G_{i-1})$. Since $u_1 x_{i-1} \in E(G_{i-1})$ but $u_1 u \notin E(G_{i-1})$, we have $x_{i-1} \neq u$, and similarly $x_{i-1} \neq v$, but then $\langle\{x_i, x_{i-1}, u, v\}\rangle_{G_{i-1}}$ is a claw. This contradiction proves the theorem. ■

Let C be a cycle in a graph G. An edge $uv \in E(G) \setminus E(C)$ with $u, v \in E(C)$ will be called a *chord of* C. A *2-chord* of a cycle C is a chord xy of C such that $x \overrightarrow{C} y$ or $x \overleftarrow{C} y$ has exactly one interior vertex. If $u_1v_1, u_2v_2 \in E(G) \setminus E(C)$ are such that $u_1, v_1 \in V(C)$ and either $\{u_2, v_2\} = \{u_1^-, v_1^+\}$ or $\{u_2, v_2\} = \{u_1^+, v_1^-\}$, then we say that the edges u_1v_1 and u_2v_2 are a *pair of parallel chords of* C.

Lemma 9. *Let G be a claw-free graph on n_G vertices such that $\mathrm{cl}(G)$ is complete and G has no cycle of length $n_G - 1$. Let C be a hamiltonian cycle in G and let $xy \in E(G) \setminus E(C)$ be a chord of C. Then there is a pair of parallel chords uv, u^-v^+ of C such that $x \in \{u^-, u\}$ and $y \in \{v, v^+\}$.*

Proof. Since G has no cycle of length $n_G - 1$, C has no 2-chord, and hence all the vertices x^-, x^+, y^-, y^+ exist and are distinct. Since $\langle\{x, x^-, x^+, y\}\rangle_G$ cannot be a claw, we have $x^-y \in E(G)$ or $x^+y \in E(G)$; from $\langle\{y, y^-, y^+, x\}\rangle_G \not\simeq K_{1,3}$ similarly $xy^- \in E(G)$ or $xy^+ \in E(G)$. If $x^-y \in E(G)$ and $xy^- \in E(G)$ or $x^+y \in E(G)$ and $xy^+ \in E(G)$, then we are done; thus suppose that $x^-y \in E(G)$ and $xy^+ \in E(G)$ or $x^+y \in E(G)$ and $xy^- \in E(G)$. In the first case, since $x^-y^- \notin E(G)$ (otherwise $xy^+ \overrightarrow{C} x^-y^- \overleftarrow{C} x$ is a cycle of length $n_G - 1$), from $\langle\{y, y^-, y^+, x^-\}\rangle_G \not\simeq K_{1,3}$ we get $x^-y^+ \in E(G)$. The second case is symmetric. ■

Lemma 10. *Let G be a claw-free graph having no cycle of length $n_G - 1$. Let C be a hamiltonian cycle in G and $\{x, y\}$ a cutset of G such that $\langle\{x^-, x, y, y^+\}\rangle_G \simeq K_4$. Then*

(i) $N_G(x) \cap (y^+ \overrightarrow{C} x^-) = N_G(y) \cap (y^+ \overrightarrow{C} x^-)$,

(ii) $\langle(N_G(x) \cap (y^+ \overrightarrow{C} x^-)) \cup \{x, y\}\rangle_G$ *is a clique.*

Proof. By symmetry, it is sufficient to show that $N_G(y) \cap (y^+ \overrightarrow{C} x^-) \subset N_G(x) \cap (y^+ \overrightarrow{C} x^-)$. Let thus $z \in N_G(y) \cap (y^+ \overrightarrow{C} x^-)$. If $z = y^+$ or $z = x^-$, then obviously $z \in N_G(x)$. Hence we may assume $z \in y^{++} \overrightarrow{C} x^{--}$. Considering $\langle\{z, z^-, z^+, y\}\rangle_G$ we have $z^-y \in E(G)$ or $z^+y \in E(G)$. Suppose without loss of generality that $z^-y \in E(G)$ (otherwise we change the notation). Since $\{x, y\}$ is a cutset, $y^-z^- \notin E(G)$ and $y^-z \notin E(G)$. From

$\langle\{y, y^-, y^+, z\}\rangle_G \not\simeq K_{1,3}$ and $\langle\{y, y^-, y^+, z^-\}\rangle_G \not\simeq K_{1,3}$ we then get $y^+z \in E(G)$ and $y^+z^- \in E(G)$, i.e., $\langle\{y, y^+, z^-, z\}\rangle_G \simeq K_4$. From $\langle\{y^+, y^{++}, z, x\}\rangle_G \not\simeq K_{1,3}$ we now get $zx \in E(G)$ (since if $y^{++}x \in E(G)$, then $xy^{++} \overrightarrow{C} x^-y \overrightarrow{C} x$, and if $y^{++}z \in E(G)$, then $y^{++}z \overrightarrow{C} yz^- \overleftarrow{C} y^{++}$ is a cycle of length $n_G - 1$). Now, since $z^+x \notin E(G)$ (otherwise $xz^+ \overrightarrow{C} x^-y^+ \overrightarrow{C} z^-y \overleftarrow{C} x$ is a cycle of length $n_G - 1$), from $\langle\{z, z^-, z^+, x\}\rangle_G \not\simeq K_{1,3}$ we get also $z^-x \in E(G)$. Hence $N_G(y) \cap (y^+ \overrightarrow{C} x^-) \subset N_G(x) \cap (y^+ \overrightarrow{C} x^-)$.

If some $u, v \in N_G(x) \cap (y^+ \overrightarrow{C} x^-)$ are nonadjacent, then $\langle\{x, x^+, u, v\}\rangle_G$ is a claw. Hence $\langle(N_G(x) \cap (y^+ \overrightarrow{C} x^-)) \cup \{x, y\}\rangle_G$ is a clique. ∎

Lemma 11. *Let G be a minimal (with respect to $n_G = |V(G)|$) claw-free graph with complete closure and without a cycle of length $n_G - 1$. Let C be a hamiltonian cycle in G and let $\{x, y\}$ be a cutset of G such that $\langle\{x, x^-, y, y^+\}\rangle_G$ is a clique. Then $|x \overrightarrow{C} y| = |y^+ \overrightarrow{C} x^-| = n_G/2$.*

Proof. Let $G_1 = \langle x \overrightarrow{C} y\rangle_G$ and $G_2 = \langle y \overrightarrow{C} x\rangle_G$. Let H_1 be the graph obtained by taking two vertex disjoint copies of G_1 and by adding the edges $x^1x^2, y^1y^2, x^1y^2, x^2y^1$ (where by x^i, y^i we denote the vertices corresponding to the vertices x and y in the i-th copy of G_1, $i = 1, 2$), and let H_2 be the graph obtained by identifying the vertices corresponding to the vertices x and y in two vertex disjoint copies of G_2. Then, by Corollary 6, both H_1 and H_2 have complete closure. If some H_i, $i \in \{1, 2\}$, has a cycle of length $n_{H_i} - 1$, then, by the construction and since $\{x, y\}$ is a cutset, we apparently have a cycle of length $n_G - 1$ in G. Hence, by the minimality of G, $|V(H_i)| \geq n_G$, $i = 1, 2$. If we show that, moreover, $|V(H_2)| \geq n_G + 2$, then we have $|V(H_1)| = 2|x \overrightarrow{C} y| \geq n_G$ and $|V(H_2)| - 2 = 2|y^+ \overrightarrow{C} x^-| \geq n_G$. Since $|x \overrightarrow{C} y| + |y^+ \overrightarrow{C} x^-| = n_G$, this implies $|x \overrightarrow{C} y| = |y^+ \overrightarrow{C} x^-| = n_G/2$.

Hence it remains to show that $|V(H_2)| \geq n_G + 2$. Suppose, to the contrary, $|V(H_2)| \leq n_G + 1$, and let $H = (H_2)'_x$. Since $\{x, y\}$ is a cutset of H_2, by Lemma 10, y is simplicial in H. The graph $\hat{H} = H - \{x, y\}$ is obviously claw-free and, by Corollary 7, $\mathrm{cl}(\hat{H})$ is complete. Since $|V(\hat{H})| = |V(H_2)| - 2 \leq n_G + 1 - 2 = n_G - 1$, by the minimality of G, $\hat{H}$ has a cycle $C_{\hat{H}}$ of length $n_{\hat{H}} - 1$. Let $B = E(H) \setminus E(H_2)$. Since $\{x, y\}$ is a cutset of H_2, $|E(C_{\hat{H}}) \cap B| \geq 2$. By Theorem 8$(i)$, $C_{\hat{H}}$ can be chosen such that $|E(C_{\hat{H}}) \cap B| = 2$. Let $e_1 = u_1v_1$, $e_2 = u_2v_2$ be these edges. Since $\{x, y\}$ is a cutset of H_2, each of e_1, e_2 has its endvertices in different components of $H_2 - \{x, y\}$. By Lemma 10(ii), replacing in $C_{\hat{H}}$ the edges u_1v_1 and u_2v_2 by the paths u_1xv_1 and u_2yv_2, we get a cycle C_{H_2} in H_2 of length $n_{H_2} - 1$. Let P be the shorter of the paths $y \overrightarrow{C}_{H_2} x$ and $y \overleftarrow{C}_{H_2} x$. Then the cycle $x \overrightarrow{C} yPx$ is a cycle in G of length $n_G - 1$. This contradiction proves the lemma. ∎

Now we can proceed to the main result of this section.

Theorem 12. *Let G be a claw-free graph such that* $\mathrm{cl}(G)$ *is complete. Then G contains a cycle of length $n_G - 1$.*

Proof. Suppose the theorem fails and let G be a counterexample with minimum $n_G = |V(G)|$. Let C be a hamiltonian cycle in G. We first make two general observations.

(i) The cycle C has no 2-chords, i.e., for any chord uv of C, both $u \overrightarrow{C} v$ and $u \overleftarrow{C} v$ have at least two interior vertices.

(ii) If a vertex x has two nonadjacent neighbors u, v lying in the same component of $\langle N_G(x)\rangle_G$, then $x \in V_{EL}(G)$ (since if x is locally disconnected, then for any z in the other component of $\langle N_G(x)\rangle_G$, $\langle x, u, v, z\rangle_G$ is a claw).

These observations will be often used implicitly throughout the proof.

For any hamiltonian cycle C and an eligible vertex x we say that the vertex x is *of the first type with respect to* C, if there is an x^-, x^+-path of length 2 in $\langle N_G(x)\rangle_G$. In the other case (i.e., if all x^-, x^+-paths in $\langle N_G(x)\rangle_G$ have length at least 3), we say that x is *of the second type with respect to* C.

First suppose that the hamiltonian cycle C can be chosen such that there is a vertex $x \in V_{EL}(G)$ of the first type with respect to C. Let y be a common neighbor of x^- and x^+ in $\langle N_G(x)\rangle_G$. If $x^-y^- \in E(G)$, then $x^-y^- \overleftarrow{C} x^+y \overrightarrow{C} x^-$ is a cycle of length $n_G - 1$; thus $x^-y^- \notin E(G)$. From $\langle\{y, y^-, y^+, x^-\}\rangle_G$ we get $x^-y^+ \in E(G)$ and, by symmetry, $x^+y^- \in E(G)$. Since $\langle\{y, y^-, y^+, x\}\rangle_G$ cannot be a claw, we have $xy^- \in E(G)$ or $xy^+ \in E(G)$. By symmetry, we can suppose that $xy^+ \in E(G)$. Then $\langle\{x^-, x, y, y^+\}\rangle_G \simeq K_4$. We consider the conditions under which $\{x, y\}$ can be a cutset of G.

By Lemma 9, it is sufficient to verify the nonexistence of all possible pairs of parallel chords uv, u^+v^- such that $u, u^+ \in y \overrightarrow{C} x$ and $v^-, v \in x \overrightarrow{C} y$.

Case	Cycle of length $n_G - 1$
$u, u^+ \in y \overrightarrow{C} x^-$; $v^-, v \in x^+ \overrightarrow{C} y^-$	$uv \overrightarrow{C} y^-x^+ \overrightarrow{C} v^-u^+ \overrightarrow{C} x^-y \overrightarrow{C} u$
$u^+ = x$; $v^-, v \in x^+ \overrightarrow{C} y^-$	$xy^+ \overrightarrow{C} x^-v \overrightarrow{C} y^-x^+ \overrightarrow{C} v^-x$
$u, u^+ \in y^+ \overrightarrow{C} x^-$; $v = y$	$uyx^+ \overrightarrow{C} y^-u^+ \overrightarrow{C} x^-y^+ \overrightarrow{C} u$

We thus have the following observation.

($*$) The only possible pair of parallel chords uv, u^+v^- such that at least one of them crosses the edge xy, is for $v^- = x$, $v = x^+$; $u, u^+ \in y^+ \overrightarrow{C} x^-$.

(This observation will be used several times in what follows.)

We show that $xy^- \notin E(G)$. Indeed, if $xy^- \in E(G)$, then, by symmetry and by the previous observations, $\{x, y\}$ is a cutset of G. But then, since $\langle\{x, y, x^+, y^-\}\rangle_G \simeq \langle\{x, y, x^-, y^+\}\rangle_G \simeq K_4$, by Lemma 11 we have $|x^+ \overrightarrow{C} y^-| = |y \overrightarrow{C} x| = n_G/2$ and $|x \overrightarrow{C} y| =$

$|y^+ \overrightarrow{C}\, x^-| = n_G/2$, from which $n_G = |x^+ \overrightarrow{C}\, y^-| + |y^+ \overrightarrow{C}\, x^-| + |\{x, y\}| = n_G/2 + n_G/2 + 2 > n_G$, a contradiction. Hence $xy^- \notin E(G)$. Considering $\langle\{x^+, x, x^{++}, y^-\}\rangle_G$ we then have $x^{++}y^- \in E(G)$.

We now prove that $x^{++}y \in E(G)$. Thus suppose, to the contrary, $x^{++}y \notin E(G)$. Then from $\langle\{y^-, y, y^{--}, x^{++}\}\rangle_G$ we have $x^{++}y^{--} \in E(G)$. We show that $\{x, y\}$ is again a cutset. Suppose, to the contrary, $u, u^+ \in y^+ \overrightarrow{C}\, x^-$ and $x^+u, xu^+ \in E(G)$ (see the observation $(*)$). If $u = y^+$, then $x^+y^+ \overrightarrow{C}\, x^-y \overleftarrow{C}\, x^+$ is a cycle of length $n_G - 1$; thus $u \neq y^+$. If $x^{++}u \in E(G)$, then $x^{++} \overrightarrow{C}\, yxu^+ \overrightarrow{C}\, x^-y^+ \overrightarrow{C}\, ux^{++}$ is a cycle of length $n_G - 1$. Thus, since $\langle\{x^+, x^{++}, y, u\}\rangle_G$ cannot be a claw, we have $yu \in E(G)$. From $\langle\{u, u^-, u^+, x^+\}\rangle_G$ then $u^-x^+ \in E(G)$ or $u^+x^+ \in E(G)$, but then in the first case $x^+ \overrightarrow{C}\, yu \overrightarrow{C}\, x^-y^+ \overrightarrow{C}\, u^-x^+$ and in the second case $x^+u^+ \overrightarrow{C}\, x^-y^+ \overrightarrow{C}\, uy \overleftarrow{C}\, x^+$ is a cycle of length $n_G - 1$. Hence $\{x, y\}$ is a cutset.

We show that x and y have no other neighbors except x^+ and y^- on $x^+ \overrightarrow{C}\, y^-$. Thus, first let, by Lemma 9, $xv \in E(G)$ and $x^+v^- \in E(G)$ for $v^-, v \in x^{++} \overrightarrow{C}\, y^{--}$. Then $xv \overrightarrow{C}\, y^{--}x^{++} \overrightarrow{C}\, v^-x^+y \overrightarrow{C}\, x$ is a cycle of length $n_G - 1$. Secondly, let $yv^- \in E(G)$ and $y^-v \in E(G)$ for some $v^-, v \in x^{++} \overrightarrow{C}\, y^{--}$. From $\langle\{y, y^+, x^+, v^-\}\rangle_G$ we have $v^-x^+ \in E(G)$. Considering $\langle\{v^-, v, v^{--}, y\}\rangle_G$ we now get $vy \in E(G)$ or $v^{--}y \in E(G)$, but then $x^{++} \overrightarrow{C}\, v^-x^+ \overleftarrow{C}\, yv \overrightarrow{C}\, y^{--}x^{++}$ in the first case and $x^{++} \overrightarrow{C}\, v^{--}y \overrightarrow{C}\, x^+v^- \overrightarrow{C}\, y^{--}x^{++}$ in the second case, respectively, is a cycle of length $n_G - 1$. Hence $N_G(x) \cap (x^+ \overrightarrow{C}\, y^-) = \{x^+\}$ and $N_G(y) \cap (x^+ \overrightarrow{C}\, y^-) = \{x^+, y^-\}$.

Since, by Lemma 10, $N_G(x) \cap (y^+ \overrightarrow{C}\, x^-) = N_G(y) \cap (y^+ \overrightarrow{C}\, x^-)$ and obviously $y \in V_{EL}(G)$, $x \in V_{SI}(G'_y)$. Then, similarly as in the proof of Lemma 11, the graph $H = G'_y - \{x, y\}$ is claw-free, $\mathrm{cl}(H)$ is complete and hence H has a cycle C_H of length $n_H - 1 = n_G - 3$ such that $E(C_H) \cap (E(G'_y) \setminus E(G)) = \{e_1, e_2\}$ for some $e_1 = u_1v_1$ and $e_2 = u_2v_2$ having endvertices in different components of $G - \{x, y\}$. Since $N_G(y) \cap (x^+ \overrightarrow{C}\, y^-) = \{x^+, y^-\}$ and $N_G(x) \cap (x^+ \overrightarrow{C}\, y^-) = \{x^+\}$, we can suppose that $u_1 = x^+$ and $u_2 = y^-$. Then, replacing u_1v_1 by u_1xv_1 and u_2v_2 by u_2yv_2, we get a cycle in G of length $n_G - 1$. This contradiction proves that $x^{++}y \in E(G)$. Hence $\langle\{x, y, y^+, x^-\}\rangle_G \simeq \langle\{x^+, x^{++}, y^-, y\}\rangle_G \simeq K_4$.

We show that $\{x, y\}$ or $\{x^+, y\}$ is a cutset of G. Indeed, if not, then, by the observation $(*)$, there are $u, u^+ \in y^+ \overrightarrow{C}\, x^-$ and $v^-, v \in x^{++} \overrightarrow{C}\, y^-$ such that $\{xv, xu^+, x^+v^-, x^+u\} \subset E(G)$, but then $xu^+ \overrightarrow{C}\, x^-y^+ \overrightarrow{C}\, ux^+v^- \overleftarrow{C}\, x^{++}y^- \overleftarrow{C}\, vx$ is a cycle of length $n_G - 1$. Thus, by symmetry, we can suppose that $\{x, y\}$ is a cutset of G.

Now, $\{x^+, y\}$ cannot be also a cutset of G, since otherwise Lemma 11 implies $|x^{++} \overrightarrow{C}\, y^-| = |y^+ \overrightarrow{C}\, x^-| = n_G/2$, from which $n_G = |x^{++} \overrightarrow{C}\, y^-| + |y^+ \overrightarrow{C}\, x^-| + |\{x, x^+, y\}| = 2n_G/2 + 3 > n_G$, a contradiction. Thus, by the observation $(*)$, there are $v^-, v \in x^{++} \overrightarrow{C}\, y^-$ such that $xv \in E(G)$ and $x^+v^- \in E(G)$. Apparently $|x^{++} \overrightarrow{C}\, v^-| \geq 4$ and $|v \overrightarrow{C}\, y^-| \geq 4$

(otherwise we easily obtain a cycle of length $n_G - 1$). If $xv^+ \in E(G)$, then $xv^+ \overrightarrow{C} y^- x^{++} \overrightarrow{C} v^- x^+ y \overrightarrow{C} x$, and if $x^+v^{--} \in E(G)$, then $xv \overrightarrow{C} y^- x^{++} \overrightarrow{C} v^{--} x^+ y \overrightarrow{C} x$ is a cycle of length $n_G - 1$. Hence both $xv^+ \notin E(G)$ and $x^+v^{--} \notin E(G)$, from which, considering $\langle\{v, v^-, v^+, x\}\rangle_G$ and $\langle\{v^-, v^{--}, v, x^+\}\rangle_G$, we have $xv^- \in E(G)$ and $x^+v \in E(G)$, i.e. $\langle\{x, x^+, v^-, v\}\rangle_G \simeq K_4$.

Let $K_1 = \langle N_G(x) \cap (x^+ \overrightarrow{C} y^-)\rangle_G$ and $K_2 = \langle N_G(y) \cap (x^+ \overrightarrow{C} y^-)\rangle_G$. Since $\{x, y\}$ is a cutset of G, both K_1 and K_2 is a clique (otherwise some two nonadjacent vertices together with x^- or y^+ form a claw centered at x or at y). Since $x^+ \in V(K_1) \cap V(K_2)$, $N_G(x^+) \cup \{x^+\} \setminus \{x, y\} \supset (V(K_1) \cup V(K_2))$.

We show that $N_G(x^+) \cup \{x^+\} \setminus \{x, y\} = (V(K_1) \cup V(K_2))$. Suppose, to the contrary, $z \in N_G(x^+) \setminus (\{x, y\} \cup V(K_1) \cup V(K_2))$. Since $\{x, y\}$ is a cutset, $z \in x^+ \overrightarrow{C} y^-$. By the definition of K_1 and K_2 and by symmetry, we can suppose that $z \in v^+ \overrightarrow{C} y^{--}$. If $z = v^+$, then $xv^- \overleftarrow{C} x^+ z \overrightarrow{C} x$, and if $z = y^{--}$, then $x^+ z \overleftarrow{C} x^{++} y \overrightarrow{C} x^+$ is a cycle of length $n_G - 1$, hence $v^+ \neq z \neq y^{--}$. From $\langle\{z, z^-, z^+, x^+\}\rangle_G$ we have $z^-x^+ \in E(G)$ or $z^+x^+ \in E(G)$. By symmetry, suppose that $z^+x^+ \in E(G)$. Then, similarly as above, $z^+ \neq y^{--}$. Since $z, y^- \notin N_G(x)$, from $\langle\{x^+, z, y^-, x\}\rangle_G$ we have $zy^- \in E(G)$. Since $z, y^{--} \notin N_G(y)$, from $\langle\{y^-, y, y^{--}, z\}\rangle_G$ we have $zy^{--} \in E(G)$, but then $x^+ z^+ \overrightarrow{C} y^{--} z \overleftarrow{C} x^{++} y \overrightarrow{C} x^+$ is a cycle of length $n_G - 1$. This contradiction proves that $N_G(x^+) \cup \{x^+\} \setminus \{x, y\} = (V(K_1) \cup V(K_2))$.

Let $H_1 = G'_{x^+}$ and $H_2 = (H_1)'_y$. Since $N_G(x^+) \cup \{x^+\} \setminus \{x, y\} = (V(K_1) \cup V(K_2))$ and, by Lemma 10, $N_G(x) \cap (y^+ \overrightarrow{C} x^-) = N_G(y) \cap (y^+ \overrightarrow{C} x^-)$, implying $N_G(x) \subset N_G(y) \cup N_G(x^+)$, we have $\{x, y, x^+\} \subset V_{SI}(H_2)$. The graph $H = H_2 - \{x, y, x^+\}$ thus has a complete closure. Let $B_1 = E(H_1) \setminus E(G)$ and $B_2 = E(H_2) \setminus E(H_1)$. Then, by the minimality of G and by Theorem 8(ii), H has a cycle C_H of length $n_H - 1 = n_G - 4$ such that either $|E(C_H) \cap B_1| \leq 2$ and $|E(C_H) \cap B_2| = 0$, or $|E(C_H) \cap B_1| \leq 1$ and $|E(C_H) \cap B_2| \leq 2$. Since $\{x, y\}$ is a cutset of G, at least two edges of $E(C_H) \cap (B_1 \cup B_2)$ have an endvertex in $y^+ \overrightarrow{C} x^-$. Since $N_G(x^+) \subset x \overrightarrow{C} y$, this implies $|E(C_H) \cap B_2| \geq 2$. Hence $|E(C_H) \cap B_1| \leq 1$ and $|E(C_H) \cap B_2| = 2$. Let $e_1 = u_1v_1$, $e_2 = u_2v_2$ be the two edges in $E(C_H) \cap B_2$ and (if nonempty), $e_3 = u_3v_3$ be the only edge in $E(C_H) \cap B_1$. By the above, we can suppose that $\{u_1, u_2, u_3, v_3\} \subset x^{++} \overrightarrow{C} y^-$ and $\{v_1, v_2\} \subset y^+ \overrightarrow{C} x^-$.

If $u_1 \in V(K_1)$ and $u_2 \in V(K_2)$, then, replacing in C_H the edge u_1v_1 by the path u_1xv_1, the edge u_2v_2 by the path $u_2x^+yv_2$ (if $E(C_H) \cap B_1 = \emptyset$) or by the path u_2yv_2 (if $E(C_H) \cap B_1 \neq \emptyset$) and the edge u_3v_3 (if any) by the path $u_3x^+v_3$, we obtain a cycle of length $n_G - 1$ in G. If $u_1, u_2 \in V(K_1)$ and $B_1 = \emptyset$, then we analogously replace in C_H the edges u_1v_1 and u_2v_2 by the paths u_1xv_1 and $u_2x^+yv_2$. Since $N_G(x^+) \cup \{x^+\} \setminus \{x, y\} = (V(K_1) \cup V(K_2))$, it remains to consider (up to symmetry) the case when $u_1, u_2 \in V(K_1)$ and $B_1 \neq \emptyset$. Since $u_3, v_3 \in N_G(x^+)$ and $u_3v_3 \notin E(G)$, we have $u_3 \in V(K_1)$ and $v_3 \in V(K_2)$, or $v_3 \in V(K_1)$ and $u_3 \in V(K_2)$. Let P_1, P_2, P_3 be the three paths that C_H splits into by deleting e_1, e_2, e_3

and suppose the notation is chosen such that, in the path system obtained by deleting e_1, e_2, e_3 from C_H, u_3 and u_1 are endvertices of the same path (this is always possible since there can be no path joining u_3, v_3 and since $\{x, y\}$ is a cutset of G). Then, replacing in C_H the edges e_1, e_2, e_3 by u_1xv_1, u_2u_3 and $v_2yx^+v_3$ if $u_3 \in V(K_1)$ and $v_3 \in V(K_2)$, or by u_2xv_2, u_1v_3 and $u_3x^+yv_1$ if $v_3 \in V(K_1)$ and $u_3 \in V(K_2)$, respectively, we obtain a cycle of length $n_G - 1$ in G.

This contradiction proves that for any choice of a hamiltonian cycle C in G, no eligible vertex of G is of the first type with respect to C.

Let now C be a hamiltonian cycle in G and x an eligible vertex (of second type with respect to C). Let P be a shortest x^-, x^+-path in $\langle N_G(x)\rangle_G$. Since G is claw-free, P is of length 3. Let $V(P) = x^-y_1y_2x^+$. Then either $y_1 \in x \overrightarrow{C} y_2$, or $y_2 \in x \overrightarrow{C} y_1$.

<u>Case 1:</u> $y_1 \in x \overrightarrow{C} y_2$. We consider $\langle\{y_1, y_1^-, y_1^+, x^-\}\rangle_G$ and $\langle\{y_2, y_2^-, y_2^+, x^+\}\rangle_G$. If both $x^-y_1^+ \in E(G)$ and $x^+y_2^- \in E(G)$, then the cycle $x^-y_1^+ \overrightarrow{C} y_2^-x^+ \overrightarrow{C} y_1y_2 \overrightarrow{C} x^-$ is a cycle of length $n_G - 1$ in G. Hence we can suppose (by symmetry) that $x^-y_1^- \in E(G)$. Then, on the cycle $C' = xy_1 \overrightarrow{C} x^-y_1^- \overleftarrow{C} x$, the predecessor of x is x^+ and the successor is y_1. Since y_1 and x^+ have a common neighbor $y_2 \in N_G(x)$, x is of type 1 with respect to C' - a contradiction.

<u>Case 2:</u> $y_2 \in x \overrightarrow{C} y_1$. We first show that x can be chosen such that y_2, y_1 are not consecutive on C. Suppose, to the contrary, that this is not the case and choose x such that $x \overrightarrow{C} y_2$ is shortest possible. Since x is of type 2, $x^+y_1 \notin E(G)$, and from $\langle\{y_2, y_1, y_2^-, x^+\}\rangle_G$ we have $x^+y_2^- \in E(G)$. Similarly $xy_2^- \notin E(G)$ (otherwise y_2 is of type 1 with respect to C) and from $\langle\{x^+, x, x^{++}, y_2^-\}\rangle_G$ we have $x^{++}y_2^- \in E(G)$. But then the path $xy_2y_2^-x^{++}$ in $\langle N_G(x^+)\rangle_G$ contradicts the choice of x. Hence we may assume that $y_2^+ \neq y_1$.

Suppose now that $x^-y_1^- \in E(G)$ and let $C' = x \overrightarrow{C} y_1^-x^- \overleftarrow{C} y_1x$. Then the predecessor y_1 and successor x^+ of x on C' have a common neighbor $y_2 \in N_G(x)$ and hence x is of type 1 with respect to C', a contradiction. Hence $x^-y_1^- \notin E(G)$ and, by symmetry, $x^+y_2^+ \notin E(G)$. Considering $\langle\{y_1, y_1^-, y_1^+, x^-\}\rangle_G$ and $\langle\{y_2, y_2^-, y_2^+, x^+\}\rangle_G$ we then get $y_1^+x^- \in E(G)$ and $y_2^-x^+ \in E(G)$.

We show that $xy_2^- \in E(G)$. If $xy_2^- \notin E(G)$, then from $\langle\{y_2, y_2^-, y_2^+, x\}\rangle_G$ we have $xy_2^+ \in E(G)$, and since we already know that $x^+y_2^+ \notin E(G)$, from $\langle\{x, x^-, x^+, y_2^+\}\rangle_G$ we get $x^-y_2^+ \in E(G)$. Considering $\langle\{y_1, y_1^-, y_1^+, y_2\}\rangle_G$ we then have $y_2y_1^- \in E(G)$ or $y_2y_1^+ \in E(G)$, but in the first case the cycle $C' = x \overrightarrow{C} y_2y_1^- \overleftarrow{C} y_2^+x^- \overleftarrow{C} y_1x$ and in the second case the cycle $C' = x \overrightarrow{C} y_2y_1^+ \overrightarrow{C} x^-y_2^+ \overrightarrow{C} y_1x$ yields a contradiction, since in both these cases x is of type 1 with respect to C'. Hence $xy_2^- \in E(G)$ and, by symmetry, $xy_1^+ \in E(G)$, which implies that $\langle\{x, x^+, y_2^-, y_2\}\rangle_G \simeq \langle\{x, x^-, y_1^+, y_1\}\rangle_G \simeq K_4$.

Now consider $\langle\{y_2, y_2^+, y_1, x^+\}\rangle_G$. If $x^+y_1 \in E(G)$, then x is of first type with respect to C; thus $x^+y_1 \notin E(G)$. Since we already know that $x^+y_2^+ \notin E(G)$, we have $y_1y_2^+ \in E(G)$.

Since $y_2^- x y_1 y_2^+$ is a path in $\langle N_G(y_2)\rangle_G$ and $y_2^- y_2^+ \notin E(G)$, by the observation (ii) we have $y_2 \in V_{EL}(G)$. Thus, by the previous argument, $\langle\{y_2, y_2^+, y_1^-, y_1\}\rangle_G \simeq K_4$.

We show that $\{y_1, y_2\}$ is a cutset of G. Suppose, to the contrary, that (recall Lemma 9) uv, u^+v^- is a pair of parallel chords such that at least one of them crosses y_1y_2, i.e. such that $u, u^+ \in y_2 \overrightarrow{C} y_1$ and $v^-, v \in y_1 \overrightarrow{C} y_2$.

Case	Cycle	Vertex of type 1
$u, u^+ \in y_2^+ \overrightarrow{C} y_1^-; v^-, v \in y_1 \overrightarrow{C} y_2$	$y_1 \overrightarrow{C} v^- u^+ \overrightarrow{C} y_1^- y_2^+ \overrightarrow{C} uv \overrightarrow{C} y_2 y_1$	y_1
$u = y_2; v = y_1^+$	C	y_1
$u = y_2; v^-, v \in y_1^+ \overrightarrow{C} x^-$	$y_1 x \overrightarrow{C} y_2 v \overrightarrow{C} x^- y_1^+ \overrightarrow{C} v^- y_2^+ \overrightarrow{C} y_1$	y_1
$u = y_2; v = x$	$x \overrightarrow{C} y_2 y_1^- \overleftarrow{C} y_2^+ x_1^- \overleftarrow{C} y_1 x$	x
$u = y_2; v = x^+$	C	y_2
$u = y_2; v^-, v \in x^+ \overrightarrow{C} y_2^-$	$x y_2 v \overrightarrow{C} y_2^- x^+ \overrightarrow{C} v^- y_2^+ \overrightarrow{C} x$	x

Since these are, up to symmetry, all possibilities, $\{y_1, y_2\}$ is a cutset of G. By symmetry, $\{x, y_1\}$ and $\{x, y_2\}$ are also cutsets of G. But then, by Lemma 11, $|x^+ \overrightarrow{C} y_2^-| = |y_2^+ \overrightarrow{C} y_1^-| = |y_1^+ \overrightarrow{C} x^-| = n_G/2$, from which $n_G = |x^+ \overrightarrow{C} y_2^-| + |y_2^+ \overrightarrow{C} y_1^-| + |y_1^+ \overrightarrow{C} x^-| + |\{x, y_1, y_2\}| = 3n_G/2 + 3 > n_G$, a contradiction. ■

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